



# A $\bar{\partial}$ -Steepest Descent Method for Oscillatory Riemann–Hilbert Problems

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## Abstract

We study the long-time asymptotic behavior of oscillatory Riemann–Hilbert problems (RHPs) arising in the mKdV hierarchy (reducing from the AKNS hierarchy). Our analysis is based on the idea of  $\bar{\partial}$ -steepest descent. We consider RHPs generated from the inverse scattering transform of the AKNS hierarchy with weighted Sobolev initial data. The asymptotic formula for three regions of the spatial- and temporal-dependent variables is presented in details.

**Keywords** MKdV hierarchy · Painlevé II hierarchy · Long-time asymptotics ·  $\bar{\partial}$ -steepest descent

**Mathematics Subject Classification** 37K10 · 35Q15

## 1 Introduction

The long-time behavior of solutions of the initial-value problem for nonlinear evolution integrable PDEs has been studied extensively. It is well known that the long-time asymptotic analysis for the integrable PDEs can be, via inverse scattering, formulated as a problem of finding asymptotics of certain oscillatory RHPs. A countless number

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of papers [see, e.g., Do (2010), Varzugin (1996) and the references therein] have been devoted to studying the asymptotic behavior of a certain type of oscillatory 2 by 2 matrix RHPs, which is also the main subject of the current study. The most influential is the nonlinear steepest descent method (or the Deift–Zhou method), which was published in *Annals of mathematics* in 1993 (Deift and Zhou 1993). Before the Deift–Zhou work, Its (1981) proposed a direct method, via an isomonodromic deformation, to study the asymptotics of a RHP arising in studying the long-time behavior of the nonlinear Schrödinger (NLS) equation. Ten years after the Deift–Zhou method was published, Deift and Zhou extended their method to study the long-time behavior of the defocusing NLS equation on some weight Sobolev space. Between 1993 and 2003, the Deift–Zhou method had been applied to not only the long-time behavior of integrable systems, but also equilibrium measure for logarithmic potentials (Deift et al. 1998), the strong asymptotics of orthogonal polynomials (Deift et al. 1999) and many other fields in mathematical physics. Shortly after the Deift–Zhou 2003 paper, McLaughlin and Miller (2004) proposed another generalization to the Deift–Zhou method: the so-called  $\bar{\partial}$ -steepest descent. This method was first applied to studying the long-time asymptotics of the defocusing NLS equation in 2008 (Dieng and McLaughlin 2008), see also its extension version (Dieng et al. 2018). Comparing to the Deift–Zhou method, the  $\bar{\partial}$ -steepest descent method provides a more elementary way and more tractable way of analyzing the error terms. Since then, the  $\bar{\partial}$ -steepest descent method has been applied to many long-time asymptotic studies for nonlinear integrable PDEs, such as the focusing NLS equation (Borghese et al. 2018), the KdV equation (Giavedoni 2017), the mKdV equation (Chen and Liu 2019), the sine-Gordon equation (Chen et al. 2020), the fifth order mKdV equation (Liu et al. 2019) and many others. It is worth mentioning that the mKdV and fifth-order mKdV equations belong to the mKdV hierarchy we will consider in the current work. In fact, by carefully checking Chen and Liu (2019) and Liu et al. (2019), we find there are many similar analyses which motivate us to study the whole mKdV hierarchy at once.

In the current paper, we will study an oscillatory 2 by 2 matrix RHP arising in studying the long-time asymptotics of the mKdV hierarchy. We will discuss the defocusing case (i.e., without solitons). The focusing case will be treated somewhere else in the future. The main analysis is based on the idea of  $\bar{\partial}$ -steepest descent (Dieng and McLaughlin 2008; McLaughlin and Miller 2004).

In the study of Cauchy initial-value problems of integrable systems by means of inverse scattering, the following RHP appears:

**Riemann–Hilbert problem 1.1** *Looking for a 2 by 2 matrix-valued function  $m(z)$  such that*

- (1)  $m$  is analytic off the real line  $\mathbb{R}$ ;
- (2) for  $z \in \mathbb{R}$ , we have

$$m_+ = m_- v_\theta(z), \quad z \in \mathbb{R}, \quad (1)$$

where  $m_{\pm}(z) = \lim_{\epsilon \rightarrow 0^+} m(z \pm i\epsilon)$ ,  $z \in \mathbb{R}$ , and the jump reads

$$v_{\theta}(z) = \begin{pmatrix} 1 - |R(z)|^2 - \bar{R}(z)e^{-2it\theta} & \\ R(z)e^{2it\theta} & 1 \end{pmatrix}, \tag{2}$$

where  $R(z)$  is the reflection coefficient in performing inverse scattering with given initial data, see, e.g., (19), and  $\theta = \theta(z; x/t)$  is a polynomial of  $z$  with coefficients depends on  $x/t$ ;

(3)  $m(z) = I + \mathcal{O}(z^{-1})$ ,  $z \rightarrow \infty$ .

In this paper, we consider the following defocusing mKdV type reduction of the AKNS hierarchy (shortly, we call it the mKdV hierarchy): Fixing  $n$  as a positive odd integer, we consider

$$\begin{aligned} \psi_x(x, t; z) &= \left( iz\sigma_3 + \begin{pmatrix} 0 & q(x, t) \\ q(x, t) & 0 \end{pmatrix} \right) \psi(x, t; z), \\ \psi_t(x, t; z) &= \left( \sum_{k=0}^n Q_k(x, t)z^k \right) \psi(x, t; z), \end{aligned} \tag{3}$$

where  $\sigma_3 = \text{diag}(1, -1)$ ,  $q(x, t)$  is the potential which solves a certain  $1 + 1$  dimensional integrable equation, and  $Q_k$  is determined by certain recursion relation [for details, see Ablowitz and Clarkson (1991)].

In the case of the mKdV hierarchy,  $Q_n(x, t)$  is a constant with respect to  $x, t$ . The corresponding nonlinear integrable PDE is worked out by the zero-curvature condition, which is also equivalent to  $\psi_{xt} = \psi_{tx}$ . In this paper, we will study the Cauchy initial-value problem for integrable PDEs generated from the defocusing mKdV hierarchy, with the initial data belonging to  $H^{n-1,1}(\mathbb{R}) = \{f \in L^2 | f^{(s)} \in L^2, s = 1, \dots, n - 1, xf \in L^2\}$ . Due to Zhou’s result (Zhou 1998), after direct scattering, the reflection coefficient  $R(z, t = 0)$  belongs to  $H^{1,n-1}(\mathbb{R})$ . By performing the time evolution, we arrive at the RHP 1.1. The first part (oscillating region) of the analysis is slightly more general than the one in the AKNS hierarchy, by making the following assumptions on the phase function:

- (1)  $\theta$  is a real polynomial of degree  $n$  with respect to  $z$ , with coefficients depends on  $x/t$ ;
- (2)  $\theta'(z_j) = 0, \theta''(z_j) \neq 0$  for  $j = 1, \dots, l$ , where  $l$  denotes the number of real stationary phase points.

**Remark 1.2** For the defocusing mKdV hierarchy case,  $n$  in the first assumption corresponds to the  $\frac{n-1}{2}$ th member of the hierarchy. Since in mKdV hierarchy,  $n$  is an odd number, say  $n = 2k - 1, k \in \mathbb{Z}_+$ , we will only need to study the phase function of the type:  $c_1z + c_2z^{2k+1}$ ,  $k \in \mathbb{Z}_+$  and  $c_1, c_2$  are some constants. The purpose of the second assumption includes the case of linear combination of several members in the mKdV hierarchy, which is again integrable. In such situation, we will see that a generic polynomial of  $z$  with coefficients depends on  $x/t$ .

### 1.1 Main Results

Before we establish the main results, we first introduce some notations. Let us denote the weighted Sobolev space by

$$H^{k,j}(\mathbb{R}) := \{f(x) \in L^2(\mathbb{R}) : \partial_x^s f \in L^2(\mathbb{R}), s = 1, \dots, k, x^j f(x) \in L^2(\mathbb{R})\},$$

with norm

$$\|f\|_{H^{k,j}} := \left( \|f\|_{L^2}^2 + \sum_{l=1}^k \|\partial_x^l f\|_{L^2}^2 + \sum_{m=1}^j \|x^m f\|_{L^2}^2 \right)^{1/2}.$$

Next, we define the meaning of the long-time behavior in the three regions we are concerned with as follows.

- (1) Long-time behavior of the potential in the oscillating region means taking the  $t \rightarrow \infty$  limit of  $q(x, t)$  along the ray  $x = -ct, c > 0, t \rightarrow \infty$ .
- (2) Long-time behavior of the potential in the fast decaying region means taking the  $t \rightarrow \infty$  limit of  $q(x, t)$  along the ray  $x = ct, c > 0, t \rightarrow \infty$ .
- (3) Long-time behavior of the potential in the Painlevé region means taking the  $t \rightarrow \infty$  limit of  $q(x, t)$  along the curve  $x = c(nt)^{1/n}, c \neq 0, t \rightarrow \infty$ , where  $n$  is the degree of the polynomial phase function  $\theta$ .

**Theorem 1.3** *For each  $n \in \mathbb{N}$ , in the oscillating region, provided that the initial data<sup>1</sup>  $q(x, 0) \in H^{n-1,1}(\mathbb{R}, dx)$ , and it evolves to satisfy the compatibility conditions for (3), then the long-time behavior for the potentials  $q(x, t)$  is given by*

$$q(x, t) = q_{as}(x, t) + \mathcal{O}(t^{-3/4}), \quad t \rightarrow \infty, \tag{4}$$

where

$$\begin{aligned} q_{as}(x, t) &= -2i \sum_{j=1}^l \frac{|\eta(z_j)|^{1/2}}{\sqrt{2t\theta''(z_j)}} e^{i\varphi(t)}, \\ \varphi(t) &= \frac{-3\pi}{4} - \arg \Gamma(-i\eta(z_j)) \\ &\quad - 2t\theta(z_j) - \frac{\eta(z_j)}{2} \log |2t\theta''(z_j)| + 2 \arg(\delta_j) + \arg(R_j), \end{aligned}$$

and the phase function  $\theta$  will depend only on  $z$  along any ray in the oscillating region,  $\{z_j\}_{j=1}^l$  are the simple real stationary phase points of the phase function, and

<sup>1</sup> Due to Zhou's theorem (Zhou 1998),  $R(z)$  belongs to  $H^{1,n-1}(dz)$ , then the time evolving reflection coefficient  $R(z)e^{\pm 2it\theta}$  will stay in  $H^{1,1}(dz)$  since the degree of  $\theta$  is  $n$ .

$$\delta(z) = \exp\left(\frac{1}{2\pi i} \int_{D_-} \frac{\log(1 - |R(s)|^2)}{s - z} ds\right), z \in \mathbb{C} \setminus D_-,$$

$$D_- = \{z \in \mathbb{R} : \theta'(z) < 0\},$$

$$\eta(z) = -\frac{1}{2\pi} \log(1 - |R(z)|^2),$$

$$R_j = R(z_j), \quad j = 1, \dots, l,$$

$$\delta_j = \lim_{\substack{z=z_j+\rho e^{i\phi}, \\ \rho \rightarrow 0, \\ \text{fix } \phi \in (0, \pi/2)}} \delta(z)(z - z_j)^{i\eta(z_j)}.$$

Here,  $R(z)$  is the reflection coefficient generated from the standard inverse scattering procedure, see Eq. (19).

**Corollary 1.4** For the case of the AKNS hierarchy, in the oscillating region, the phase function  $\theta(z) = \frac{x}{t}z + cz^n$ ,  $c > 0$ ,  $n$  is odd number and has just two real stationary phase points:  $z_{\pm} = \pm \left| -\frac{x}{nct} \right|^{\frac{1}{n-1}}$ , and then the long-time asymptotics for the potentials in the AKNS hierarchy are merely a special case of Theorem 1.3.

**Theorem 1.5** In the fast decay region, the potential which belongs to the defocusing mKdV flow has the following long-time asymptotic behavior:

$$q(x, t) = \mathcal{O}(t^{-1}), \quad t \rightarrow \infty. \tag{5}$$

**Theorem 1.6** For each  $n \in \mathbb{N}$ , in the Painlevé region, the potential which belongs to the defocusing mKdV flow has the following long-time asymptotic behavior:

$$q(x, t) = (nt)^{-\frac{1}{n}} u_n(x(nt)^{-\frac{1}{n}}) + \mathcal{O}(t^{-\frac{3}{2n}}), \quad t \rightarrow \infty, \tag{6}$$

where  $u_n$  solves the  $n^{\text{th}}$  member of the Painlevé II hierarchy.

### 1.2 Outline

In Sect. 2, we simply review the inverse scattering for the AKNS hierarchy. In Sect. 3, we summarize the idea of the  $\bar{\partial}$ -steepest descent method following Dieng et al. (2018) and Chen and Liu (2019). In the following sections, we first discuss the long-time behavior of the potential in the oscillating region. The general workflow is shown in Fig. 5. The first step (see Sect. 4) is so-called conjugation by which one can simultaneously factorize the jump matrix to lower/upper triangle and upper/lower triangle. The next step (see Sect. 5) is so-called lenses opening. In each interval where  $\theta$  is monotonic, we can deform those intervals into new contours which are off the real line and the exponential terms will decay as  $t$  goes to infinity on the new contours. The core idea of this step of the  $\bar{\partial}$ -steepest descent is to use Stokes' theorem to transfer

contour integrals to double integrals, while in the original Deift–Zhou’s method, this step is done by first performing rational approximation then analytic continuation. After lenses opening, we will end up at a mixed  $\bar{\partial}$ -RHP. Next, from Sects. 6 and 7, we will first approximate the pure RHP. Three main steps of approximating the pure RHP are so-called localization, phase reduction and contribution separation, which lead to an exact solvable model RHP (also called its isomonodromy problem). Due to the exact solvability of the model RHP and the small norm theory, one can establish the existence and uniqueness of the pure RHP part of the mixed  $\bar{\partial}$ -RHP. The last step (see Sect. 8) is to estimate the errors by analyze the pure  $\bar{\partial}$ -problem which dominates the errors generated by approximating the pure RHP. Undo all the steps, we will eventually prove Theorem 1.3. Then, in Sect. 10, we will study the long-time behavior of the potential in the fast decaying region. Following similar analysis, we end up proving Theorem 1.5. The final section (see Sect. 11) is devoted to proving Theorem 1.5. In that section, we first give an algorithm to generate the Painlevé II hierarchy. Following the method of  $\bar{\partial}$ -steepest descent, we represent the long-time behavior of the potential by the solution to a member of the Painlevé II hierarchy.

## 2 Inverse Scattering Transform and Riemann–Hilbert Problem in $L^2$

In this section, we simply review the inverse scattering transform for the AKNS hierarchy in a certain weighted  $L^2$  Sobolev space. For more details, we direct readers to Zhou’s paper (Zhou 1998).

The AKNS hierarchy is the integrable hierarchy associated with the following spectral problem:

$$\psi_x(x, t; z) = (iz\sigma_3 + Q(x, t))\psi(x, t; z), \tag{7}$$

where  $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  and  $Q(x, t) = \begin{pmatrix} 0 & q(x, t) \\ r(x, t) & 0 \end{pmatrix}$ .

In the current paper, we only consider the defocusing type reduction:

$$r(x, t) = q(x, t) \in \mathbb{R},$$

and we assume  $q(x, t = 0) \in H^{n-1,1}(\mathbb{R}, dx)$ .<sup>2</sup> For  $t = 0$ , we are looking for solutions (so-called Jost solutions) of equation (7) in  $H^{1,1}(\mathbb{R}, dx)$ , which satisfy the following boundary conditions at infinity:

$$\psi_{\pm} = e^{ixz\sigma_3} + o(1), \quad x \rightarrow \pm\infty. \tag{8}$$

<sup>2</sup> This guarantees the time evolving of the initial data will stay in  $H^{1,1}$ . Roughly speaking, from Zhou’s work, we know  $q(x, 0) \in H^{n-1,1} \subset H^{1,1}$  is mapped to  $R(z) \in H^{1,n-1}$ . Time evolution of the reflection coefficient gives  $R(z)e^{itz^n}$ , which belongs to  $H^{1,1}$  due to the fact that  $R(z) \in H^{1,n-1}$ , and then the inverse scattering leads to  $q(x, t) \in H^{1,n-1}$ .

The scattering matrix  $S(z)$  is then defined as

$$S(z) := \psi_-^{-1} \psi_+. \tag{9}$$

It is well known that  $S$  enjoys the following properties: for  $z \in \mathbb{R}$ ,

$$S(z) = \begin{pmatrix} a(z) & \bar{b}(z) \\ b(z) & \bar{a}(z) \end{pmatrix}, \tag{10}$$

where  $a, b$  can be represented in terms of the initial data and the eigenfunctions  $\psi$ . To find such representations, we consider

$$\mu^{(\pm)} = \psi_{\pm} e^{-izx\sigma_3}. \tag{11}$$

Then, the spectral problem (7) becomes:

$$(\mu^{(\pm)})_x = iz[\sigma_3, \mu^{(\pm)}] + Q\mu^{(\pm)}. \tag{12}$$

Due to the initial setting of the potential matrix  $Q$ , the standard inverse scattering theory tell us that one can analytically extend the first column  $\mu^{(+)}$  and the second column  $\mu^{(-)}$  to the upper half plane while the first column  $\mu^{(-)}$  and the second column  $\mu^{(+)}$  to the lower half plane. Moreover, the reflection coefficients enjoy the following representations:

$$a(z) = \mu_{11}^{(+)}(x \rightarrow -\infty) = 1 - \int_{\mathbb{R}} q(y) \mu_{21}^{(+)}(y, z) dy, \tag{13}$$

$$b(z) = \mu_{12}^{(+)}(x \rightarrow -\infty) = - \int_{\mathbb{R}} e^{2iyz} q(y) \mu_{22}^{(+)}(y, z) dy. \tag{14}$$

From the above representations, it is straightforward to show that  $a(z) = 1 + \mathcal{O}(1/z)$  and  $b(z) = \mathcal{O}(1/z)$  as  $z \rightarrow \infty$ , and  $a(z)$  can be analytically extended to the upper half plane.

Now, with the analyticity of the column solutions to equation (12), by setting

$$m_+(x, z) = (\mu_1^{(+)}(x, z)/a(z), \mu_2^{(-)}(x, z)), \quad \text{Im } z \geq 0, \tag{15}$$

$$m_-(x, z) = (\mu_1^{(-)}(x, z), \mu_2^{(+)}(x, z)/\bar{a}(z)), \quad \text{Im } z \leq 0, \tag{16}$$

we can then define the jump matrix on the real line by

$$v(z) = e^{-izx \text{ ad}\sigma_3} (m_-^{-1} m_+). \tag{17}$$

A direct computation shows

$$v(z) = \begin{pmatrix} 1 - |R(z)|^2 & -\bar{R}(z) \\ R(z) & 1 \end{pmatrix}, \tag{18}$$

where

$$R(z) = \frac{b(z)}{a(z)}. \tag{19}$$

It also worth mentioning that (again from standard inverse scattering theory)  $|a|^2 - |b|^2 = 1$ , from where we have

$$0 \leq |R(z)| < 1. \tag{20}$$

The deformation of the spectral problem (7) with respect to  $t$  is governed by the following equation:

$$\psi_t(x, t; z) = \left( \sum_{k=0}^n Q_k(x, t) z^k \right) \psi(x, t; z). \tag{21}$$

To generate isospectral flow,  $\psi$  need to satisfy the compatibility condition, i.e.,  $\psi_{xt} = \psi_{tx}$ . By this condition, one can uniquely determine  $Q_k$  if the integration constants are assumed to be all zeros. One can systematically determines the  $Q_k$ 's via associated Lie algebra techniques, see for example Ma (2013). Through the Lie algebra, one can show the AKNS hierarchy is integrable, i.e., there are infinite many conservation laws. Moreover, using the powerful trace identity (Tu 1989), one can easily show the bi-Hamiltonian structure of the AKNS hierarchy. Moreover, under the same framework, one can show that any linear combinations of the time-evolution problem are also integrable.

The compatibility condition of (7) and (21) generates integrable PDEs, including the defocusing nonlinear Schrödinger equation, the modified KdV equation, the fifth-order modified KdV equation. Due to the decaying of the potential  $Q$ , it is easy to show the time evolution of the jump matrix  $v$  is trivial. In fact, if the potential decays as  $|x| \rightarrow \infty$ , then  $\psi_{\pm} e^{Q_n z^n t}$  will be the simultaneous fundamental solution to the system (3). By the time-evolution condition in (3), one get the  $t$  - differential equation for the Jost solution  $\psi_{\pm}$  as follows:

$$\frac{\partial}{\partial t} \psi_{\pm} = V \psi_{\pm} - \psi_{\pm} Q_n z^n.$$

Now by the definition of the scattering matrix, we have

$$\begin{aligned} \frac{\partial S}{\partial t} &= (\psi_-^{-1} \psi_+)_t \\ &= \psi_-^{-1} (-\psi_+ Q_n z^n + V \psi_+) - \psi_-^{-1} (-\psi_- Q_n z^n + V \psi_-) \psi_-^{-1} \psi_+ \\ &= [Q_n z^n, S] \end{aligned}$$



It is of our current interest that  $Q_n = -ic\sigma_3$  for some positive constant  $c$ . Therefore, we have the time evolution for the scattering matrix

$$S(z; t) = e^{-icz^n t \operatorname{ad} \sigma_3} S(z; t = 0), \tag{22}$$

where  $e^{\operatorname{ad} \sigma_3}(\cdot) := e^{\sigma_3}(\cdot)e^{-\sigma_3}$ . This implies the time evolution of the jump matrix  $v(z)$  [see (18)], and we have

$$v_\theta(z) := e^{-it\theta(z;x,t) \operatorname{ad} \sigma_3} v(z), \tag{23}$$

where (in the case of the defocusing AKNS hierarchy)  $\theta(z; x, t) = \frac{x}{t}z + cz^n$  for some positive constant  $c$ .

Finally, we formulate the inverse scattering problem as a Riemann–Hilbert problem as follows:

**Riemann–Hilbert problem 2.1** *Looking for a 2 by 2 matrix-valued functions  $m(z; x, t)$  such that*

- (1)  $m(z)$  is analytic in  $\mathbb{C} \setminus \mathbb{R}$ ;
- (2)  $m_+ = m_- v_\theta, \quad z \in \mathbb{R}$ ;
- (3)  $m = I + m_1(x, t)/z + \mathcal{O}(1/z^2), \quad z \rightarrow \infty$ ;

where  $v_\theta$  is defined in Eq. (23) and  $m_\pm = \lim_{\epsilon \downarrow 0} m(z \pm i\epsilon)$ .

From Eq. (12), and the definition of the jump matrix  $v$ , we can recover the potential by

$$\begin{aligned} q(x, t) &= -2i \lim_{z \rightarrow \infty} [z(m - I)]_{12} \\ &= -2i(m_1(x, t))_{12}. \end{aligned} \tag{24}$$

In the following sections, we will perform the  $\bar{\partial}$ -steepest descent method and study the asymptotic behavior for  $t$  being sufficiently large.

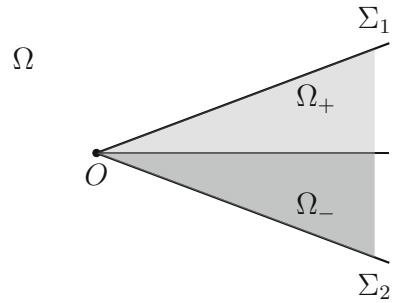
### 3 Overview of the Strategies

In this section, we will simply review the idea of Deift–Zhou’s nonlinear steepest descent method and its variation, the  $\bar{\partial}$ -steepest descent method. In general, the key step in both methods is to deform the RHP. After the deformation, the new RHP is expected to be approximable locally as  $t$  goes to  $\infty$ . Next, we will explain the main ideas of both methods. The notations in this section are used in this section only.

Let us consider the following RHP on  $\mathbb{R}_+$ :

$$\begin{aligned} M_+(z) &= M_-(z)e^{-it\theta(z) \operatorname{ad} \sigma_3} V(z), \quad z \in \mathbb{R}_+, \\ M(z) &\rightarrow I, \quad z \rightarrow \infty. \end{aligned}$$

Fig. 1 Contour deformation



The main idea<sup>3</sup> of Deift–Zhou’s method is to find a factorization of  $V(z)$ , say,  $V(z) = V_-(z)V_+(z)$ , such that  $V_{\pm}(z)$  can be approximated by  $\tilde{V}_{\pm}(z)$  which are analytic in the sectors  $\Omega_+$  and  $\Omega_-$  respectively, see Fig. 1.

By introducing a new analytic function  $\tilde{M}$  as follows:

$$\begin{aligned} \tilde{M}|_{\Omega} &= M, \\ \tilde{M}|_{\Omega_+} &= M\tilde{V}_+^{-1}, \\ \tilde{M}|_{\Omega_-} &= M\tilde{V}_-, \end{aligned}$$

we arrive at a new RHP:

$$\begin{aligned} \tilde{M}_+ &= \tilde{M}_- \tilde{V}, \quad z \in \mathbb{R}_+ \cup \Sigma_1 \cup \Sigma_2, \\ \tilde{V} &= \begin{cases} \tilde{V}_+, & z \in \Sigma_1, \\ \tilde{V}_-, & z \in \Sigma_2, \\ \tilde{V}_-^{-1} V_- V_+ \tilde{V}_+^{-1}, & z \in \mathbb{R}_+. \end{cases} \end{aligned}$$

Also, we want to guarantee that based on the signatures of the phase function  $\text{Re}(i\theta(z))$  in each sector, the new jumps converge rapidly to the identity away from  $O$  as  $t \rightarrow \infty$ . Usually, one needs to deform the RHP several times. Eventually, the initial RHP can be approximated locally by the following fairly simple model RHP:

$$\begin{aligned} M_+^{\sharp} &= M_-^{\sharp} e^{-i\tilde{\theta}(z) \text{ad } \sigma_3} V(0), \quad z \in \mathbb{R}_+, \\ M^{\sharp} &\rightarrow I, \quad z \rightarrow \infty, \end{aligned}$$

where  $\tilde{\theta}(z)$  is a certain rational approximation to  $\theta(z)$  near  $z = 0$ . This model RHP can be solved explicitly and by undoing all deformations, one can track all errors in the middle steps.

The Deift–Zhou method of analyzing errors is heavily based on the harmonic analysis for the Cauchy operators on contours; however, the  $\bar{\partial}$ -steepest descent method transfers the error estimation to some fairly simple estimations of certain double integrals. A natural way of connecting the contour integrals to the double integrals is to

<sup>3</sup> A good summary of this method can be found in Deift et al. (1993).

use Stokes’ theorem (or the Cauchy–Green theorem): for any  $C^1(\mathbb{R}^2 \rightarrow \mathbb{C})$  function  $f(z) := f(z, \bar{z})$ , we have

$$\int_{\partial\Omega} f(z)dz = 2i \int_{\Omega} \frac{\partial f(z)}{\partial \bar{z}} dx dy,$$

where  $z = x + iy$ . So in the  $\bar{\partial}$ -steepest descent theory, we try to find an interpolation, say  $E(z)$ , between the old contour and the new one. Such an  $E$  satisfies

$$E(z) = \begin{cases} V_+(0), & z \in \Sigma_1^-, \\ V_-(0), & z \in \Sigma_2^+, \\ V_+(z), & z \in \mathbb{R}_+^+, \\ V_-(z), & z \in \mathbb{R}_+^-, \\ I, & z \in \Sigma_1^+ \cup \Sigma_2^-, \end{cases}$$

where all contours are orientated from  $O$  to  $\infty$  and  $\Gamma^\pm$  mean the limit from left/right, and it is  $C^1$  in  $\overline{\Omega_+ \cup \Omega_-}$ . Also, we want  $e^{it\bar{\theta}(z) \text{ ad } \sigma_3} V_\pm(0)$  go to  $I$  as  $t \rightarrow \infty$ . Now, let us set  $\hat{M} = M(z)E(z)$ , we obtain the so-called  $\bar{\partial}$ -RHP:

1. (The RHP)  $\hat{M}_+ = \hat{M}_- e^{-it\theta(z) \text{ ad } \sigma_3} \hat{V}(z)$ ,  $z \in \Sigma_1 \cup \Sigma_2$ , where

$$\hat{V}(z) = \begin{cases} e^{-it\theta(z) \text{ ad } \sigma_3} V_+(0), & z \in \Sigma_1, \\ e^{-it\theta(z) \text{ ad } \sigma_3} V_-(0), & z \in \Sigma_2. \end{cases}$$

2. (The  $\bar{\partial}$ -problem) For any  $z \in \mathbb{C}$ , we have

$$\bar{\partial} \hat{M} = \hat{M} E^{-1} \bar{\partial} E.$$

The deformation of the RHP follows from Deift–Zhou’s method, but the error estimations here are transferred to a  $\bar{\partial}$  problem, which turns out to be equivalent to some singular integral equation with respect to the area measure. Then through some fairly simple estimates on the double integrals, one will obtain the same error estimates as the Deift–Zhou method. In the following sections, we will apply the  $\bar{\partial}$ -steepest descent to the defocusing mKdV type reduction of the AKNS hierarchy.

### 4 Conjugation

In this section, we will factorize the jump matrix (as defined by Eq. (2)) in a way that it can be used for deforming the RHP. It is easy to see that the jump matrix enjoys the

following two kinds of factorization:

$$v_\theta(z) = \begin{cases} \begin{pmatrix} 1 - \bar{R}(z)e^{-2it\theta} & \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ R(z)e^{2it\theta} & 1 \end{pmatrix}, \\ \begin{pmatrix} 1 & 0 \\ \frac{R(z)}{1-|R(z)|^2}e^{2it\theta} & 1 \end{pmatrix} (1 - |R|^2)^{\sigma_3} \begin{pmatrix} 1 - \frac{\bar{R}(z)}{1-|R|^2}e^{-2it\theta} & \\ 0 & 1 \end{pmatrix}. \end{cases} \tag{25}$$

In the light of the main ideas we described in the last section, we want to remove the middle term in the second factorization. By doing so, we can eventually find the proper factorization based on the signatures of the  $\text{Re}(i\theta)$ . Due to our assumptions on  $\theta$ , near a stationary phase point (say  $|z - z_j| < \epsilon$ , for some small positive  $\epsilon$ ),  $\theta = \theta(z_j) + \frac{\theta''(z_j)}{2}(z - z_j)^2 + \mathcal{O}(|z - z_j|^3)$ . If  $\theta''(z_j) > 0$ , then  $\text{Re}(i\theta(z))$  is negative in the line (I):  $\{z = z_j + re^{i\alpha}, r \in (-\epsilon, \epsilon)$  with fixed  $\alpha \in (0, \pi/2)\}$ , and it is positive in the line (II):  $\{z = z_j + re^{i\alpha}, r \in (-\epsilon, \epsilon)$  with fixed  $\alpha \in (-\pi/2, 0)\}$ . On the line (I), notice that  $e^{2it\theta}$  decays to 0 as  $t \rightarrow \infty$ , we can deform the jump on the contour right to the stationary phase point using the first factorization. With the same argument on the line (II), we can deform the jump on the contour left to the stationary phase point using the second factorization. If  $\theta''(z_j) < 0$ , notice now  $e^{2it\theta}$  decays to 0 as  $t \rightarrow \infty$  on the line (II), and thus we need the second factorization for the jump on the contour right to the stationary phase point and the first factorization for the jump on the contour left to the stationary phase point. Motivated by the above arguments, we denote  $D_\pm = \{z \in \mathbb{R} : \pm\theta'(z) > 0\}$

To eliminate the diagonal matrix in the second factorization, we introduce a scalar RHP:

$$\begin{aligned} \delta_+ &= \delta_-[(1 - |R|^2)\chi_{D_-} + \chi_{D_+}], \quad z \in \mathbb{R}, \\ \delta(z) &= 1 + \mathcal{O}(z^{-1}), \quad z \rightarrow \infty. \end{aligned} \tag{26}$$

Then by conjugating the initial RHP, we arrive at a new RHP:

**Riemann–Hilbert problem 4.1** *Looking for a 2 by 2 matrix-valued function  $m^{[1]}(z; x, t)$  such that*

- (1)  $m_\pm^{[1]} = m_-^{[1]}\delta_-^{\sigma_3}v_\theta\delta_+^{-\sigma_3}$ ,  $z \in \mathbb{R}$ ;
- (2)  $m^{[1]} = I + \mathcal{O}(z^{-1})$ ,  $z \rightarrow \infty$ .

By denoting  $v_\theta^{[1]} := \delta_-^{\sigma_3}v_\theta\delta_+^{-\sigma_3}$ , the new jump matrix reads

$$v_\theta^{[1]}(z) = \begin{cases} \begin{pmatrix} 1 - \bar{R}(z)\delta_-^2(z)e^{-2it\theta} & \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ R(z)\delta_-^{-2}e^{2it\theta} & 1 \end{pmatrix}, & z \in D_+, \\ \begin{pmatrix} 1 & 0 \\ \frac{R(z)\delta_-^{-2}e^{2it\theta}}{1-|R|^2} & 1 \end{pmatrix} \begin{pmatrix} 1 - \frac{\bar{R}(z)\delta_+^2(z)e^{-2it\theta}}{1-|R|^2} & \\ 0 & 1 \end{pmatrix}, & z \in D_-. \end{cases} \tag{27}$$

The scalar RHP (26) has been carefully studied in the literature [see for example Beals et al. (1988, Lemma 23.3), Deift and Zhou (2003), Varzugin (1996) and Do (2010)]. Here, we just list some of the properties. First, the solution to the RHP (26) can be represented as follows:

$$\log(\delta(z)) = (C_{D_-}(\log(1 - |R|^2)))(z), \quad z \in \mathbb{C} \setminus D_-, \tag{28}$$

where the Cauchy operator  $(C_{D_-}f)(z) = \frac{1}{2\pi i} \int_{D_-} \frac{f(s)}{s-z} ds$ . Since the initial data are within  $H^{n-1,1}(\mathbb{R}, dx)$ , we have  $R(z) \in H_1^{1,n-1}(\mathbb{R}, dz) = H^{1,1} \cap \{f : |f| < 1\}$  [see also Eq. (20)]. Thus, one can show  $\log(1 - |R|^2)$  is in  $H^{1,0}$ , and then by the Sobolev embedding, we know it is also Hölder continuous with index 1/2. Then, by the Privalov–Plemelj theorem, which says that Cauchy operator perseveres Hölder continuity with index less than 1, one can show  $\log(\delta(z))$  is Hölder continuous with index 1/2 except for the end points. Next, we study the behavior near those points.

Let us denote

$$\eta(z) = -\frac{1}{2\pi} \log(1 - |R(z)|^2), \quad z \in \mathbb{R}. \tag{29}$$

We will prove the following proposition.

First, we define a tent function supported on the interval  $[-\epsilon, \epsilon]$ ,

$$s_\epsilon(z) = \begin{cases} 0, & |z| \geq \epsilon \\ -\frac{1}{\epsilon}z + 1, & 0 < z < \epsilon, \\ \frac{1}{\epsilon}z + 1, & -\epsilon < z \leq 0. \end{cases} \tag{30}$$

**Proposition 4.2** *For each  $\epsilon > 0$ , and  $\epsilon \leq \frac{1}{3} \min_{j \neq k} |z_j - z_k|$ , there exists a neighborhood  $I = I(\epsilon)$  such that the identity*

$$\begin{aligned} \log(\delta(z)) &= i \int_{D_- \setminus I} \frac{\eta(s)}{s-z} ds + i \sum_{j=1}^l [\eta(z_j)(1 + \log(z - z_j))] \epsilon_j \\ &\quad + i \sum_{j=1}^l \int_{I \cap D_-} \frac{\eta(s) - \eta_j(s)}{s-z} ds \\ &\quad + i \sum_{j=1}^l \frac{1}{\epsilon} \eta(z_j) [(z - z_j) \log(z - z_j) - (z - z_j + \epsilon_j \epsilon) \log(z - z_j + \epsilon_j \epsilon)] \end{aligned}$$

is true, where  $\epsilon_j = \text{sgn}(\theta''(z_j))$ ,  $\eta_j(z) = \eta(z_j)s_\epsilon(z - z_j)$  and see (29) for the definition of  $\eta$ . As for the logarithm function, the branch is chosen such that argument  $\in (-\pi, \pi)$ .

**Proof** Let  $I = \cup_{j=1}^l (I_{j+} \cup I_{j-})$ , where  $I_{j\pm} = \{z : 0 < \pm(z - z_j) < \epsilon\}$ . Now we have

$$\begin{aligned} \log(\delta(z)) &= i \int_{D_- \setminus I} \frac{\eta(s)}{s - z} ds \\ &\quad + i \sum_{j=1}^l \left( \int_{I_{j-} \cap D_-} + \int_{I_{j+} \cap D_-} \frac{\eta(s)}{s - z} ds \right). \end{aligned}$$

For each  $j$ , we have

$$\int_{I_{j-}} \frac{\eta(s)}{s - z} ds = \int_{I_{j-}} \frac{\eta(s) - \eta_j(s)}{s - z} ds + \int_{I_{j-}} \frac{\eta_j(s)}{s - z} ds.$$

The first integral on the right hand side is the non-tangential limit as  $z \rightarrow z_j$  and the second one generates a logarithm singularity near  $z_j$ . In fact, direct computation shows

$$\begin{aligned} \int_{I_{j-}} \frac{\eta_j(s)}{s - z} ds &= \eta(z_j) + \frac{1}{\epsilon} [(z - z_j) \log(z - z_j) - (z - z_j + \epsilon) \log(z - z_j + \epsilon)] \eta(z_j) \\ &\quad + \eta(z_j) \log(z - z_j). \end{aligned}$$

Similarly, for  $I_{j+}$ ,

$$\begin{aligned} \int_{I_{j+}} \frac{\eta_j(s)}{s - z} ds &= -\eta(z_j) + \frac{1}{\epsilon} [(z - z_j) \log(z - z_j) - (z - z_j - \epsilon) \log(z - z_j - \epsilon)] \\ &\quad - \eta(z_j) \log(z - z_j). \end{aligned}$$

And note that only one of the  $I_{j\pm} \cap D_-$  is nonempty, which depends on the sign of the second derivative of the phase function  $\theta$ . By assembling all together, the proof is done. □

**Remark 4.3** The proposition tells us how the function  $\delta(z)$  behavior near the saddle points. Near the saddle points  $z_j$ ,  $\delta(z)$  has a mild singularity  $(z - z_j)^{i\eta(z_j)}$ . Fortunately, those singularities are bounded along any ray off  $\mathbb{R}$ , and hence, in some sense they do not affect asymptotics much. It is worth mentioning that one can ignore the mild singularity by introducing an auxiliary function, see Lemma 3.1 in Dieng et al. (2018).

### 5 Lenses Opening

The purpose of lens opening is to deform the RHP on the real line to a new RHP on new contours such that jumps on the new contours will rapidly decay to  $I$  as  $t \rightarrow \infty$ . We first study the signature of  $\text{Im } \theta$  near the saddle point  $z_j$ .

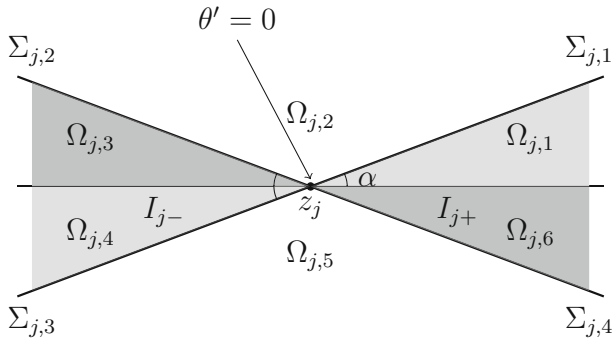


Fig. 2 Notations for studying signatures of  $\text{Im}(\theta(z))$  near  $z_j$

Let us denote  $I_{j+} = [z_j, \frac{z_j+z_{j+1}}{2}]$  and  $I_{j-} = [\frac{z_j+z_{j-1}}{2}, z_j]$ . Two cases need to be discussed. The first case is  $\theta''(z_j) > 0$ , and so we have  $I_{j\pm} \subset D_{\pm}$ . The second case is  $\theta''(z_j) < 0$ , and then we have  $I_{j\pm} \subset D_{\mp}$ .

Recall the factorization of the conjugated jump matrix  $v_{\theta}^{[1]}$ , to deform it from  $I_{j+}$  to  $\Sigma_{j,1}$ , we need make sure the exponential term  $e^{2it\theta(z)}$  decays rapidly to  $I$  on  $\Sigma_{j,1}$ , and thus, we need to discuss  $\text{Re}(i\theta)$  on  $\Sigma_{j,1}$ . Considering a Taylor approximation of  $\theta(z)$  near  $z_j$ , we have  $\theta(z) = \theta(z_j) + \varepsilon_j A_j (z - z_j)^2 + O(|z - z_j|^3)$ , where  $A_j = \left| \frac{\theta''(z_j)}{2} \right|$  and  $\varepsilon_j = \text{sgn}\{\theta''(z_j)\}$ .

Let  $z = z_j + u + iv = z_j + \rho e^{i\phi}$ . Then,  $\text{Im}(\theta(z)) = \varepsilon_j A_j \rho^2 \sin(2\phi) + O(\rho^3)$ , where  $\phi \in (0, \alpha]$  is fixed. Now we define the regions  $\Omega_{j,n}$ ,  $n = 1, \dots, 6$ , as follows:

$$\begin{aligned}
 \Omega_{j,1} &= \left\{ z = z_j + \rho e^{i\phi}, \phi \in (0, \alpha], \rho \in \left( 0, \frac{|z_j - z_{j+1}|}{2 \cos \alpha} \right), \text{Re } z \in I_{j+}^{\varepsilon_j} \right\}, \\
 \Omega_{j,3} &= \left\{ z = z_j + \rho e^{i\phi}, \phi \in [\pi - \alpha, \pi), \rho \in \left( 0, \frac{|z_j - z_{j-1}|}{2 \cos \alpha} \right), \text{Re } z \in I_{j-}^{\varepsilon_j} \right\}, \\
 \Omega_{j,2} &= \mathbb{C}^+ \setminus (\Omega_{j,1} \cup \Omega_{j,3}), \\
 \Omega_{j,4} &= \left\{ z = z_j + \rho e^{i\phi}, \phi \in (\pi, \pi + \alpha], \rho \in \left( 0, \frac{|z_j - z_{j-1}|}{2 \cos \alpha} \right), \text{Re } z \in I_{j-}^{\varepsilon_j} \right\}, \\
 \Omega_{j,6} &= \left\{ z = z_j + \rho e^{i\phi}, \phi \in [-\alpha, 0), \rho \in \left( 0, \frac{|z_j - z_{j+1}|}{2 \cos \alpha} \right), \text{Re } z \in I_{j+}^{\varepsilon_j} \right\}, \\
 \Omega_{j,5} &= \mathbb{C}^- \setminus (\Omega_{j,4} \cup \Omega_{j,6}),
 \end{aligned} \tag{31}$$

where

$$I_{j\pm}^{\varepsilon_j} = \begin{cases} I_{j\pm}, & \varepsilon_j = 1, \\ I_{j\mp}, & \varepsilon_j = -1. \end{cases}$$

Since the number of real saddle points is finite, we can always choose a sufficiently small  $\alpha$ , such that for each  $j$ ,  $e^{2it\theta}$  decays to 0 in  $\Omega_{j,1} \cup \Omega_{j,4}$  and  $e^{-2it\theta}$  decays to 0 in  $\Omega_{j,3} \cup \Omega_{j,6}$ .

Now we are in the position to open the lenses. First, we introduce a bounded smooth function  $\mathcal{K}$  defined on  $\mathbb{R}$  such that

$$\begin{aligned} \mathcal{K}(0) &= 1, \\ \mathcal{K}(\alpha) &= 0, \\ \text{Period of } \mathcal{K} &\text{ is } \pi, \\ \mathcal{K} &\text{ is even function.} \end{aligned} \tag{32}$$

Consider  $\varepsilon_j = 1$  first. Then, the  $\bar{\delta}$  extension functions are as follows. Let  $z - z_j = u + iv = \rho e^{i\phi}$ , and for the case  $\varepsilon_j = 1$ , we set

$$\begin{aligned} E_{j,1}(z) &= \mathcal{K}(\phi) R(u + z_j) \delta^{-2}(z) \\ &\quad + [1 - \mathcal{K}(\phi)] R(z_j) \delta_j^{-2} (z - z_j)^{-2i\varepsilon_j\eta(z_j)}, \quad z \in \Omega_{j,1}, \\ E_{j,3}(z) &= \mathcal{K}(\pi - \phi) \left( -\frac{\bar{R}(u + z_j)}{1 - |R(u + z_j)|^2} \delta_+^2(z) \right) \\ &\quad + [1 - \mathcal{K}(\pi - \phi)] \left( -\frac{\bar{R}(z_j)}{1 - |R(z_j)|^2} \delta_j^2 (z - z_j)^{2i\varepsilon_j\eta(z_j)} \right), \quad z \in \Omega_{j,3}, \\ E_{j,4}(z) &= \mathcal{K}(\pi + \phi) \left( \frac{R(z_j + u)}{1 - |R(z_j + u)|^2} \delta_-^2(z) \right) \\ &\quad + [1 - \mathcal{K}(\pi + \phi)] \left( \frac{R(z_j)}{1 - |R(z_j)|^2} \delta_j^{-2} (z - z_j)^{-2i\varepsilon_j\eta(z_j)} \right), \quad z \in \Omega_{j,4}, \\ E_{j,6}(z) &= \mathcal{K}(-\phi) \left( -\bar{R}(z_j + u) \delta^2(z) \right) \\ &\quad + [1 - \mathcal{K}(-\phi)] \left( -\bar{R}(z_j) \delta_j^2 (z - z_j)^{2i\varepsilon_j\eta(z_j)} \right), \quad z \in \Omega_{j,6}, \end{aligned} \tag{33}$$

where

$$\delta_j = \lim_{\substack{z=z_j+\rho e^{i\phi}, \\ \rho \rightarrow 0, \\ \phi \in (0, \pi/2)}} \delta(z)(z - z_j)^{i\eta(z_j)}.$$

For the case  $\varepsilon_j = -1$ , one only needs to switch the index 1 with 3 and 4 with 6. For the sake of simplicity, in what follows, we focus just on the case  $\varepsilon_j = 1$ . The extension functions can be considered as interpolations between jumps on the old and new contours. Using the extension functions  $E_{j,k}$ ,  $k = 1, 3, 4, 6$ , we can construct the



lens-opening matrices  $O(z)$  as follows:

$$O(z) = \begin{cases} O_{j,n}(z) = \begin{pmatrix} 1 & 0 \\ (-1)^n E_{j,n} e^{2it\theta(z)} & 1 \end{pmatrix}, & z \in \Omega_{j,n}, \quad n = 1, 4, \\ O_{j,m}(z) = \begin{pmatrix} 1 & (-1)^m E_{j,m} e^{-2it\theta(z)} \\ 0 & 1 \end{pmatrix}, & z \in \Omega_{j,m}, \quad m = 3, 6, \\ O_{j,k}(z) = I, & z \in \Omega_{j,k}, \quad k = 2, 5. \end{cases} \tag{34}$$

Then lens opening is performed by multiplying  $O(z)$  to the right of the matrix  $m^{[1]}$ . Let us denote  $m^{[2]}(z) = m^{[1]}(z)O(z)$ ,  $z \in \mathbb{C} \setminus \mathbb{R}$ . Due to the lacking of analyticity of  $O(z)$  (in fact, since we only assume  $R(z) \in C^1(\mathbb{R})$ ,  $O(z)$  is also just in  $C^1(\mathbb{R}^2)^4$ ), we arrive at the following mixed  $\bar{\partial}$ -Riemann–Hilbert problem( $\bar{\partial}$ -RHP):

**Mixed  $\bar{\partial}$ -Riemann–Hilbert problem 5.1**

Looking for a 2 by 2 matrix-valued function  $m^{[2]}$  such that

(1) The RHP

- $m^{[2]}(z) \in C^1(\mathbb{R}^2 \Sigma)$ ;
- $m_+^{[2]} = m_-^{[2]} v_\theta^{[2]}$ ,  $z \in \cup_{j=1, \dots, l, k=1, 2, 3, 4} \Sigma_{j,k}$ , where the jump matrices read

$$v_\theta^{[2]} = \begin{cases} O_{j,1}^{-1}, & z \in \Sigma_{j,1}, \\ O_{j,3}^{-1}, & z \in \Sigma_{j,2}, \\ O_{j,4}, & z \in \Sigma_{j,3}, \\ O_{j,6}, & z \in \Sigma_{j,4}; \end{cases} \tag{35}$$

- $m^{[2]}(z) = I + \mathcal{O}(z^{-1})$ ,  $z \rightarrow \infty$ .

(2) The  $\bar{\partial}$ -problem

For  $z \in \mathbb{C}$ , we have

$$\bar{\partial} m^{[2]}(z) = m^{[2]}(z) \bar{\partial} O(z). \tag{36}$$

To close this section, we state a bound estimate for  $\bar{\partial} E_{j,k}$ , which will be used in later sections.

**Lemma 5.2** For  $j = 1 \dots l$ ,  $k = 1, 2, 3, 4$ , and  $z \in \Omega_{j,k}$ ,  $u = \text{Re}(z - z_j)$ ,

$$|\bar{\partial} E_{j,k}(z)| \leq c(|z - z_j|^{-1/2} + |R'(u + z_j)|). \tag{37}$$

<sup>4</sup> Here,  $R(z) \in C^1(\mathbb{R})$  means  $R(z)$  is a function defined on the real line with continuous first-order derivative. While since  $O(z)$  is a matrix-valued function defined on the complex plan, so  $O(z) \in C^1(\mathbb{R}^2)$  means all the entries have continuous first-order derivatives with respect to  $z$  and  $\bar{z}$ .

**Proof** In the polar coordinates,  $\bar{\partial} = \frac{e^{i\phi}}{2}(\partial_\rho + i\rho^{-1}\partial_\phi)$ . For  $z$  in any ray starting from  $z_j$  and off the real line, we have

$$\begin{aligned} \bar{\partial}E_{j,1}(z) &= \frac{ie^{i\phi}\mathcal{K}'(\phi)}{2\rho}[R(u+z_j)\delta^{-2}(z) - R(z_j)\delta_j^{-2}(z-z_j)^{-2i\eta(z_j)}] \\ &\quad + \mathcal{K}(\phi)R'(u+z_j)\delta^{-2}(z). \end{aligned}$$

From Proposition 4.2, we know  $|\delta(z) - \delta_j(z-z_j)^{i\eta(z_j)}| \leq c|z-z_j|^{1/2}$ . Also since

$$\delta(z)^{-1} = e^{-C_{D_-}(\log(1-|R|^2))},$$

it is evident that  $\delta(z)^{-1}$  is bounded. Therefore,

$$|\delta^{-2}(z) - \delta_j^{-2}(z-z_j)^{-2i\eta(z_j)}| \leq c|z-z_j|^{1/2}.$$

And we have<sup>5</sup>

$$\begin{aligned} &|R(u+z_j)\delta^{-2}(z) - R(z_j)\delta_j^{-2}(z-z_j)^{-2i\eta(z_j)}| \\ &\leq |R(u+z_j) - R(z_j)||\delta^{-2}(z)| \\ &\quad + |\delta^{-2}(z) - \delta_j^{-2}(z-z_j)^{-2i\eta(z_j)}||R(z_j)| \\ &\leq c\left|\int_{z_j}^{u+z_j} R'(s)ds\right| + c|z-z_j|^{1/2} \\ &\text{by Cauchy-Schwartz inequality} \\ &\leq c\|R'\|_{L^2}|z-z_j|^{1/2} + c|z-z_j|^{1/2} \\ &\leq c|z-z_j|^{1/2}. \end{aligned}$$

Therefore

$$\begin{aligned} |\bar{\partial}E_{j,1}(z)| &\leq c\rho^{-1}|z-z_j|^{1/2} + c|R'(u+z_j)| \\ &\leq c(|z-z_j|^{-1/2} + |R'(u+z_j)|). \end{aligned} \tag{38}$$

Here, we have use the fact that  $u \leq \rho$ , which implies  $|z-z_j|^{1/2}/\rho = u^{1/2}/\rho \leq u^{-1/2}$ . Noting also that  $\sup |R| < 1$ , we have  $\frac{R}{1-|R|^2} \leq \frac{R}{1-\sup |R|}$ , and thus all the estimates for  $E_{j,1}$  can be smoothly moved to  $E_{j,k}$ ,  $k = 3, 4, 6$ . □

## 6 Separate Contributions and Phase Reduction

The RHP and the mixed  $\bar{\partial}$ -RHP we have discussed above are global. In this section, we shall approximate the global RHP by performing two steps: (1) separate contributions

<sup>5</sup> In the middle steps,  $c$  means a generic positive constant.

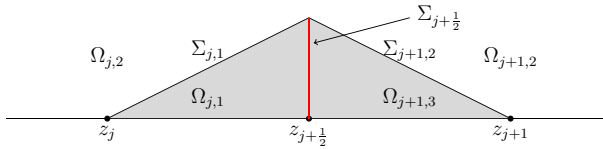


Fig. 3 Jumps in a small triangular region

from each stationary phase point, (2) phase reduction. Before that, let us first consider two saddle points  $z_j, z_{j+1}$ , and discuss  $\varepsilon_j = 1 = -\varepsilon_{j+1}$  for example. We will first remove the vertical segments, see Fig. 3:

$$\Sigma_{j+\frac{1}{2}} := \Omega_{j,1} \cap \Omega_{j+1,3} \cup \Omega_{j,6} \cap \Omega_{j+1,4} \setminus \mathbb{R},$$

where  $\Omega_{j,\cdot}$ 's are defined in 31.

Recall the constructions of  $E_{j,1}$  and  $E_{j+1,3}$  [see (33)], the boundary value of  $m^{[2]}(z)$  on  $\Sigma_{j+\frac{1}{2}}$  from  $\Omega_{j,1}$  is

$$m^{[1]}(z_{j+1/2} + iv)O_{j,1}(z_{j+1/2} + iv),$$

while from  $\Omega_{j+1,3}$  it is

$$m^{[1]}(z_{j+1/2} + iv)O_{j+1,3}(z_{j+1/2} + iv).$$

Both correspond to locally increasing parts of the phase function, and thus correspond to an upper/lower factorization. So the jump on the new contour  $\Sigma_{j+1/2}$  is  $O_{j+1,3}O_{j,1}^{-1}(z), z = z_{j+1/2} + iv$ , where the nontrivial entry is [regarding the property of  $\mathcal{K}$  and definitions of those matrix  $O_{j,k}$ , see (32) and (34)]:

$$(1 - \mathcal{K}(\phi))[R(z_j)\delta_j^{-2}(z_{j+1/2} - z_j + iv)^{-2i\eta(z_j)} - R(z_{j+1})\delta_{j+1}^{-2}(z_{j+1/2} - z_{j+1} + iv)^{-2i\eta(z_{j+1})}]e^{2it\theta(z_{j+1/2}+iv)},$$

with  $v \in (0, (z_{j+1/2} - z_j) \tan(\alpha))$  and  $\phi = \arg(z - z_j)$ .

Note that

$$|(z_{j+1/2} - z_j + iv)^{-2i\eta(z_j)}| = e^{2\eta(z_j)\phi} \leq e^{2\eta(z_j)\alpha}.$$

and

$$|e^{2it\theta(z_{j+1/2}+iv)}| \leq ce^{-2tdv}, \quad d = (z_{j+1} - z_j)/2.$$

Thus we have, for any  $z \in \Sigma_{j+\frac{1}{2}}$ ,

$$O_{j+1,3}O_{j,1}^{-1} - I = \mathcal{O}(e^{-ct \operatorname{Im} z}), \quad t \rightarrow \infty,$$

where  $c$  is some generic positive constant. Since the jump is close to  $I$ , by a small norm theory, the solution will also be close to  $I$ . In fact, we have the following estimate for the potential

$$\begin{aligned}
 & \left| \lim_{z \rightarrow \infty} z(m^{[2]} - I) \right| \\
 &= \left| \lim_{z \rightarrow \infty} z \mathcal{C}_{\Sigma_{j+\frac{1}{2}}}(m^{[2]}) \right| \\
 &\leq \frac{1}{2\pi} \int_0^{d \tan(\alpha)} \left| m_-^{[2]}(z_{j+1/2} + is) \right| \left| O_{j+1,3} O_{j,1}^{-1}(z_{j+1/2} + is) - I \right| ds \\
 &\leq \frac{1}{2\pi} \int_0^{d \tan(\alpha)} \left| m_-^{[2]}(z_{j+1/2} + is) \right| e^{-2tsd} ds \\
 &\leq \frac{1}{2\pi} \|m^{[1]}\|_{\Sigma_{j+\frac{1}{2},3}} \|\infty\| O_{j+1,3} \|\infty\| \int_0^{d \tan(\alpha)} e^{-2tsd} ds \\
 &= \mathcal{O}(t^{-1}),
 \end{aligned}$$

where  $\mathcal{C}_{\Sigma_{j+\frac{1}{2}}}(\cdot)$  means the Cauchy transform of a function over the contour  $\Sigma_{j+1/2}$ . Also, here we have assumed  $m^{[2]}$ , as a solution to the conjugated RHP, exists.<sup>6</sup> So it is analytic in a neighborhood of  $\Sigma_{j+\frac{1}{2}}$ , and hence, it is bounded on  $\Sigma_{j+\frac{1}{2}}$ . By the definition (see (33)) of  $O_{j+1,3}$ , it is continuous in  $\Sigma_{j+\frac{1}{2}}$  and does not blow up at the endpoints of  $\Sigma_{j+\frac{1}{2}}$ . So  $\|O_{j+1,3}\|_\infty$  is also finite.<sup>7</sup> Therefore, we can remove all those vertical segments by paying a price of error  $\mathcal{O}(t^{-1})$ , which will be dominated by the error generated by the  $\partial$ -problem (it is  $\mathcal{O}(t^{-3/4})$ , we will show it in a moment.) Let us denote the new RHP by  $\tilde{m}^{[2]}$ . To make it clear, we note that the jumps for  $\tilde{m}^{[2]}$  are

$$\tilde{v}^{[2]}(z) = \begin{cases} v^{[2]}(z), & z \in \cup_{j=1,\dots,l, k=1,2,3,4} \Sigma_{j,k}, \\ I, & z \in \cup_{j=1,\dots,l} \Sigma_{j+\frac{1}{2}} \cup \mathbb{R}. \end{cases}$$

Next, we will show that the RHP for  $\tilde{m}^{[2]}$  can be localized to each saddle point. For example, near  $z_j$ , along the segment  $\Sigma_{j,1} : z = z_j + u + iv, \arg z = \alpha$ , we have

$$|E_{j,1} e^{2it\theta}| \leq ce^{-2t \tan(\alpha)u^2}$$

It is well known (Deift and Zhou 1993; Do 2010) that the  $|E_{j,1} e^{2it\theta}| \leq ce^{-2t \tan(\alpha)u^2}$ , where let  $u \geq u_0 > 0$ , and then, the jump matrix will go to  $I$  with decaying rate at  $\mathcal{O}(e^{-ct}), c > 0$ , as  $t \rightarrow \infty$ . The RHP is localized in the small neighborhoods of those stationary phase points. Note that near each  $z_j$ , we have

$$\theta(z) = \theta(z_0) + \frac{\theta''(z_0)(z - z_0)^2}{2} + \mathcal{O}(|z - z_j|^3).$$

<sup>6</sup> The existence and uniqueness will be discussed later.

<sup>7</sup> Here the  $L^\infty(\Sigma)$  norm  $\|f(z)\|_\infty$  means  $\sup_{z \in \Sigma} |f(z)|$ , where  $|f(z)| = \max_{i,j=1,2,z \in \Sigma} |f_{i,j}(z)|$ .

By a similar argument of Lemma 3.35 in Deift et al. (1993) or subsection 8.2 in Do (2010) for the phase reduction, the error generated by reducing the phase function  $\theta$  to  $\theta(z_0) + \frac{\theta''(z_0)(z-z_0)^2}{2}$  will be bounded by  $\mathcal{O}(t^{-1})$ . Both analyses of the mentioned references are based on the analysis of the so-called Beals–Coifman operator (Beals and Coifman 1984). Now we shall simply describe it here. For the sake of simplicity, we only consider the RHP on the contour  $\Sigma_{j,1}$  (for more details, we direct the interested reader to Deift and Zhou 2003):

**Riemann–Hilbert problem 6.1** *Looking for 2 by 2 matrix-valued function  $\tilde{m}^{[2]}$  such that*

- (1)  $\tilde{m}(z)$  is analytic off  $\Sigma_{j,1}$ ;
- (2)  $\tilde{m}_+ = \tilde{m}_- v^{[2]}$ ,  $z \in \Sigma_{j,1}$ ;
- (3)  $\tilde{m} = I + \mathcal{O}(z^{-1})$ ,  $z \rightarrow \infty$ .

Since  $E_{j,1}|_{\Sigma_{j,1}}$  is analytic near  $\Sigma_{j,1}$  for  $z$  away from  $z_j$ , and enjoys a factorization:<sup>8</sup>

$$(I - w^-)^{-1}(I + w^+),$$

where

$$\begin{aligned} w^- &= I - (v^{[2]})^{-1} = (v^{[2]}) - I, \\ w^+ &= 0, \end{aligned}$$

and the superscribes  $\pm$  indicate the analyticity in the left/right neighborhood of the contour.

Following the definition in Beals and Coifman (1984), we define the Beals–Coifman operator, for any  $f \in L^2(\Sigma_{j,1})$ , as follows:

$$C_w(f) = C_+(f w^-) + C_-(f w^+),$$

where  $C$  means the usual Cauchy operator, i.e.,

$$Cf(z) = \frac{1}{2\pi i} \int_{\Sigma_{j,1}} \frac{f(s)}{s - z} ds,$$

and  $C_{\pm}$  means the non-tangential limits from left/right side.

The following proposition, which plays a fundamental role in Deift–Zhou’s method, is well known.

**Proposition 6.2** (see also proposition 2.11 in Deift and Zhou (2003)). *If  $\mu \in I + L^2$  solves the singular integral equation:*

$$\mu = I + C_w(\mu). \tag{39}$$

<sup>8</sup> ( $w^-, w^+$ ) will be called the factorization data for the jump matrix.

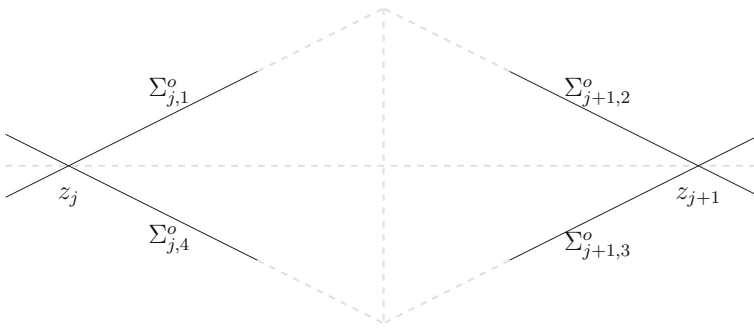


Fig. 4 New contours, dashed line segments are those deleted parts

Then, the (unique) solution to the RHP for  $\tilde{m}$  reads<sup>9</sup>

$$\tilde{m} = I + C(\mu w). \tag{40}$$

Then follow the localization principle in Deift and Zhou (1993), Do (2010) and Varzugin (1996), and the simple argument on the vertical segments, we arrive at a new RHP on the new contours: fixing  $\rho_0 > 0$  small, define

$$\begin{aligned} \Sigma_{j,1}^o &:= \{z : z = z_j + \rho e^{i\alpha}, \rho \in (0, \rho_0)\}, \\ \Sigma_{j,2}^o &:= \{z : z = z_j + \rho e^{i(\pi-\alpha)}, \rho \in (0, \rho_0)\}, \\ \Sigma_{j,3}^o &:= \{z : z = z_j - \rho e^{i\alpha}, \rho \in (0, \rho_0)\}, \\ \Sigma_{j,4}^o &:= \{z : z = z_j + \rho e^{i(\alpha-\pi)}, \rho \in (0, \rho_0)\}. \end{aligned}$$

Then with the new contour (see Fig. 4)  $\Sigma^o = \cup_{j=1,\dots,l,k=1,2,3,4} \Sigma_{j,k}^o$ , the new RHP reads as follows:

**Riemann–Hilbert problem 6.3** Looking for a 2 by 2 matrix-valued function  $\hat{m}^{[2]}$  such that

- (1)  $\hat{m}_+^{[2]} = \hat{m}_-^{[2]} \hat{v}^{[2]}$ ,  $z \in \Sigma^o$ , with  $\hat{v}^{[2]} = \tilde{v}^{[2]}|_{\Sigma^o}$ ;
- (2)  $\hat{m}^{[2]} = I + \mathcal{O}(z^{-1})$ ,  $z \rightarrow \infty$ .

Moreover, since the potential of the mKdV hierarchy can be recovered by formula (24), which can also be written as the Beals–Coifman solution:

$$q_{\text{RHP}}(x, t) = -\frac{1}{2\pi i} \int_{\Sigma} ((I - C_w)^{-1} I) w(s) ds. \tag{41}$$

Then, by localization, we have

$$\int_{\Sigma} ((I - C_w)^{-1} I) w(s) ds = \int_{\Sigma^o} ((I - C_w)^{-1} I) w(s) ds + \mathcal{O}(t^{-1}), \quad t \rightarrow \infty, \tag{42}$$

<sup>9</sup> Here,  $w = w^+ + w^-$ .

where  $\Sigma$  is the contour before localization and  $w$  can be easily defined in each cross since the jumps are all triangle matrices and all entries in the diagonal are one. Let us denote

$$q_{\text{RHP}}^o(x, t) = -\frac{1}{2\pi i} \int_{\Sigma^o} ((I - C_w)^{-1} I)w(s)ds. \tag{43}$$

Then from the localization principal, we have

$$q_{\text{RHP}}(x, t) = q_{\text{RHP}}^o(x, t) + \mathcal{O}(t^{-1}), \quad t \rightarrow \infty. \tag{44}$$

Moreover, we define the RHP ( $m^{[3]}$ ) which corresponds to the local Beals–Coifman solution (i.e.,  $q_{\text{RHP}}^o$ ) as follows:

**Riemann–Hilbert problem 6.4** *Looking for a 2 by 2 matrix-valued function  $m^{[3]}$  such that*

- (1)  $m_{\pm}^{[3]} = m_{\pm}^{[3]}v^{[3]}(z), \quad z \in \Sigma^o$ , with jump matrix reads  $v^{[3]} = \hat{v}^{[2]} \upharpoonright_{\Sigma^o}$ ;
- (2)  $m^{[3]} = I + \mathcal{O}(z^{-1}), \quad z \rightarrow \infty$ .

However, the integral  $\int_{\Sigma^o} ((I - C_w)^{-1} I)w(s)ds$  is still hard to compute, and following the Deift–Zhou method, we need to separate the contributions from each stationary phase point. Thus, we need the following important lemma.

**Lemma 6.5** (see equation (3.64) or proposition 3.66 in Deift and Zhou (1993)). *As  $t \rightarrow \infty$ ,*

$$\int_{\Sigma^o} ((1 - C_w)^{-1} I)w = \sum_{j=1}^l \int_{\Sigma_j^o} ((1 - C_{w_j})^{-1} I)w_j + \mathcal{O}(t^{-1}), \tag{45}$$

where  $w_j$  is the factorization data supported on  $\Sigma_j^o = \cup_{k=1}^4 \Sigma_{j,k}^o$ ,  $w = \sum_{j=1}^l w_j$  and  $\Sigma^o = \cup_j^l \Sigma_j^o$ .

**Proof** First, recall the following observation by Varzugin (1996),

$$(1 - C_w) \left( 1 + \sum_j C_{w_j} (1 - C_{w_j})^{-1} \right) = 1 - \sum_{j \neq k} C_{w_j} C_{w_k} (1 - C_{w_k})^{-1}.$$

With the hints from this observation, we need to estimate the norms of  $C_{w_j} C_{w_k}$  from  $L^\infty$  to  $L^2$  and from  $L^2$  to  $L^2$ . Also from next section (with a small norm argument), we know  $(1 - C_{w_j})^{-1}$  are uniformly bounded in  $L^2$  sense. Now let us focus on the contour  $\Sigma_{j,1}^o$ , and  $\varepsilon = 1$ . Then, the nontrivial entry of the factorization data is  $E_{j,1}(z)e^{-2it\theta(z)}$ ,  $z \in \Sigma_{j,1}^o$ , and thus we have

$$|w_j \upharpoonright_{\Sigma_{j,1}^o}| \leq ce^{-2t \tan(\alpha)u^2},$$

which implies that  $\|w_j \upharpoonright_{\Sigma_{j,1}^o}\|_{L^1} = \mathcal{O}(t^{-1/2})$  and  $\|w_j \upharpoonright_{\Sigma_{j,1}^o}\|_{L^2} = \mathcal{O}(t^{-1/4})$ . Then following exactly the same steps in the proof of Deift and Zhou (1993, Lemma 3.5), we have for  $j \neq k$

$$\begin{aligned} \|C_{w_j} C_{w_k}\|_{L^2(\Sigma^o)} &= \mathcal{O}(t^{-1/2}), \\ \|C_{w_j} C_{w_k}\|_{L^\infty \rightarrow L^2(\Sigma^o)} &= \mathcal{O}(t^{-3/4}). \end{aligned}$$

Then use the resolvent identities and the Cauchy–Schwartz inequality,

$$\begin{aligned} ((1 - C_w)^{-1} I) &= I + \sum_{j=1}^l C_{w_j} (1 - C_{w_j})^{-1} I \\ &+ \left[ 1 + \sum_{j=1}^l C_{w_j} (1 - C_{w_j})^{-1} \right] \left[ 1 - \sum_{j \neq k} C_{w_j} C_{w_k} (1 - C_{w_k})^{-1} \right]^{-1} \\ &\times \left( \sum_{j \neq k} C_{w_j} C_{w_k} (1 - C_{w_k})^{-1} \right) I \\ &= I + \sum_{j=1}^l C_{w_j} (1 - C_{w_j})^{-1} I + ABDI, \end{aligned}$$

where

$$\begin{aligned} A &:= 1 + \sum_{j=1}^l C_{w_j} (1 - C_{w_j})^{-1}, \\ B &:= \left[ 1 - \sum_{j \neq k} C_{w_j} C_{w_k} (1 - C_{w_k})^{-1} \right]^{-1}, \\ D &:= \sum_{j \neq k} C_{w_j} C_{w_k} (1 - C_{w_k})^{-1}, \end{aligned}$$

and thus

$$\begin{aligned} \left| \int_{\Sigma^o} ABDI w \right| &\leq \|A\|_{L^2} \|B\|_{L^2} \|D\|_{L^\infty \rightarrow L^2} \|w\|_{L^2} \\ &\leq ct^{-3/4} t^{-1/4} = \mathcal{O}(t^{-1}). \end{aligned}$$



Then applying the restriction lemma (Deift and Zhou 1993, Lemma 2.56), we have

$$\begin{aligned} \int_{\Sigma^o} (I + C_{w_j}(1 - C_{w_j})^{-1}I)w \upharpoonright_{\Sigma_j^o} &= \int_{\Sigma_j^o} (I + C_{w_j}(1 - C_{w_j})^{-1}I)w \\ &= \int_{\Sigma_j^o} ((1 - C_{w_j})^{-1}I)w_j. \end{aligned}$$

Therefore, the proof is done. □

### 7 A Model Riemann–Hilbert Problem

In the previous section, we have reduce the global RHP to  $l$  local RHPs near each stationary phase point due to Lemma 6.5. In fact, near each stationary phase point, we need to compute the integral  $\int_{\Sigma_j^o} ((1 - C_{w_j})^{-1}I)w_j$ , which is equivalent to a local RHP. In this section, we will approximate the local RHPs by a model RHP which can be solved explicitly by solving a parabolic-cylinder equation. Consider the following RHP:

**Riemann–Hilbert problem 7.1** *Looking for a 2 by 2 matrix-valued function  $P(\xi; R)$  such that*

(1)  $P_+(\xi; R) = P_-(\xi; R)J(\xi)$ ,  $\xi \in \mathbb{R}$ , where

$$J(\xi) = \begin{pmatrix} 1 - |R|^2 & -\bar{R} \\ R & 1 \end{pmatrix}$$

*is a constant matrix with respect to  $\xi$  and the constant  $R$  satisfies  $|R| < 1$ ;*

(2)  $P(\xi; R) = (I + P_1\xi^{-1} + \mathcal{O}(\xi^{-2}))\xi^{i\eta\sigma_3}e^{-i\frac{\xi^2}{4}\sigma_3}$ ,  $\xi \rightarrow \infty$ , where  $P_1 = \begin{pmatrix} 0 & \beta \\ \bar{\beta} & 0 \end{pmatrix}$ .<sup>10</sup>

Then by Liouville’s argument,  $P'P^{-1}$  is analytic and thus

$$P'(\xi) = \left( -\frac{i\xi}{2}\sigma_3 + \frac{i}{2}[\sigma_3, P_1] \right) P(\xi), \tag{46}$$

which can be solved in terms of the parabolic-cylinder equation, and apply the asymptotics formulas we can eventually determine that <sup>11</sup>

$$\beta = -\frac{\sqrt{2\pi}e^{i\pi/4}e^{-\pi\eta/2}}{R\Gamma(-a)}, \tag{47}$$

<sup>10</sup> The fact is nontrivial, but the derivation is in Deift and Zhou (1993), see equation (4.47)

<sup>11</sup> Compare with formulas (4.27) in Deift and Zhou (1993), we only differ by a sign, but since the sign does not affect the parabolic-cylinder equation, we can pass the sign to (47), where we have borrowed the result (4.45) in Deift and Zhou (1993).

where

$$a = i\eta, \tag{48}$$

with  $\eta = -\frac{1}{2\pi} \log(1 - |R|^2)$ .

The above result has been presented in the literature<sup>12</sup> in many ways. Here, we follow the representations in Deift and Zhou (1993). Next, we will connect this model RHP to the original RHP. Recall, near stationary phase point  $z_j$ , we need to estimate integral  $\int_{\Sigma_j^o} ((1 - C_{w_j})^{-1} I)(w_{j+} + w_{j-})$ , which is equivalent to solve the following RHP ( $m^{[3,j]}$ ,  $j = 1, \dots, l$ ):

(1)  $m_+^{[3,j]}(z) = m_-^{[3,j]}(z)v^{[3,j]}(z)$ ,  $z \in \Sigma_j^o$ . The jump matrix reads

$$v^{[3,j]}(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ R_j^\sharp(z - z_j)^{-2i\eta(z_j)} e^{-it\theta''(z_j)(z - z_j)^2} & 1 \end{pmatrix}, & z \in \Sigma_{j,1}^o, \\ \begin{pmatrix} 1 - \frac{\bar{R}_j^\sharp}{1 - |R_j^\sharp|^2} (z - z_j)^{2i\eta(z_j)} e^{it\theta''(z_j)(z - z_j)^2} & \\ 0 & 1 \end{pmatrix}, & z \in \Sigma_{j,2}^o, \\ \begin{pmatrix} 1 & 0 \\ \frac{R_j^\sharp}{1 - |R_j^\sharp|^2} (z - z_j)^{2i\eta(z_j)} e^{-it\theta''(z_j)(z - z_j)^2} & 1 \end{pmatrix}, & z \in \Sigma_{j,3}^o, \\ \begin{pmatrix} 1 - \bar{R}_j^\sharp (z - z_j)^{-2i\eta(z_j)} e^{it\theta''(z_j)(z - z_j)^2} & \\ 0 & 1 \end{pmatrix}, & z \in \Sigma_{j,4}^o, \end{cases} \tag{49}$$

where  $R_j^\sharp = R_j \delta_j^{-2} e^{-2it\theta(z_j)}$ ;

(2)  $m^{[3,j]} = I + \mathcal{O}(z^{-1})$ ,  $z \rightarrow \infty$ .

Set  $\xi = (2t\theta''(z_j))^{1/2}(z - z_j)$  and by closing lenses, we arrive at an equivalent RHP on the real line:

(1)  $m_+^{[4,j]}(\xi) = m_-^{[4,j]}v^{[4,j]}(\xi)$ ,  $\xi \in \Sigma_j^p$ . The new jump is

$$v^{[4,j]}(\xi) = (2\theta''(z_j)t)^{-\frac{i\eta(z_j)}{2}} \text{ad } \sigma_3 \xi^{i\eta(z_j)} \text{ad } \sigma_3 e^{-\frac{i\xi^2}{4}} \text{ad } \sigma_3 \begin{pmatrix} 1 - |R_j^\sharp|^2 - \bar{R}_j^\sharp & \\ R_j^\sharp & 1 \end{pmatrix}; \tag{50}$$

(2)  $m^{[4,j]} = I + \mathcal{O}(\xi^{-1})$ ,  $\xi \rightarrow \infty$ .

<sup>12</sup> The first description of this model RHP was presented by Its Its (1981). Later examples of the model can be found in Deift and Zhou (1993), Deift and Zhou (2003), Dieng and McLaughlin (2008), Do (2010), Varzugin (1996), Ma (2019) and Ma (2020).

Comparing with the model RHP, we observe that  $m^{[4,j]}(\xi)(2\theta''(z_j)t)^{-\frac{i\eta(z_j)}{2}}\sigma_3\xi^{i\eta(z_j)}\sigma_3 e^{-\frac{i\xi^2}{4}\sigma_3}$  solves the model RHP, which leads to

$$m_{1,12}^{[4]} = \frac{-\sqrt{2\pi}e^{i\pi/4}e^{-\pi\eta(z_j)/2}}{R_j^\# \Gamma(-i\eta(z_j))}, \tag{51}$$

$$m_{1,21}^{[4]} = \frac{\sqrt{2\pi}e^{-i\pi/4}e^{-\pi\eta(z_j)/2}}{\bar{R}_j^\# \Gamma(i\eta(z_j))}. \tag{52}$$

Changing the variable  $\xi$  back to  $z$ , we have

$$m_{1,12}^{[3,j]}(t) = -(2t\theta''(z_j))^{-\frac{1}{2}-\frac{i\eta(z_j)}{2}} \frac{\sqrt{2\pi}e^{i\pi/4}e^{-\pi\eta(z_j)/2}}{R_j^\# \Gamma(-i\eta(z_j))},$$

$$m_{1,21}^{[3,j]}(t) = (2t\theta''(z_j))^{-\frac{1}{2}+\frac{i\eta(z_j)}{2}} \frac{\sqrt{2\pi}e^{-i\pi/4}e^{-\pi\eta(z_j)/2}}{\bar{R}_j^\# \Gamma(i\eta(z_j))}.$$

Noting that  $R_j^\# = R_j\delta_j^{-2}e^{2it\theta(z_j)}$ , one can rewrite it in a neat way:

$$m_{1,12}^{[3,j]}(t) = \frac{|\eta(z_j)|^{1/2}}{\sqrt{2t\theta''(z_j)}}e^{i\varphi(t)},$$

$$m_{1,21}^{[3,j]}(t) = \frac{|\eta(z_j)|^{1/2}}{\sqrt{2t\theta''(z_j)}}e^{-i\varphi(t)},$$

where the phase is

$$\varphi(t) = \frac{-3\pi}{4} - \arg \Gamma(-i\eta(z_j)) - 2t\theta(z_j) - \frac{\eta(z_j)}{2} \log |2t\theta''(z_j)| + 2 \arg(\delta_j) + \arg(R_j).$$

Here, we have used the fact that  $|\beta|^2 = \eta$ . Denoting

$$q_{as}(x, t) = -2i \sum_{j=1}^l \frac{|\eta(z_j)|^{1/2}}{\sqrt{2t\theta''(z_j)}}e^{i\varphi(t)}, \tag{53}$$

then the connection formula (43) and Lemma 6.5 lead to

$$q_{\text{RHP}}^o(x, t) = q_{as}(x, t) + \mathcal{O}(t^{-1}), \quad t \rightarrow \infty. \tag{54}$$

### 8 Errors from the Pure $\bar{\partial}$ -Problem

In this section, we will discuss the error generated from the pure  $\bar{\partial}$ -problem of  $m^{[2]}$ . Let us denote

$$E(z) = m^{[2]}(m_{\text{RHP}}^{[2]})^{-1}, \tag{55}$$

where  $m_{\text{RHP}}^{[2]}$  denotes the solution to the pure RHP part of  $m^{[2]}$ . Assuming the existence (which we will be provided in the next section), and by the normalization condition, we have

$$E(z) = 1 + (m_1^{[2]} - m_{\text{RHP},1}^{[2]})z^{-1} + \mathcal{O}(z^{-2}), \quad z \rightarrow \infty. \tag{56}$$

Due to the procedure of localization and separation of the contributions, we can approximate  $m_{\text{RHP}}^{[2]}$  by  $\hat{m}^{[2]}$ , and the error of approximating the potential is of  $\mathcal{O}(t^{-1})$  as  $t \rightarrow \infty$ . Thus, by the Eq. (24),

$$q(x, t) = q_{\text{RHP}}(x, t) + \mathcal{O}(t^{-1}) + \lim_{z \rightarrow \infty} z(E - I), \quad t \rightarrow \infty. \tag{57}$$

Moreover, from this construction [Eq. (55)], there is no jump on the contours  $\Sigma_{j,k}$ ,  $k = 1, 2, 3, 4$ , but only a pure  $\bar{\partial}$ -problem is left due to the non-analyticity. The  $\bar{\partial}$ -problem reads

$$\bar{\partial} E = E W, \tag{58}$$

where

$$W(z) = m_{\text{RHP}}^{[2]} \bar{\partial} O(z) (m_{\text{RHP}}^{[2]})^{-1}. \tag{59}$$

From the normalization condition of  $m_{\text{RHP}}^{[2]}$ , we see it is uniformly bounded by  $\frac{c}{1 - \sup R}$ . And to estimate the errors of recovering the potential, one actually needs to estimate  $\lim_{z \rightarrow \infty} z(E - I)$ , where the limit can be chosen along any rays that are not parallel to  $\mathbb{R}$ . For simplicity, we will take the imaginary axis. The  $\bar{\partial}$ -problem is equivalent to the following Fredholm integral equation by a simple application of the generalized Cauchy integral formula:

$$E(z) = I - \frac{1}{\pi} \int_{\mathbb{C}} \frac{E(s)W(s)}{s - z} dA(s). \tag{60}$$

In the following, we will show for each fixed  $z \in \mathbb{C}$ ,  $\mathcal{K}_W(E)(z) := \int_{\mathbb{C}} \frac{E(s)W(s)}{s - z} dA(s)$  is bounded and then by the dominated convergence theorem, we will show  $\lim_{z \rightarrow \infty} z(E - I) = \mathcal{O}(t^{-3/4})$ . First of all, since  $m^{[3]}$  is uniformly bounded, upon setting  $z = z_j +$

$u + iv$ , we have

$$\|W\|_\infty \lesssim \begin{cases} |\bar{\partial} E_{j,k}| e^{-2t\theta''(z_j)uv}, & z \in \Omega_{j,k}, k = 1, 4, \\ |\bar{\partial} E_{j,k}| e^{2t\theta''(z_j)uv}, & z \in \Omega_{j,k}, k = 3, 6, \end{cases}, \tag{61}$$

where  $0 \leq a \lesssim b$  means there exists  $C > 0$  such that  $a \leq Cb$ . Then, we have

$$\mathcal{K}_W(E) \leq \|E\|_\infty \int_{\mathbb{C}} \frac{\|W(s)\|_\infty}{|s - z|} dA(s). \tag{62}$$

We claim the following lemma:

**Lemma 8.1** *Let  $\Omega = \{s : s = \rho e^{i\phi}, \rho \geq 0, \phi \in [0, \pi/4]\}$ , and  $z \in \Omega$ . Then,*

$$\int_{\Omega} \frac{|u^2 + v^2|^{-1/4} e^{-tuv}}{|u + iv - z|} dudv = \mathcal{O}(t^{-1/4}). \tag{63}$$

**Proof** Since there are two singularities of the integrand at  $z$  and  $(0, 0)$ . In the first case, set  $z \neq 0$ , and let  $d = \text{dist}(z, 0)$ . We split  $\Omega$  into three parts:  $\Omega_1 \cup \Omega_2 \cup \Omega_3$ , where  $\Omega_1 = \{s : |s| < d/3\} \cap \Omega$ ,  $\Omega_2 = \{s : |s - z| < d/3\} \cap \Omega$  and  $\Omega_3 = \Omega \setminus (\Omega_1 \cup \Omega_2)$ . In the region  $\Omega_1$ ,  $|s - z| \geq 2d/3$ , and thus

$$\begin{aligned} \left| \int_{\Omega_1} \frac{|u^2 + v^2|^{-1/4} e^{-tuv}}{|u + iv - z|} dudv \right| &\leq \frac{3}{2d} \int_0^\infty \int_0^u \frac{e^{-tuv}}{(u^2 + v^2)^{1/4}} dvdu \\ &\text{substituted } v = wu \\ &\leq \frac{3}{2d} \int_0^\infty \int_0^1 \frac{e^{-tu^2w}}{(1 + w^2)^{1/4}} u^{1/2} dwdu \\ &\leq \frac{3}{2d} \int_0^\infty \int_0^1 e^{-tu^2w} u^{1/2} dwdu \\ &= \frac{3}{2d} \int_0^\infty \frac{1 - e^{-tu^2}}{tu^{3/2}} du \\ &= \frac{3}{2d} \frac{1}{2} t^{-3/4} \int_0^\infty \frac{1 - e^{-u}}{u^{5/4}} du \\ &= \frac{3}{d} \Gamma(3/4) t^{-3/4}. \end{aligned} \tag{64}$$

In the region  $\Omega_2, |s|^{-1/2} \leq (2d/3)^{-1/2}$ , we have

$$\begin{aligned} \left| \int_{\Omega_1} \frac{|u^2 + v^2|^{-1/4} e^{-tuv}}{|u + iv - z|} du dv \right| &\leq \sqrt{\frac{3}{2d}} \int_{\Omega_2} \frac{e^{-tuv}}{((u-x)^2 + (v-y)^2)^{1/2}} dv du \\ &\leq \sqrt{\frac{3}{2d}} \int_0^{d/3} \int_0^{2\pi} e^{-t(x+\rho \cos(\theta))(y+\rho \sin(\theta))} d\theta d\rho \\ &\leq \frac{2\pi}{3} \sqrt{\frac{3d}{2}} e^{-txy}. \end{aligned}$$

While in the region  $\Omega_3$ ,

$$\left| \int_{\Omega_3} \frac{|u^2 + v^2|^{-1/4} e^{-tuv}}{|u + iv - z|} du dv \right| \leq \int_0^\infty \int_0^u e^{-tuv} dv du = \mathcal{O}(t^{-1}).$$

Now consider  $z = 0$ . We have

$$\begin{aligned} \left| \int_{\Omega} \frac{e^{-tuv}}{(u^2 + v^2)^{3/4}} dA(u, v) \right| &= \int_0^\infty \int_0^u \frac{e^{-tuv}}{(u^2 + v^2)^{3/4}} dv du \\ &= \int_0^\infty \int_0^1 \frac{e^{-tu^2w}}{(1 + w^2)^{3/4} u^{1/2}} dw du \\ &\leq \int_0^\infty \int_0^1 \frac{e^{-tu^2w}}{u^{1/2}} dw du \\ &= \int_0^\infty \frac{1 - e^{-tu^2}}{tu^{5/2}} du \\ &= \int_0^\infty \frac{1 - e^{-u}}{t t^{-5/4} u^{5/4}} t^{-1/2} \frac{1}{2} u^{-\frac{1}{2}} du \\ &= \frac{1}{2} t^{-1/4} \int_0^\infty \frac{1 - e^{-u}}{u^{7/4}} du \\ &= \frac{3}{8} t^{-1/4} \Gamma(1/4). \end{aligned}$$

By assembling all together, the proof is done. □

**Remark 8.2** The essential fact that makes the above true is the rapid decay of the exponential factor in the region. And the lemma also tells us that those mild singularities, which have rational order growth, can be absorbed by the exponential factor. Back to our situation, after some elementary transformations (translation and rotation), the estimation of  $\int_{\mathbb{C}} \frac{\|W(s)\|_\infty}{|s-z|} dA(s)$  will eventually reduce to a similar situation discussed in the above lemma.

Based on Lemma 8.1, we know that when  $t$  is sufficiently large,  $\|\mathcal{K}_W\| < 1$  and thus the resolvent is uniformly bounded, and we obtain the following estimate by taking a

standard Neumann series, for some sufficiently large  $t_0$ ,

$$\|E - I\|_\infty = \|\mathcal{K}_W(1 - \mathcal{K}_W)^{-1}I\|_\infty \leq \frac{ct^{-1/4}}{1 - ct^{-1/4}} \leq ct^{-1/4}, \quad t > t_0. \quad (65)$$

Now since for each  $z \in \Omega_{j,k}$ , we have  $|\bar{\partial}E_{j,k}(z)| \leq c(|z - z_j|^{-1/2} + |R'(u + z_j)|)$ , and apply the dominated convergence theorem, we have

$$\lim_{z \rightarrow \infty} |z(E - I)| \leq \frac{1}{\pi} \sum_{j=1}^l \sum_{k=1}^4 \|E\|_{L^\infty} \int_{\Omega_{j,k}} \|W\|_\infty ds,$$

and use Lemma 8.1 again, we will eventually have:

$$E_1 = \lim_{z \rightarrow \infty} |z(E - I)| = \mathcal{O}(t^{-3/4}). \quad (66)$$

### 9 Asymptotics Representation

First, we summarize all the steps as following (see Fig. 5):

- (1) Initial RHP  $m^{[0]} = m$ , see RHP 2.1.
- (2) Conjugate initial RHP to obtain  $m^{[1]} = m^{[0]}\delta^{\sigma_3}$ , see RHP 4.1.
- (3) Open lenses to obtain a mixed  $\bar{\partial}$ -RHP 5.1.
- (4) Approximate the RHP part  $m_{\text{RHP}}^{[2]}$  of  $m^{[2]}$  by removing  $\Sigma_{j+\frac{1}{2}}$  (see RHP 6.1), localization (see RHP 6.3), reducing the phase function and separating the contributions (see RHP 6.4). The error term is  $\mathcal{O}(t^{-1})$ . Note those exponential decaying errors are absorbed by  $\mathcal{O}(t^{-1})$ .
- (5) Comparing  $m^{[2]}$  and  $m_{\text{RHP}}^{[2]}$  and computing the error by analysis a pure  $\bar{\partial}$ -problem. The error term is  $\mathcal{O}(t^{-3/4})$ .

Now by undoing all the steps, we arrive at:

$$m^{[0]}(z) = E(z)m_{\text{RHP}}^{[2]}(z)O^{-1}(z)\delta^{-\sigma_3}.$$

From the definition of the lens-opening matrices, it is easy to see (e.g., Dieng et al. 2018, Lemma 3.2) that  $O(z) = I + o(1/z)$  as  $z \rightarrow \infty$  along any straight line except the real line (that is fine since when we recover the potential, we can choose the limit of  $z \rightarrow \infty$  not along the real line.). Also, since  $\delta^{-\sigma_3}$  is a diagonal matrix, they do not

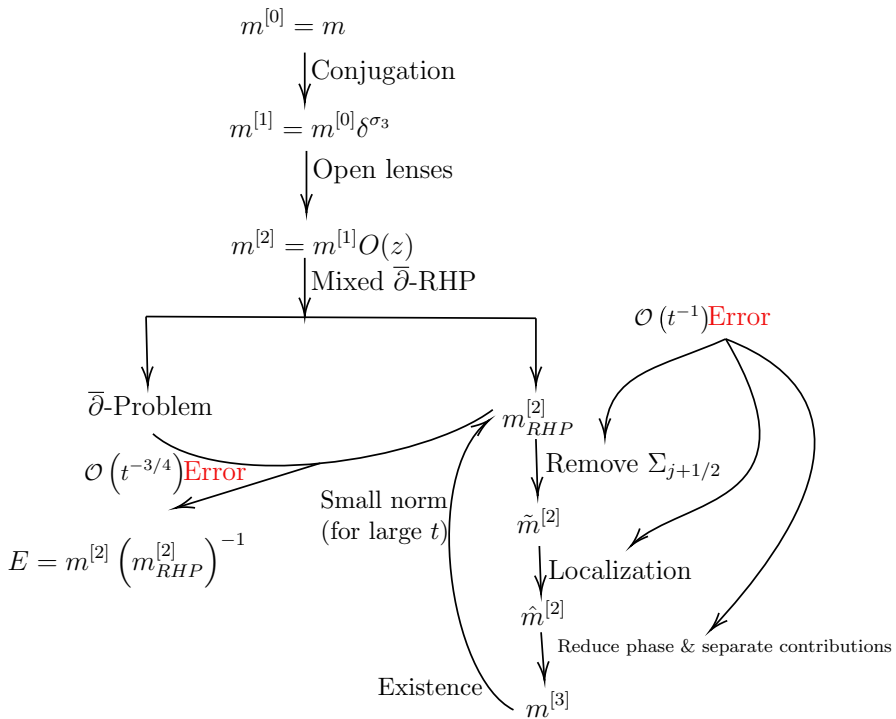


Fig. 5 Steps of the  $\bar{\partial}$ -steepest method

affect the recovering of the potential. Thus, we obtain

$$\begin{aligned}
 q(x, t) &= -2i(m_{RHP,1,12}^{[2]} + E_{1,12}) \\
 &= q_{RHP}(x, t) - 2iE_{1,12} \\
 &\text{by (44), (54)} \\
 &= q_{as}(x, t) + \mathcal{O}(t^{-1}) - 2iE_{1,12} \\
 &\text{by (66)} \\
 &= q_{as}(x, t) + \mathcal{O}(t^{-1}) + \mathcal{O}(t^{-3/4}) \\
 &= q_{as}(x, t) + \mathcal{O}(t^{-3/4}),
 \end{aligned}$$

where  $q_{as}(x, t)$  is given by Eq. (53).

**Remark 9.1** Note that due to the analysis in section 7, according to Proposition 2.6 and Proposition 2.11 of Deift and Zhou (2003), together with the small norm theory, the existence and uniqueness of the model RHP implies, via the estimates of the corresponding Beals–Coifman operators, the existence and uniqueness of RHP 6.4. Similarly, we obtain the existence and uniqueness of  $\tilde{m}^{[2]}$ ,  $\hat{m}^{[2]}$  and eventually  $m_{RHP}^{[2]}$ .



**Remark 9.2** From Eq. (53), we know  $q_{as}$  is  $\mathcal{O}(t^{-1/2})$  as  $t \rightarrow \infty$  in the region  $x < 0$  and consider the limit along the ray  $x = -ct$  for some positive constant  $c$ .

### 10 Fast Decaying Region

In this section and the next section, we will focus only on the case of the defocusing mKdV flow. In this case, the phase function reads

$$\theta(z; x, t) = \frac{x}{t}z + cz^n, \quad n \text{ is an odd positive integer.}$$

In the previous sections, we have derived the asymptotic solutions to the defocusing mKdV flow in the oscillating region, namely along the ray  $x = -vt, v > 0, t \rightarrow \infty$ . In this section, we consider the long-time behavior along the ray  $x = vt, v > 0, t \rightarrow \infty$ , which we call it the fast decaying region as we will soon prove in this region, the solution decay like  $\mathcal{O}(t^{-1})$ , which is faster than the leading term in the oscillating region, i.e.,  $\mathcal{O}(t^{-1/2})$ , as  $t \rightarrow \infty$ .

In the fast decaying region, the phase function enjoys the following properties:

- (1) There exists  $\epsilon = \epsilon(n, v) > 0$  such that  $\pm \text{Im}(\theta) > 0$  in the strips  $\{z : \pm \text{Im}(z) \in (0, \epsilon)\}$ , respectively.
- (2) There exists  $M \in (0, 1/\epsilon)$  such that  $\text{Im}(\theta) \geq nvu^{n-1}$  for  $|u| \geq M\epsilon$  and  $\text{Im}(\theta) \geq v(1 - (M\epsilon)^2)$  for  $|u| \leq M\epsilon$ . Here  $z = u + iv$ .

First, we will formulate the RHP as follows:

**Riemann–Hilbert problem 10.1** Given  $R(z) \in H^{1,1}(\mathbb{R})$ , looking for a 2 by 2 matrix-value function  $m$  such that

- (1)  $m_+ = m_- e^{-it\theta(z)} \text{ad } \sigma_3 v(z), z \in \mathbb{R}$ , where the jump matrix is given by

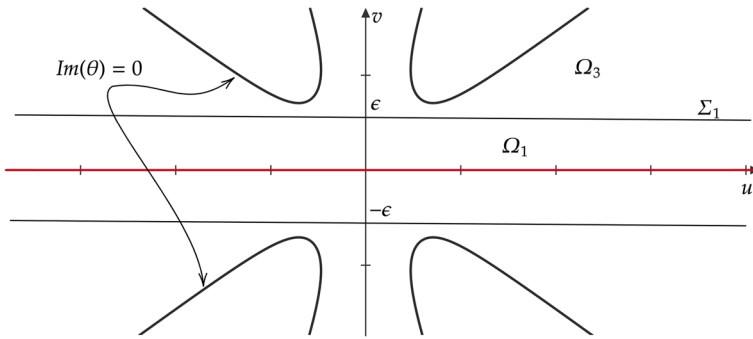
$$v(z) = \begin{pmatrix} 1 - |R|^2 & -\bar{R} \\ R & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\bar{R} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ R & 1 \end{pmatrix}; \tag{67}$$

- (2)  $m = I + \mathcal{O}(z^{-1}), z \rightarrow \infty$ .

**Theorem 10.2** For the above RHP, the solution  $m$  enjoys the following asymptotics as  $t \rightarrow \infty$ :

$$m_1(t) = \mathcal{O}(t^{-1}). \tag{68}$$

where  $m = I + m_1(t)/z + \mathcal{O}(z^{-2}), z \rightarrow \infty$ .



**Fig. 6**  $\bar{\partial}$ -extension for the case of the fast decaying region. Here, we only draw the case when  $n = 5$ . For generic odd  $n$ , there are  $\frac{n-1}{2}$  curves of  $\text{Im } \theta = 0$  in the upper and in the lower half plane

**Proof** In the light of  $\bar{\partial}$ -steepest descent, to open the lens, we multiply a smooth function  $O(z)$  to  $m$ , where  $O(z)$  is given by

$$O(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ \frac{-R(\text{Re } z)e^{2it\theta(z)}}{1+(\text{Im } z)^2} & 1 \end{pmatrix}, & z \in \Omega_1, \\ \begin{pmatrix} 1 & \frac{\bar{R}(\text{Re } z)e^{-2it\theta(z)}}{1+(\text{Im } z)^2} \\ 0 & 1 \end{pmatrix}, & z \in \Omega_1^*, \\ I, & z \in \mathbb{C} \setminus (\Omega_1 \cup \Omega_1^*), \end{cases}$$

where (see Fig. 6)

$$\begin{aligned} \Omega_1 &= \{z : \text{Im } z \in (0, \epsilon)\}, \\ \Omega_1^* &= \{z : \text{Im } z \in (-\epsilon, 0)\}. \end{aligned}$$

Let us denote  $\Sigma_1 = \{z : \text{Im } z = \epsilon\}$ , see Fig. 6, and let

$$\tilde{m} = mO, \quad z \in \mathbb{C}.$$

Now as usual, we obtain a  $\bar{\partial}$ -RHP, due to the exponential decaying of the off-diagonal term, and the jump matrix of the RHP part will approach  $I$ . Hence by a small norm argument, we know the solution will close to  $I$  as  $z \rightarrow \infty$ . Denote the solution to the pure RHP by  $m^\sharp$ , and small norm theory leads to  $m^\sharp = I + \mathcal{O}(e^{-c(\epsilon)t})$ ,  $c(\epsilon) > 0$ ,  $z \rightarrow \infty$ . Next, consider

$$E = \tilde{m}(m^\sharp)^{-1}. \tag{69}$$

By direct computation, one can show  $E$  does not have any jump on  $\Sigma_1$  and it satisfies a pure  $\bar{\partial}$ -problem:

$$\bar{\partial}E = EW, \tag{70}$$

where

$$W = \begin{cases} \begin{pmatrix} 0 & m^\sharp e^{-2it\theta(z)} \bar{\partial} \left( \frac{\bar{R}(\operatorname{Re} z)}{1+(\operatorname{Im} z)^2} \right) (m^\sharp)^{-1} \\ 0 & 0 \end{pmatrix}, & z \in \Omega_1, \\ \begin{pmatrix} 0 & 0 \\ m^\sharp e^{2it\theta(z)} \bar{\partial} \left( \frac{-R(\operatorname{Re} z)}{1+(\operatorname{Im} z)^2} \right) (m^\sharp)^{-1} & 0 \end{pmatrix}, & z \in \Omega_1^*, \\ 0, & z \in \mathbb{C} \setminus (\Omega_1 \cup \Omega_1^*), \end{cases}$$

where  $\bar{\partial} = \frac{1}{2}(\partial_{\operatorname{Re} z} + i\partial_{\operatorname{Im} z})$ .

Since  $R, \bar{R} \in H^{1,1}$ ,  $\bar{\partial} \left( \frac{\bar{R}(\operatorname{Re} z)}{1+(\operatorname{Im} z)^2} \right), \bar{\partial} \left( \frac{-R(\operatorname{Re} z)}{1+(\operatorname{Im} z)^2} \right)$  are uniformly bounded by some nonnegative  $L^2(\mathbb{R})$  function  $f(\operatorname{Re} z)$ . Note that  $m^\sharp$  is uniformly close to  $I$ , and setting  $z = u + iv$ , and considering  $z \in \Omega_1$  first, we have

$$\|W\|_\infty \leq f(u)e^{-t \operatorname{Im} \theta(u,v)}, \forall u \in \mathbb{R}, v \in (0, \epsilon).$$

By the same procure as the one in section 8, the error of approximating  $m$  by the identity matrix is given by the following integral (since there is only one nontrivial entry of  $W$ ):

$$\Delta := \int_0^\epsilon \int_{\mathbb{R}} f(u)e^{-t \operatorname{Im} \theta} du dv. \tag{71}$$

Split the  $u$  into two regions: (1)  $|u| \leq M\epsilon$ , (2)  $|u| \geq M\epsilon$ . And denote them by  $\Delta_1, \Delta_2$  respectively. Then,  $\Delta = \Delta_1 + \Delta_2$ . And

$$\begin{aligned} \Delta_1 &\leq \int_0^\epsilon \int_{-M\epsilon}^{M\epsilon} f(u)e^{-tv(1-M^2\epsilon^2)} du dv \\ &\text{by Cauchy-Schwartz} \\ &\leq \|f\|_{L^2(\mathbb{R})} (2M\epsilon)^{1/2} \frac{1 - e^{-t\epsilon(1-M^2\epsilon^2)}}{t(1 - M^2\epsilon^2)} \\ &= \mathcal{O}(t^{-1}). \end{aligned}$$

On the other hand,

$$\begin{aligned} \Delta_2 &\leq \int_0^\epsilon \int_{|u| \geq M\epsilon} f(u)e^{-ntvu^{n-1}} du dv \\ &= \int_{|u| \geq M\epsilon} f(u) \int_0^\epsilon e^{-ntvu^{n-1}} dv du \\ &\leq t^{-1} \|f\|_{L^2} \left( \int_{|u| \geq M\epsilon} \left( \frac{1 - e^{-ntvu^{n-1}}}{nu^{n-1}} \right)^2 du \right)^{1/2} \\ &\leq t^{-1} \|f\|_{L^2} \frac{n}{n-2} (M\epsilon)^{-(n-2)} \end{aligned}$$

$$= \mathcal{O}(t^{-1}).$$

Similarly, we can prove that for  $z \in \Omega_1^*$ , we also have the error estimate  $\mathcal{O}(t^{-1})$ . Assembling all together, we conclude that the error term is  $\mathcal{O}(t^{-1})$ , and  $m_1 = \mathcal{O}(t^{-1})$ , as  $t \rightarrow \infty$ . □

## 11 Painlevé Region

In this section, we first derive the Painlevé II hierarchy based on some RHP. Then, we will connect the long-time behavior of the mKdV hierarchy in the so-called Painlevé region to solutions of the Painlevé II hierarchy.

### 11.1 Painlevé II Hierarchy

As mentioned in Ablowitz and Segur (1977), the mKdV equation is can be transferred to the Painlevé II equation. The authors in Ablowitz and Segur (1977) also suggest the connection between integrable PDEs with Painlevé equations. In Clarkson et al. (2006), the authors explicitly derived the Painlevé II hierarchy from self-symmetry reduction of the mKdV hierarchy [see page 59 of Clarkson et al. (2006). And also Clarkson et al. (1999)]. In this section, we will provide a slight different [as comparing to Clarkson et al. (1999)] algorithm based on Riemann–Hilbert problems to generate the Painlevé II hierarchy. Let us denote  $\Theta(x, z) = xz + \frac{c}{n}z^n$ , and suppose  $Y$  solves the following RHP:

$$Y_+ = Y_- e^{i\Theta\sigma_3} v_0 e^{-i\Theta\sigma_3}, \quad z \in \Sigma_n,$$

$$Y = I + \mathcal{O}(z^{-1}), \quad z \rightarrow \infty.$$

where the contour  $\Sigma_n$  consists of all stokes lines  $\{z : \text{Im } \Theta(z) = 0\}$  and  $v_0$  is a constant  $2$  by  $2$  matrix that is independent of  $x, z$ .

Now let  $\tilde{Y} = Y e^{i\Theta\sigma_3}$ , and we arrive at a new RHP:

$$\tilde{Y}_+ = \tilde{Y}_- v_0, \quad z \in \Sigma_n,$$

$$\tilde{Y} = (I + \mathcal{O}(z^{-1})) e^{i\Theta\sigma_3}, \quad z \rightarrow \infty.$$

Since  $v_0$  is constant, it is easily to check, by Louisville’s argument, that both  $\partial_z \tilde{Y} \tilde{Y}^{-1}$  and  $\partial_x \tilde{Y} \tilde{Y}^{-1}$  are polynomial of  $z$ . Hence, we obtain the following two differential equations:

$$\partial_x \tilde{Y} \tilde{Y}^{-1} = A(x, z), \tag{72}$$

$$\partial_z \tilde{Y} \tilde{Y}^{-1} = B(x, z). \tag{73}$$

If we assume

$$Y = I + \sum_{j=1}^{n-1} Y_j(x)z^{-j} + \mathcal{O}(z^{-n}), \quad z \rightarrow \infty, \tag{74}$$

$$\underline{Y} = Y^{-1} = I + \sum_{j=1}^{n-1} \underline{Y}_j(x)z^{-j} + \mathcal{O}(z^{-n}), \quad z \rightarrow \infty, \tag{75}$$

then a direct computation shows

$$\begin{aligned} A &= i[Y_1, \sigma_3] + iz\sigma_3, \\ B &= ix\sigma_3 + icz^{n-1}\sigma_3 + icz^{n-2}[Y_1, \sigma_3] \\ &\quad + \sum_{k=2}^{n-1} icz^{n-1-k} \left( Y_k\sigma_3 + \sigma_3\underline{Y}_k + \sum_{j=1}^{k-1} Y_{k-j}\sigma_3\underline{Y}_j \right). \end{aligned}$$

Since  $Y_{x,z} = Y_{z,x}$ , we have

$$A_z - B_x + [A, B] = 0. \tag{76}$$

Set

$$Y_j = \begin{pmatrix} p_j(x) & u_j(x) \\ v_j(x) & q_j(x) \end{pmatrix}, \quad j = 1, \dots, n - 2, \tag{77}$$

where  $p_j, q_j, u_j, v_j$  are smooth functions of  $x$ . To guarantee (76), all the coefficients of  $z$  must vanish. Those equations can be solved recursively. Eventually, by eliminating  $u_j, v_j, j = 2, \dots, n - 2$ , and let  $v_1 = u_1$ , we will arrive at a nonlinear ODE of  $u_1$ ,<sup>13</sup> which turns out to be a member of the hierarchy of Painlevé II equations. We list the first few of them:

$$\begin{aligned} n = 3 : & -8cu^3 + cu_{xx} - 4xu = 0, \\ n = 5 : & -24cu^5 + 10cu^2u_{xx} + 10cuu_x^2 - \frac{c}{4}u_{xxxx} - 4xu = 0, \\ n = 7 : & -80cu^7 + 70cu^4u_{xx} + 140cu^3u_x^2 - \frac{7cu^2u_{xxxx}}{2} \\ & + \left( -\frac{21}{2}cu_{xx}^2 - 14cu_xu_{xxx} - 4x \right) u + \frac{c}{16}u_{xxxxxx} - \frac{35}{2}cu_x^2u_{xx} = 0. \end{aligned}$$

In the current article, we focus only on the odd members. In fact,  $n = 3$  corresponds to the mKdV equation,  $n = 5$  corresponds to the 5th order mKdV, and so on. In the following subsection, we will show how to connect the long-time asymptotics behavior of the mKdV hierarchy to the solutions to the Painlevé II hierarchy.

<sup>13</sup> Surprisingly, the dependence on  $p_j, q_j$  will disappear.

### 11.2 Painlevé Region

Recall the phase functions of the AKNS hierarchy of mKdV type equations are

$$\theta(z; x, t) = xz + ctz^n, \quad n \text{ is odd.} \tag{78}$$

By the Painlevé region we mean a collection of all the curves  $x = s(nt)^{1/n}$ ,  $s \neq 0$ , by rescaling  $z \rightarrow (nt)^{-\frac{1}{n}}\xi$ , we have

$$\Theta(\xi) = s\xi + \frac{c}{n}\xi^n. \tag{79}$$

Now the modulus of the stationary phase points of (78) is

$$|z_0| = \left| -\frac{x}{ct} \right|^{\frac{1}{n-1}} = \mathcal{O}\left(t^{-\frac{1}{n}}\right),$$

and however, after scaling, the modulus of the stationary phase points of  $\Theta(\xi)$  is

$$\xi_0 = |z_0|t^{\frac{1}{n}}, \tag{80}$$

which is fixed as  $t \rightarrow \infty$ . A direct computation shows for any odd  $n$ , one can always perform lens opening to the rays  $\{z \in \mathbb{R} : |z| > |\xi_0|\}$ , due to the signature of  $\text{Re}(i\theta)$ , see Fig. 7.

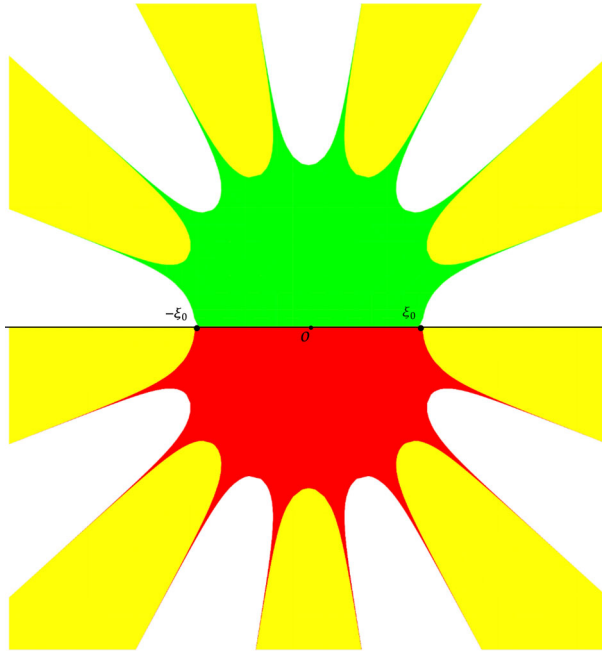
Note that

$$\begin{aligned} e^{-i\theta(z)} \text{ad } \sigma_3 v(z) &= e^{-i\Theta(\xi)} \text{ad } \sigma_3 v(\xi) \\ &= \begin{pmatrix} 1 - |R|^2 & -\bar{R}e^{-2i\Theta} \\ Re^{2i\Theta} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 - \bar{R}e^{-2i\Theta} & \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Re^{2i\Theta} & 1 \end{pmatrix}. \end{aligned}$$

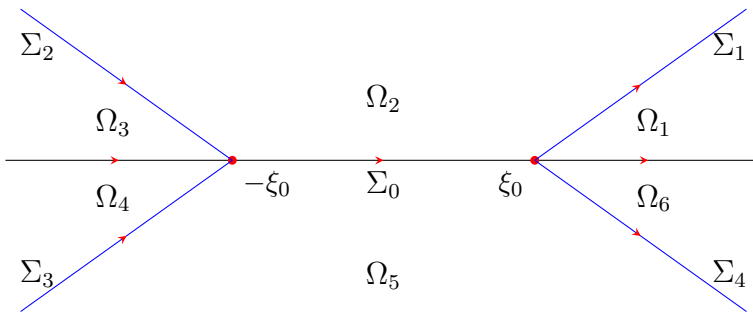
We can deform the contour  $\{z \in \mathbb{R} : |z| > |\xi_0|\}$  as before and get the deformed contour as follows (see Fig. 8): Fix a positive constant  $\alpha < \frac{\pi}{n}$ ,<sup>14</sup>

$$\begin{aligned} \Sigma_0 &= \{z \in \mathbb{R} : -\xi_0 \leq z \leq \xi_0\}, \\ \Sigma_1 &= \{z : z = \xi_0 + \rho e^{i\alpha}, \rho \in (0, \infty)\}, \\ \Sigma_2 &= \{z : z = -\xi_0 + \rho e^{-i\alpha}, \rho \in (-\infty, 0)\}, \\ \Sigma_3 &= \{z : z = -\xi_0 + \rho e^{i\alpha}, \rho \in (-\infty, 0)\}, \\ \Sigma_4 &= \{z : z = \xi_0 + \rho e^{-i\alpha}, \rho \in (0, \infty)\}, \end{aligned}$$

<sup>14</sup> Such a choice of  $\alpha$  guarantees that the new contours will stay within the regions where the corresponding exponential term will decay (considering Fig. 7).



**Fig. 7** Signature of  $\text{Re}(i\theta)$ . The green region:  $\text{Re}(i\theta) > 0$  when  $x < 0$ ; The red region:  $\text{Re}(i\theta) > 0$  when  $x > 0$ ; The yellow region: the overlapping region of red and green; The white region:  $\text{Re}(i\theta) < 0$ . Here, we only plot the signatures of  $\text{Re}(i\theta)$  for  $n = 9$ . Other odd  $n$ , the region plot looks very similar (Color figure online)



**Fig. 8** Contour for  $\bar{\partial}$ -RHP

and we define the regions as follows:

$$\begin{aligned} \Omega_1 &= \{z : z = \xi_0 + \rho e^{i\phi}, \rho \in (0, \infty), \phi \in (0, \alpha)\}, \\ \Omega_2 &= \mathbb{C}^+ \setminus (\Omega_1 \cup \Omega_3), \\ \Omega_3 &= \{z : z = -\xi_0 + \rho e^{-i\phi}, \rho \in (-\infty, 0), \phi \in (-\alpha, 0)\}, \\ \Omega_4 &= \{z : z = -\xi_0 + \rho e^{i\phi}, \rho \in (-\infty, 0), \phi \in (0, \alpha)\}, \\ \Omega_5 &= \mathbb{C}^- \setminus (\Omega_4 \cup \Omega_6), \\ \Omega_6 &= \{z : z = \xi_0 + \rho e^{i\phi}, \rho \in (0, \infty), \phi \in (-\alpha, 0)\}. \end{aligned}$$

As before, set the original RHP as  $m^{[1]}$  with jump  $e^{-i\theta(z)} \text{ad } \sigma_3 v(z)$ . After re-scaling and  $\bar{\partial}$ -lenses opening, we set  $m^{[2]}(\xi) = m^{[1]}O(\gamma)$ , where the lenses opening matrix is

$$O(\gamma) = \begin{cases} \begin{pmatrix} 1 & 0 \\ -E_+ e^{2i\Theta(\gamma)} & 1 \end{pmatrix}, & \gamma \in \Omega_1 \cup \Omega_3, \\ \begin{pmatrix} 1 - E_- e^{-2i\Theta(\gamma)} & \\ 0 & 1 \end{pmatrix}, & \gamma \in \Omega_4 \cup \Omega_6, \\ I, & \gamma \in \Omega_2 \cup \Omega_5, \end{cases} \tag{81}$$

where

$$\begin{aligned} E_+(\gamma) &= \mathcal{K}(\phi)R \left( (nt)^{-\frac{1}{n}} \xi \right) + (1 - \mathcal{K}(\phi))R(\tilde{\xi}_0(nt)^{-\frac{1}{n}}), \\ E_-(\gamma) &= \overline{E_+(\gamma)}, \\ \gamma &= \begin{cases} \xi_0 + \rho e^{i\phi}, & \text{if } \gamma \in \Omega_1 \cup \Omega_6, \\ -\xi_0 + \rho e^{i\phi}, & \text{if } \gamma \in \Omega_3 \cup \Omega_4, \end{cases} \\ \xi &= \text{Re}(\gamma), \\ \tilde{\xi}_0 &= \begin{cases} \xi_0, & \text{if } \gamma \in \Omega_1 \cup \Omega_6, \\ -\xi_0, & \text{if } \gamma \in \Omega_3 \cup \Omega_4. \end{cases} \end{aligned}$$

Now we arrive at the following  $\bar{\partial}$ -RHP:

**Mixed  $\bar{\partial}$ -Riemann–Hilbert problem 11.1**

Looking for a 2 by 2 matrix-valued function  $m^{[2]}$  such that

(1) *The RHP:*

(1.a)  $m^{[2]}(\gamma) \in C^1(\mathbb{R}^2 \setminus \Sigma)$  and  $m^{[2]}(z) = I + \mathcal{O}(\gamma^{-1})$ ,  $\gamma \rightarrow \infty$ ;

(1.b) the jumps on  $\Sigma_1$  and  $\Sigma_2$  are  $e^{-i\Theta(\xi)} \text{ad } \sigma_3 v_+$ , and the jumps on  $\Sigma_3$  and  $\Sigma_4$  are  $e^{-i\Theta(\xi)} \text{ad } \sigma_3 v_-$ , where

$$v_- = \begin{pmatrix} 1 & \bar{R} \\ 0 & 1 \end{pmatrix}, \quad v_+ = \begin{pmatrix} 1 & 0 \\ R & 1 \end{pmatrix}.$$

The jump on  $\Sigma_0$  is  $e^{-i\Theta} \text{ad } \sigma_3 v((nt)^{-\frac{1}{n}} \xi)$ , and the jumps on  $\{z \in \mathbb{R} : |z| > |\xi_0|\}$  is  $I$ .

(2) *The  $\bar{\partial}$ -problem:*

For  $z \in \mathbb{C}$ , we have

$$\bar{\partial} m^{[2]}(\xi) = m^{[2]}(\xi) \bar{\partial} O(\xi). \tag{82}$$

Again, we will need the following lemma in order to estimate errors from the  $\bar{\partial}$ -problem.



**Lemma 11.2** For  $\gamma \in \Omega_{1,3,4,6}$ ,  $\xi = \text{Re } \gamma$ ,

$$|\bar{\partial} E_{\pm}(\gamma)| \leq (nt)^{-\frac{1}{n}} |(nt)^{-\frac{1}{n}} (\xi - \xi_0)|^{-\frac{1}{2}} \|R\|_{H^{1,0}} + (nt)^{-\frac{1}{n}} |R'((nt)^{-\frac{1}{n}} \xi)|. \tag{83}$$

**Proof** For brevity, we only prove for the region  $\Omega_1$ . Using the polar coordinates, we have

$$\begin{aligned} |\bar{\partial} E_+(\gamma)| &= \left| \frac{ie^{i\phi}}{2\rho} \mathcal{K}'(\phi) \left[ R \left( (nt)^{-\frac{1}{n}} \xi \right) - R(\xi_0 (nt)^{-\frac{1}{n}}) \right] + \mathcal{K}(\phi) R' \left( (nt)^{-\frac{1}{n}} \xi \right) (nt)^{-\frac{1}{n}} \right| \\ &\text{by Cauchy-Schwartz inequality} \\ &\leq \left| \frac{\|R\|_{H^{1,0}} |(nt)^{-\frac{1}{n}} \xi - \xi_0 (nt)^{-\frac{1}{n}}|^{1/2}}{\gamma - \xi_0} \right| + (nt)^{-\frac{1}{n}} \left| R' \left( (nt)^{-\frac{1}{n}} \xi \right) \right| \\ &\leq (nt)^{-\frac{1}{n}} |(nt)^{-\frac{1}{n}} (\xi - \xi_0)|^{-\frac{1}{2}} \|R\|_{H^{1,0}} + (nt)^{-\frac{1}{n}} |R'((nt)^{-\frac{1}{n}} \xi)|. \end{aligned}$$

Similarly, we can prove for other regions. □

Next, consider a pure RHP  $m^{[3]}$  which satisfies exactly the RHP part of  $\bar{\partial}$ -RHP( $m^{[2]}$ ).  $m^{[3]}$  can be approximated by the RHP corresponding to a special solution of the Painlevé II hierarchy.<sup>15</sup> Since for  $\gamma \in \Omega_1$ ,

$$\begin{aligned} \left| \left( R(\xi (nt)^{-\frac{1}{n}}) - R(0) \right) e^{2i\Theta(\gamma)} \right| &\leq |\xi (nt)^{-\frac{2}{n}}|^{\frac{1}{2}} \|R\|_{H^{1,0}} e^{2\text{Re } i\Theta(\gamma)} \\ &\leq (nt)^{-\frac{1}{n}} |\text{Re } \gamma|^{\frac{1}{2}} \|R\|_{H^{1,0}} e^{2\text{Re } i\Theta(\gamma)}, \end{aligned}$$

it is evident that

$$\|R e^{2i\Theta} - R(0) e^{2i\Theta}\|_{L^\infty \cap L^1 \cap L^2} \leq c (nt)^{-\frac{1}{n}}. \tag{84}$$

Let  $m^{[4]}$  solves the RHP formed by replacing  $R(\pm \xi_0 (nt)^{-1/n})$  and its complex conjugate in the jumps of  $m^{[3]}$  along  $\Sigma_k$ ,  $k = 1, 2, 3, 4$  by  $R(0)$  and  $\bar{R}(0)$  respectively. Then, by the small norm theory, the errors between the corresponding potential are given by

$$\begin{aligned} \text{error}_{3,4} &= \lim_{\gamma \rightarrow \infty} |\gamma (m_{12}^{[4]} - m_{12}^{[3]})| \\ &\leq c \int_{\Sigma} |(R(\text{Re}(s) (nt)^{-\frac{1}{n}}) - R(0)) e^{2i\Theta(s)}| ds \\ &\leq c (nt)^{-\frac{1}{n}}. \end{aligned}$$

Then since now the jumps are all analytic, we can perform an analytic deformation and arrive at the green contours as show in Fig. 9. Let us denote the new RHP by  $m^{[5]}(\gamma)$ , and we arrive at the following RHP:

<sup>15</sup> As for the existence of the RHP  $m^{[3]}$ , which is not completely trivial due to the fact that solutions to the Painlevé II equations have poles, we refer the readers to the book (Fokas et al. 2006) for the details

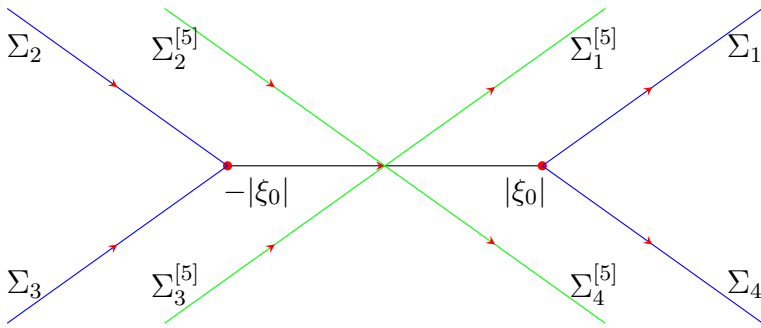


Fig. 9 Contour for  $m^{[4]}$ (Green part) (Color figure online)

**Riemann–Hilbert problem 11.3** Looking for a 2 by 2 matrix-valued function  $m^{[5]}$  such that

- (1)  $m^{[5]}$  is analytic off the contours  $\cup_{k=1,2,3,4} \Sigma_k^{[5]}$ ;
- (2)  $m_+^{[5]} = m_-^{[5]} v^{[5]}$ ,  $z \in \cup_{k=1,2,3,4} \Sigma_k^{[5]}$ , where

$$v^{[5]} = \begin{cases} \begin{pmatrix} 1 & 0 \\ R(0)e^{2i\Theta(\gamma)} & 1 \end{pmatrix}, & \gamma \in \Sigma_1^{[5]} \cup \Sigma_2^{[5]}, \\ \begin{pmatrix} 1 & \bar{R}(0)e^{-2i\Theta(\gamma)} \\ 0 & 1 \end{pmatrix}, & \gamma \in \Sigma_3^{[5]} \cup \Sigma_4^{[5]}. \end{cases}$$

Here the new contours (see Fig. 9) are

$$\begin{aligned} \Sigma_1^{[5]} &= \{z : z = \rho e^{i\alpha}, \rho \in (0, \infty)\}, \\ \Sigma_2^{[5]} &= \{z : z = \rho e^{-i\alpha}, \rho \in (-\infty, 0)\}, \\ \Sigma_3^{[5]} &= \{z : z = \rho e^{i\alpha}, \rho \in (-\infty, 0)\}, \\ \Sigma_4^{[5]} &= \{z : z = \rho e^{-i\alpha}, \rho \in (0, \infty)\}. \end{aligned}$$

Then according to the previous subsection, the (1, 2) entry of the solution  $m^{[5]}$ , similarly the solution  $m^{[4]}$ , is the solution to the Painlevé II hierarchy, i.e.,

$$m_{12}^{[4]}(\gamma) = m_{12}^{[5]}(\gamma), \tag{85}$$

Hence we have  $P_k^{II}(s) = \lim_{\gamma \rightarrow \infty} \gamma m_{12}^{[5]}$  where  $P_k^{II}$  solves the  $k$ th equation in the Painlevé II hierarchy, where  $k = \frac{n-1}{2}$ .

Now let us consider the error generated from the  $\bar{\partial}$ -extension. Recall that the error  $E$  satisfies a pure  $\bar{\partial}$ -problem:

$$\begin{aligned} \bar{\partial} E &= EW, \\ W &= m^{[3]} \bar{\partial} O(m^{[3]})^{-1}. \end{aligned}$$

As before, the  $\bar{\partial}$ -equation is equivalent to an integral equation which reads

$$E(z) = I + \frac{1}{\pi} \int_{\mathbb{C}} \frac{E(s)W(s)}{z - s} dA(s) = I + \mathcal{K}(E).$$

As before, we can show that the resolvent always exists for large  $t$ . So we only need to estimate the true error which is:  $\lim_{z \rightarrow \infty} z(E - I)$ . In fact, we have

$$\begin{aligned} \lim_{z \rightarrow \infty} |z(E - I)| &= \left| \int_{\mathbb{C}} EW ds \right| \\ &\leq c \|E\|_{\infty} \int_{\Omega} |\bar{\partial} O| ds. \end{aligned}$$

For the sake of simplicity, we only estimate the integral on the right hand side in the region of the top right corner. Note there is only one entry which is nonzero in  $\bar{\partial} O$ , which is one of the  $E_{\pm}$  and we split the integral into two parts in the obvious way, i.e.,

$$\begin{aligned} \int_{\Omega} |\bar{\partial} O| ds &\leq I_1 + I_2 \\ &= \int_{\Omega} (nt)^{-\frac{1}{2n}} |\operatorname{Re} s - \xi_0| \|R\|_{H^{1,0}} e^{2\operatorname{Re} i\Theta(s)} ds \\ &\quad + \int_{\Omega} (nt)^{-\frac{1}{n}} |R'((nt)^{-\frac{1}{n}} s)| e^{2\operatorname{Re} i\Theta(s)} ds. \end{aligned}$$

As we know from previous sections,  $e^{\operatorname{Re} 2i\Theta(s)} \leq ce^{-2|\Theta''(\xi_0)|uv}$  in the region  $\{z = u + iv : u > \xi_0, 0 < v < \alpha u\}$  for some small  $\alpha$ , where  $s = u + iv + \xi_0$ . Then, we have

$$\begin{aligned} I_1 &\leq (nt)^{-\frac{1}{2n}} \int_{\Omega} |\operatorname{Re} s - \xi_0|^{-1/2} e^{-cuv} du dv \\ &\leq (nt)^{-\frac{1}{2n}} \int_0^{\infty} \int_0^{\alpha u} u^{-1/2} e^{-cuv} du dv \\ &\leq C(nt)^{-\frac{1}{2n}} \int_0^{\infty} \frac{1 - e^{-2\alpha|\Theta''(\xi_0)|}}{u^{3/2}} du \\ &= \mathcal{O}\left((nt)^{-\frac{1}{2n}}\right), \end{aligned}$$

and

$$\begin{aligned}
 I_2 &\leq (nt)^{-\frac{1}{n}} \int |R'((nt)^{-\frac{1}{2n}} \operatorname{Re} s)| e^{-cuv} \, du \, dv \\
 &\text{by Cauchy–Schwartz inequality} \\
 &\leq (nt)^{-\frac{1}{n}} \|R\|_{H^{1,0}} \int_0^\infty \left( \int_{\alpha v}^\infty e^{-2cuv} \, du \right)^{1/2} \, dv \\
 &\leq (nt)^{-\frac{1}{n}} \|R\|_{H^{1,0}} \int_0^\infty \frac{e^{-c\alpha v^2}}{\sqrt{2\alpha cv}} \, dc \\
 &= \mathcal{O}((nt)^{-\frac{1}{n}}).
 \end{aligned}$$

Thus, we arrive at

$$\bar{\delta} \text{Error} = \mathcal{O}((nt)^{-\frac{1}{2n}}). \tag{86}$$

And we undo all the deformations, we obtain

$$\begin{aligned}
 m^{[1]} \left( (nt)^{-\frac{1}{n}} \gamma \right) &= m^{[2]}(\gamma) \mathcal{O}^{-1}(\gamma) \\
 &= \left( 1 + \frac{\mathcal{O}\left(t^{\frac{1}{2n}}\right)}{\gamma} \right) m^{[3]}(\gamma) \mathcal{O}^{-1}(\gamma) \\
 &= \left( 1 + \frac{\mathcal{O}\left(t^{\frac{1}{2n}}\right)}{\gamma} \right) \left( 1 + \frac{\mathcal{O}\left(t^{\frac{1}{2n}}\right)}{\gamma} \right) m^{[4]}(\gamma) \mathcal{O}^{-1}(\gamma) \\
 &= \left( 1 + \frac{\mathcal{O}\left(t^{\frac{1}{2n}}\right)}{\gamma} \right) \left( 1 + \frac{\mathcal{O}\left(t^{\frac{1}{2n}}\right)}{\gamma} \right) m^{[5]}(\gamma) \mathcal{O}^{-1}(\gamma).
 \end{aligned}$$

It can also be rewritten in terms of the variable  $z$ :

$$m^{[1]}(z) = \left( 1 + \frac{\mathcal{O}(t^{-1/(2n)})}{z(nt)^{1/n}} \right) m^{[5]}((nt)^{1/n} z) + \mathcal{O}(z^{-2}), \quad z \rightarrow \infty.$$

Since  $m^{[5]}$  corresponds to the RHP for the Painlevé II hierarchy, we have

$$m^{[5]}(\gamma) = I + \frac{m_1^{[5]}(s)}{\gamma} + \mathcal{O}(\gamma^{-1}),$$

where  $\gamma = z(nt)^{1/n}$ .

Thus,

$$\begin{aligned} m^{[1]}(z) &= \left(1 + \frac{\mathcal{O}(t^{-\frac{1}{2n}})}{z(nt)^{1/n}}\right) \left(1 + \frac{m_1^{[5]}(s)}{z(nt)^{1/n}} + \mathcal{O}(z^{-2})\right) \\ &= I + \frac{m_1^{[5]}(s)}{z(nt)^{1/n}} + \frac{\mathcal{O}(t^{-\frac{1}{2n}})}{z(nt)^{1/n}} + \mathcal{O}(z^{-2}). \end{aligned}$$

Since  $m_1^{[5]}(s)$  is connected to solutions of the Painlevé II hierarchy, we conclude that

$$\begin{aligned} q(x, t) &= \lim_{z \rightarrow \infty} z(m^{[1]} - I) \\ &= (nt)^{-\frac{1}{n}} u_n(x(nt)^{-\frac{1}{n}}) + \mathcal{O}(t^{-\frac{3}{2n}}), \end{aligned}$$

where  $u_n$  solves the  $\frac{n-1}{2}$ th equation of the Painlevé II hierarchy. The odd integer  $n$  corresponds to the  $\frac{n-1}{2}$ th member in the mKdV hierarchy.

**Remark 11.4** As for the asymptotics for the Painlevé II equation, we refer the readers to the classical book (Fokas et al. 2006). There are also some recent works related to Painlevé II hierarchy, see for example Miller and Sheng (2017), Claeys et al. (2010) and Cafasso et al. (2019).

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