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Lump solutions to nonlinear PDEs involving Hirota derivative $D_t^2 D_x D_y$

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This paper aims to study lump solutions to a class of $(2+1)$-dimensional nonlinear PDE systems, which involve the fourth-order Hirota derivative term: $D_t^2 D_x D_y$. This Hirota derivative term generates higher-order derivatives of the temporal variable. Lump solutions to the resulting new class of nonlinear PDE systems are studied in detail via the Hirota bilinear method.

Keywords: Lump solution; Hirota bilinear derivative; dispersion relation.

1. Introduction

As is well known, Hirota's bilinear method[1] is one of the most powerful tools to search for multiple-soliton solutions (stable, localized, exponentially decaying at infinity). Recently, applying Hirota's bilinear method to search for lump solutions is an ad hoc problem and the results are fruitful. Lump solutions (see, e.g. Ref. 2) are a kind of spatially localized and rationally decaying solutions.

An early example of constructing lump solution is due to Manakov and his collaborators,[3] they obtained special lumps of the KPI equation by first constructing multi-solitons and then taking the long-wave limit. Based on this method, many integrable equations are found to have lump solutions such as the Davey–Stewartson II equation[4] and the Ishimori-I equation.[5] Recently, in 2015, one of the authors[6] applied Hirota's bilinear method to construct a complete set of second-order lump solutions for the KP equation in a straightforward and effective way.

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Based on Hirota's bilinear method, lump solutions of many famous integrable equations are algorithmically obtained. Those examples include the BKP equation[7,8] and its generalization,[9] the generalized Bogoyavlensky–Konopelchenko equation,[10]the Hirota–Satsuma–Ito equation[11] just to name a few. Besides those (2+1)-dimensional equations, there also exist higher-dimensional examples, such as (3+1)-dimensional KP equation and its generalizations[12,13] and (3+1)-dimensional BKP–Boussinesq equation.[14] Lump solutions are a kind of solutions which are stable and detectable and the peaks of a solution move along a straight line. All these properties enable us to study the interaction between lump solutions and solitons, hence lump solutions play a significant role in physics. Moreover, it is also important to study the time evolution of an integrable equation with given a lump initial data.[15]

In this study, we will mainly focus on the computation part, and lump solutions to a new type of $(2+1)$-dimensional nonlinear PDEs will be construed algorithmically. We first simply describe the main idea of the method, for more details with rigorous proofs, see Ref. 2. Given a nonlinear PDE or a coupled PDE system in general,

$$P(u) = 0\,,$$

where P is a differential polynomial with respect to spatial variables. Then via the bilinear transform or Hopf transform, $u = 2\log(f)_{xx}$ or $u = 2\log(f)_x$, we arrive at a bilinear differential equation

$$B(f) = 0\,.$$

The relation between the original differential equation(s) $P(u)$ and the bilinearized differential equation(s) $B(f)$ is given by

$$P(u) = (B(f)/f^2)_x \text{ or } B(f)/f^2\,.$$

The most difficult part of the method is to find Hirota's representation:

$$B(f) = P(D_x, D_y, D_t)f \cdot f\,,$$

where $P(D_x, D_y, D_t)$ is a polynomial of the Hirota derivatives.

In many integrable equations, Hirota's representation involves terms like: $D_tD_x^3$, $D_x^2D_y^2$ and D_x^4 and so on, and we find that there are no terms for higher-order Hirota derivative with respect to t. This gives us the motivation to study a type of PDEs which involves Hirota's derivatives $D_t^2D_xD_y$. After one finds Hirota's representation, following the theorems in Ref. 2, all lower degree lump solutions share the following form:

$$f = (a_1t + a_2x + a_3y + a_4)^2 + (b_1t + b_2x + b_3y + b_4)^2 + e\,.$$

The rest of the work is to determine the involved coefficients. Lastly, to ensure that the solution is a lump solution, one needs to check that $a_2b_3 - a_3b_2 \neq 0$.

In this paper, we would like to study a class of coupled PDE systems in $(2+1)$ dimensions. Lump solutions are constructed via the method described in the paper.[6] In Sec. 3, we give some relations between the coefficients of the solution and the Hirota derivatives, which are used to find the explicit solutions. The notation introduced there can significantly reduce the length of the expression. In Sec. 5, an illustrative example for a reduced case is presented, along with 3D plots for each component in the coupled system.

2. A Family of Nonlinear PDEs with 4th Order Hirota's Form

In this paper, we will focus on nonlinear PDEs in $\mathbb{R}^3$ whose corresponding Hirota bilinear forms are of fourth-order. Let us assume that the independent variables are $x_1, x_2, x_3 \in \mathbb{R}$. And we define the Hirota derivatives with respect to the jth independent variable as follows:

$$D_j^{k_j} f \cdot f = \sum_{n=0}^{k_j} \frac{k_j!}{n!(k_j - n)!(-1)^n} (\partial_j f)^n (\partial_j f)^{k_j - n} ,$$

where $\partial_j f = \frac{\partial f(x_1,x_2,x_3)}{\partial x_j}, j = 1, 2, 3$. In the following contents, we set $x_1 = t, x_2 = y, x_3 = y$.

Due to the cyclic symmetry of x_1, x_2, x_3, there are actually only four types of 4th powers Hirota derivatives we need to discuss, which are $D_x^4, D_t D_x^3, D_x^2 D_y^2, D_t^2 D_x D_y$.

Now, we would like to introduce a class of nonlinear PDEs:

$$\begin{aligned} P(u) = {} & \alpha_1(6uu_x + u_{xxx}) + \alpha_2(3us_x + u_{tx})_x + \alpha_3(2v_x^2 + uv_y + u_{yy})_x \\ & + \alpha_4(v_x s_t + 2s_x s_y + s_{txy})_x + d_1 u_x + d_2 v_{tx} , \end{aligned} \tag{1}$$

where $u = w_x, s_x = w_t, v_x = w_y$,and the coefficients $\alpha_j, j = 1, 2, 3, 4$ and $d_j, j = 1, 2$ are arbitrary. Then by applying the Hopf transform $w = 2(\log(f))_x$, we arrive at the following Hirota bilinear form:

$$B(f) = (\alpha_1 D_x^4 + \alpha_2 D_t D_x^3 + \alpha_3 D_x^2 D_y^2 + \alpha_4 D_t^2 D_x D_y d_1 D_x^2 + d_2 D_t D_y) f \cdot f = 0 . \tag{2}$$

Remark 1. In our class of NLPDEs, the fourth order term such as $D_x^3 D_y$ is not included, which has been analyzed in previous work[16,17] by one of the authors. In other words, the main purpose of this paper is to study how the Hirota bilinear form generated by types $(2, 1, 1)$, i.e. $D_t^2 D_x D_y$, affects the process of solving for lump solutions.

3. Relations between Hirota's Form and the Coefficients of the Quadratic Solution

In this section, we will study the relations between the coefficients of the quadratic function and the Hirota bilinear form.

First of all, following the theorems in Ref. 2, the low order general lump solution to $(2 + 1)$-dimensional NLPDEs is by setting f in the bilinear form as

$$f(x, y, t) = (a_1 t + a_2 x + a_3 y + a_4)^2 + (b_1 t + b_2 x + b_3 y + b_4)^2 + e . \tag{3}$$

Due to Corollary 3.5 in Ref. 2, if f solves $B(f) = 0$, then a_4, b_4 are arbitrary. Hence, the main computation is to represent a_1, b_1, e in terms of the rest of the coefficients.

Now, introduce the following compact notations:

$$\begin{aligned} N_{\pm}(i,j) &= a_i a_j \pm b_i b_j \,, \\ M_{\pm}(i,j) &= a_i b_j \pm b_i a_j \,. \end{aligned} \tag{4}$$

Then by direct calculating, we obtain

$$\begin{aligned} D_x^4 f \cdot f &= 24N_+^2(2,2)\,, \\ D_t D_x^3 f \cdot f &= 24N_+(2,2)N_+(1,2)\,, \\ D_x^2 D_y^2 f \cdot f &= 8M_-^2(2,3) + 24M_+^2(2,3)\,, \\ D_t^2 D_x D_y f \cdot f &= 8N_+(2,3)N_+(1,1) + 16a_2a_3a_1^2 + 16b_1M_+(2,3)a_1 + 16b_2b_3b_1^2\,. \end{aligned} \tag{5}$$

As we can see, the fourth-order Hirota derivative acting on the quadratic functions, f, only generates some constant terms. Moreover, the degree of the coefficients is exactly the order of the corresponding order of the Hirota derivative, i.e. in the second equation, the order of a_2 equals the order of D_x which is 3.

Next, we compute the terms generated by the second-order Hirota derivative, in fact, there are only two types $(2,0)$ or $(1,1)$, hence, we list the coefficients related to D_x^2 and D_xD_y.

For $D_x^2 f \cdot f$, we obtain a new quadratic function which has 9 coefficients, namely

$$\begin{aligned} x^2 &: -4N_+^2(2,2)\,, \\ y^2 &: -4N_-(2,2)N_-(3,3)\,, \\ t^2 &: -4N_-(1,1)N_-(2,2) - 16a_1a_2b_1b_2\,, \\ tx &: -8N_+(2,2)N_+(1,2)\,, \\ ty &: -8N_-(1,3)N_-(2,2) - 16M_+(1,3)a_2b_2\,, \\ xy &: -8N_+(2,2)N_+(2,3)\,, \\ x &: -8N_+(2,2)N_+(2,4)\,, \\ y &: -8N_-(3,4)N_-(2,2) - 16M_+(3,4)a_2b_2\,, \\ t &: -8N_-(1,4)N_-(2,2) - 16M_+(1,4)a_2b_2\,, \\ 1 &: [-4N_-(4,4) + 4e]N_-(2,2) - 16a_2b_2a_4b_4\,. \end{aligned} \tag{6}$$

For $D_x D_y f \cdot f$, we have

$$\begin{aligned}
x^2 &: -4N_+(2,2)N_+(2,3)\,,\\
y^2 &: -4N_-(2,3)N_-(3,3)\,,\\
t^2 &: -4N_-(1,1)N_-(2,3) - M_+(2,3)a_1b_1\,,\\
tx &: -8N_+(2,2)N_+(1,3)\,,\\
ty &: -8N_-(3,3)N_-(1,2)\,,\\
xy &: -8N_+(2,2)N_+(3,3)\,,\\
x &: -8N_+(2,2)N_+(3,4)\,,\\
y &: -8N_-(3,3)N_-(2,4)\,,\\
t &: -(8a_2-b_2)N_+(3,4) - 8M_-(3,4)M_-(1,2)\,,\\
1 &: -4N_-(4,4)N_-(2,3) + 4eN_+(2,3) - 8M_-(2,3)a_4b_4\,.
\end{aligned} \tag{7}$$

For the rest of the second order terms, the coefficients can be carried out by using cyclic symmetry of the independent variables (x, y, t).

Now, let us consider the following differential polynomial:

$$\begin{aligned}
P(u) &= \alpha_1(6uu_x + u_{xxx}) + \alpha_2(3us_x + u_{tx})_x + \alpha_3(2v_x^2 + uv_y + u_{yy})_x\\
&\quad + \alpha_4(v_x s_t + 2s_x s_y + s_{txy})_x + d_1u_x + d_2v_{tx}\,.
\end{aligned} \tag{8}$$

By a straightforward symbolic computation, we have a set of solutions for the parameters as:

$$\begin{aligned}
a_1 &= -\frac{d_1(N_-(2,2)a_3 + 2a_2b_2b_3)}{d_2N_+(3,3)}\,,\\
b_1 &= -\frac{d_1(N_-(2,2)b_3 - 2a_2b_2a_3)}{d_2N_+(3,3)}\,.
\end{aligned} \tag{9}$$

The constant e can also be represented in terms of $a_2, a_3, a_4, b_2, b_3, b_4$ and the coefficients coming from the original equation. Here, we introduce a way using the notation giving in the last section. Since the equations related to e appear only at the constant terms, that means we need to solve the following linear equation to get e:

$$\begin{aligned}
&\alpha_1(24N_+^2(2,2)) + \alpha_2(24N_+(2,2)N_+(1,2)\\
&\quad + \alpha_3(8M_-^2(2,3) + 24M_+^2(2,3)) + \alpha_4(8N_+(2,3)N_+(1,1) + 16a_2a_3a_1^2\\
&\quad + 16b_1M_+(2,3)a_1 + 16b_2b_3b_1^2)\\
&\quad + d_1([-4N_-(4,4) + 4e]N_-(2,2) - 16a_2b_2a_4b_4)\\
&\quad + d_2(-4N_-(4,4)N_-(2,3) + 4eN_+(2,3) - 8M_-(2,3)a_4b_4) = 0\,.
\end{aligned} \tag{10}$$

Then replacing a_1, b_1 using the equations we get in (9), we finally arrive at an expression in terms of $a_2, a_3, a_4, b_2, b_3, b_4$ and the coefficients coming from the original equation. In fact, we see that e equals

$$
\begin{aligned}
&[4d_1 N-(2,2)+4d_2N_+(2,3)]^{-1}\{\alpha_1(24N_+^2(2,2))\\
&\quad -\alpha_2\frac{d_1}{d_2N_+(3,3)}(24N_+(2,2)(N_-(2,2)N_+(2,3)+2a_2b_2M_-(2,3))\\
&\quad +\alpha_3(8M_-^2(2,3)+24M_+^2(2,3))\\
&\quad +\alpha_4(\frac{8N_+(2,3)d_1^2}{d_2^2N_+^2(3,3)}([N_-(2,2)a_3+2a_2b_2b_3]^2+[N_-(2,2)b_3+2a_2b_2a_3]^2)\\
&\quad +\frac{16a_2a_3d_1^2}{d_2^2N_+^2(3,3)}[N_-(2,2)a_3+2a_2b_2b_3]^2\\
&\quad +\frac{16M_+(2,3)d_1^2}{d_2^2N_+^2(3,3)}(N_-^2(2,2)a_3b_3+4a_2^2b_2^2a_3b_3+2N_-(2,2)N_+(3,3)a_2b_2)\\
&\quad +\frac{16b_2b_3d_1^2}{d_2^2N_+^2(3,3)}[N_-(2,2)b_3+2a_2b_2a_3]^2)\\
&\quad +d_1(-4N_-(4,4)N_-(2,2)-16a_2b_2a_4b_4)\\
&\quad +d_2(-4N_-(4,4)N_-(2,3)-8M_-(2,3)a_4b_4)\}\,. \qquad (11)
\end{aligned}
$$

Though (11) is very complicated to read, the symbolic computer algebra system really brings many benefits for figuring out the exact solutions, especially, the lump solutions.

Remark 2. In theory, by the algorithm we introduced above, we can compute all the lump solutions to the class of equations given by (1).

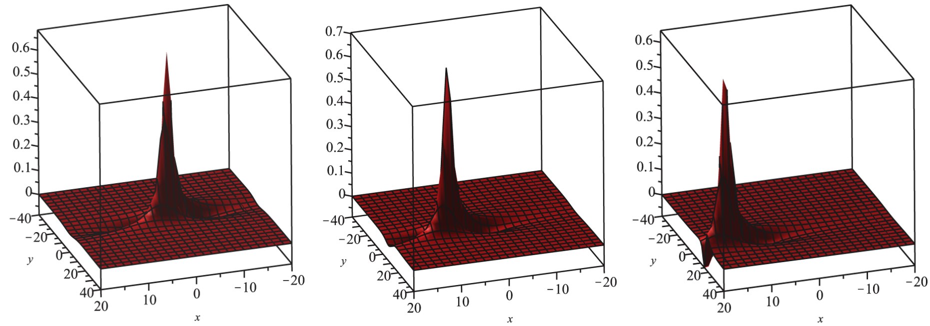
4. An Illustrative Example

Let us consider the situation when

$$\alpha_1=1,\alpha_2=2,\alpha_3=1,\alpha_4=2,d_1=-1,d_2=1\,, \qquad (12)$$

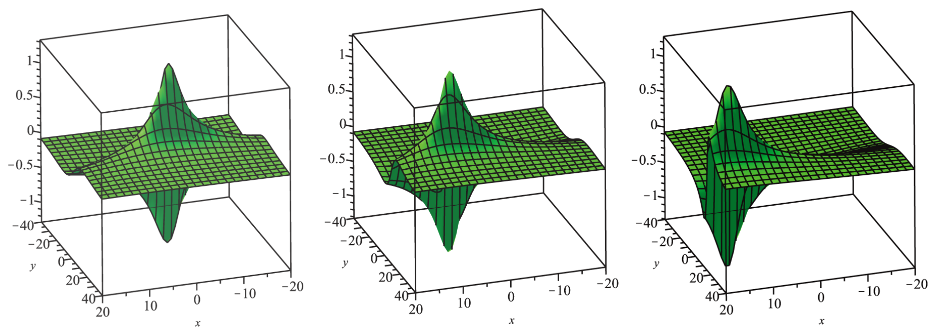
which leads to the following coupled nonlinear PDEs:

$$
\begin{cases}
(6uu_x+u_{xxx})+2(3us_x+u_{tx})+(2v_x^2+uv_y+u_{yy})\\
\quad +2(v_xs_t+2s_xs_y+s_{txy})-w_{xx}+w_{ty}=0\,,\\
u=w_x\,,\\
s_x=w_t\,,\\
v_x=w_y\,.
\end{cases} \qquad (13)
$$



(a) u at $t = 0$. (b) u at $t = 20$. (c) u at $t = 40$.

Fig. 1. Profiles of u.



(a) v at $t = 0$. (b) v at $t = 20$. (c) v at $t = 40$.

Fig. 2. Profiles of v.

Then based on the previous computation, since $a_j, b_j, j = 2, 3, 4$ can be chosen arbitrarily, we take

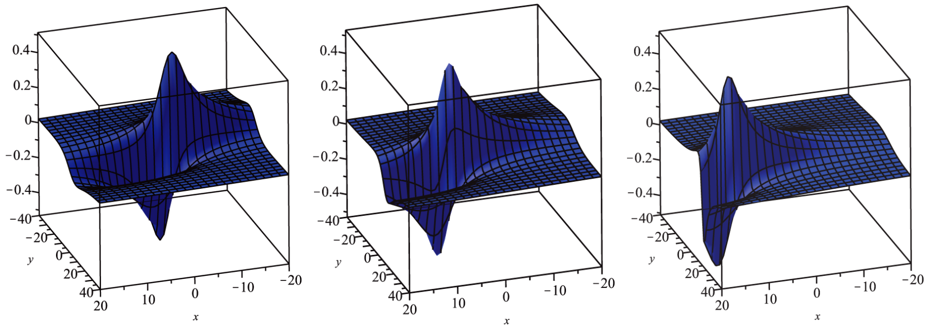
$$\{a_2 = 1, a_3 = 1, a_4 = 2, b_2 = -1, b_3 = 2, b_4 = 2\}\ . \tag{14}$$

Plug into Eqs. (10) and (12), we can determine coefficients a_1, b_1, e, which are

$$\left\{a_1 = -\frac{4}{5}, \quad b_1 = -\frac{2}{5}, \quad e = \frac{56}{5}\right\}.$$

Via the relation between the solution of bilinear equation and the original equation, one obtains the solution to the coupled nonlinear system as follows:

$$\begin{aligned} u(x,y,t) &= 16\frac{8t^2 + 10tx - 50ty - 25x^2 + 25xy + 50y^2 - 60t + 150y + 240}{(4t^2 - 4tx - 16ty + 10x^2 - 10xy + 25y^2 - 24t + 60y + 96)^2}\,, \\ s(x,y,t) &= \frac{-8x - 32y - 48 + 16t}{4t^2 + (-4x - 16y - 24)\,t + 25y^2 + (-10x + 60)\,y + 10x^2 + 96}\,, \\ v(x,y,t) &= \frac{-20x + 100y + 120 - 32t}{4t^2 + (-4x - 16y - 24)\,t + 25y^2 + (-10x + 60)\,y + 10x^2 + 96}\,. \end{aligned} \tag{15}$$



(a) s at $t = 0$. (b) s at $t = 20$. (c) s at $t = 40$.

Fig. 3. Profiles of s.

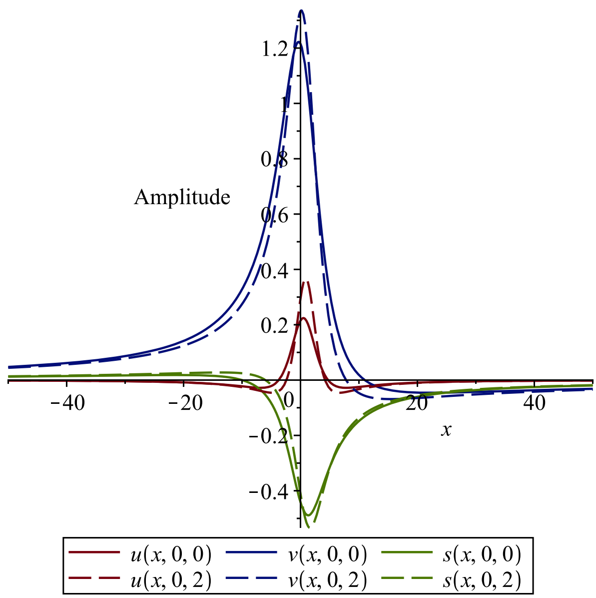


Fig. 4. A 2D plot for u, v, s at $y = 0$ and (solid) $t = 0$, (dash) $t = 2$.

The three-dimensional plots for lump solutions of all three functions u, v, s are presented in Figs. 1–3. Also, Fig. 4 shows profiles of u, v, s by fixing y and varying t.

5. Conclusion and Remarks

In this paper, we explores a class of nonlinear PDE systems, whose Hirota representations involve $D_t^2 D_x D_y$. Such a class of PDE systems has not been studied before, to the best of our knowledge. From the computation point of view, the higher-order of the t derivative will increase the complexity of the representation of the constant e. And also since many famous nonlinear evolutions do not involve this high-order t derivative, our work will bring some new thoughts to the study of nonlinear evolutionary PDEs, especially lump solutions to the field.

In general, nonlinear PDEs are rarely solvable. However, Hirota's bilinear method, along with the idea by one of the authors, it is possible to explore some class of nonlinear PDEs. This paper provides an example and a brand-new nonlinear PDEs which can be explicitly solved by using the bilinear method. However, there are many studies which need to be done in the field of lump solutions. To list some of them: (1) how to efficiently construct multiple-lump solution? (2) how to construct lump solutions for a hierarchy of equations, say KP hierarchy?.

Lump solutions are also important in studying the initial-boundary value problems of nonlinear evolution equations. For example, a study about how initial lump data evolute has been presented by Smyth and his collaborates;[15] their studies show that lump initial data play a crucial role in computing the numerical solutions to nonlinear equations such as the KP equation, the nonlinear Schrödinger equation and the mKdV equation.

Recently, the study for the rational solutions of integrable equations such as Painleve-type equations, attract much interest. It would also be interesting to study rational solutions to higher-dimensional equations. Definitely, lump solutions as a special type of rational solutions will enrich the theory of integrable equations.

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