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RESEARCH

Integrability characteristics and exact solutions of an extended $(3+1)$-dimensional variable-coefficient shallow water wave model

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Abstract This paper is concerned with an extended $(3+1)$-dimensional shallow water wave equation with variable coefficients, which is used to describe the interaction of nonlinear waves in ocean dynamics, shallow water waves, etc. Hereby, it is of further value to investigate the integrability characteristics of this model. Firstly, we conduct the Painlevé analysis and find it can pass the Painlevé test. Then, the one- and two-soliton solution are obtained by virtue of the Hirota bilinear method. Bäcklund transformation, Lax pair and infinitely many conservation laws are derived through the Hirota bilinear method and Bell polynomial approach. Particularly, we generate two type of interaction solutions in terms of a combination of quadratic function, exponential function and trigonometric function, namely, the lump-kink solution and the periodic lump solution. Finally, dynamics characteristics and evolution behaviors are exhibited for the obtained solution waves through particular plots with proper choices of different values for the parameters.

Keywords Painlevé analysis · Soliton solution · Bäcklund transformation · Lax pair · Infinite conservation laws

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1 Introduction

Nonlinear evolution equations (NLEEs) are used to simulate a variety of frequent phenomena in the physical world. The equations have been used in numerous fields, including fluids mechanics, condensed matter physics, atmospheric physics, and a variety of scientific fields [1–10]. The dynamics and derivation of exact solutions for NLEEs play a significant role in the theory of solitons. There are many efficient ways to find the exact solutions as the long wave limit method [11], the (G'/G)-expansion [12], the Hirota bilinear method [13], the tanh-coth [14], the exp-function [15], the Wronskian formulation [16]. These techniques allow for the calculation of NLEEs solutions in many different forms. The Hirota bilinear method is a simple and direct method that has been widely and successfully used to solve NLEEs and obtain their multiple soliton solutions, lump solutions, interaction solutions, Bäcklund transformations (BTs) and so on [17,18].

Shallow water wave equations are one of the important models of NLEEs, which have been used to simulate the dynamic behavior of water wave propagation in the field of oceanography and the atmosphere [19]. In the shallow water wave equations, the water depth is substantially less than the wavelength of the free surface disturbance [20]. The shallow water wave equations, along with their exact solutions, serve as powerful tools for investigating various aspects of water wave dynamics, for understanding internal waves within the ocean and gor studying the characteristics and interactions of waves near beaches, such as velocity, wavelength and amplitude. Studying these exact solutions can reveal the essential phenomena associated with nonlinear waves, including their propagation characteristics, interactions, stability, and responses to disturbances [21–24]. In 1990, Ref. [25] has discussed a (2 + 1)-dimensional shallow water wave equation, is shown to be integrable and has an equivalent Lax representation. The symmetry reductions and exact solutions for two cases of the (2 + 1)-dimensional shallow water wave equation have been studied by Ref. [26]. Ref. [27] has obtained the Grammian and Pfaffian solutions for a (3 + 1)-dimensional generalized shallow water equation in the Hirota bilinear form. The lump-type solutions and their interaction solutions with one- or two-stripe solutions have been generated and the analyticity and localization of the resulting solutions in the space have been analyzed [28]. And Ref. [29] has obtained spatiotemporal breather soliton solutions and exact extended breather wave solutions by Bell polynomial method for the shallow water equation with variable coefficients.

Ref. [30] has studied two new extended (3+1)-dimensional shallow water wave equations with constant coefficients and time-dependent coefficients. Active researches on the equations have been done. For the constant-coefficient equation, the hyperbolic cosine-function solution and cosine-function solution have been obtained and five linear superposition formulas have been proved [31]. Ref. [32] has investigated rational and exponential traveling wave solutions to obtain kink and singular kink solitons. Based on the polynomial-expansion method and the Hirota method, travelling-wave, mixed-lump-kink and mixed-rogue-wave-kink solutions have been obtained [33]. In addition, an auto-Bäcklund transformation for the variable-coefficient equation has been derived in Ref. [34], the breather and lump wave solutions have been analyzed [35], and the similarity reductions have been constructed out [36].

This paper will investigate the extended (3 + 1)-dimensional variable-coefficient shallow water wave equation:

$$u_{yt} + \alpha_1(t)u_{xxxy} + \beta\alpha_1(t)(u_{xy})_x + \gamma\alpha_1(t)u_{xx} + \alpha_2(t)u_{yy} + \alpha_3(t)u_{xy} + \alpha_4(t)u_{yz} = 0, \quad (1)$$

where $u = u(x, y, z, t)$ are analytic functions of the three scaled spatial variables x, y, z and time variable t. The functions $\alpha_i(t)$ $(i = 1, 2, 3, 4)$ demonstrate the influence of time-varying disturbances on dispersion along distinct spatial directions [37]. Models incorporating time-dependent variable coefficients are uniquely equipped to capture the temporal evolution of a system. This capability enables them to track how various factors within the system develop and interact over time, rather than being confined to spatial variations alone. Additionally, the variable coefficients and the random constants β, γ represent bit deeper physical meaning and more possible cases in nonlinear shallow water wave. Equation (1) includes several key features: nonlinear terms, dispersion effects, and mass and energy conservation.It is a simulation tool that is applied to the propagation of water waves in different scenarios, including floods, tsunami prediction, weather simulations, tidal waves, river and irrigation flows [38–40].

Consider the transformation

$$u = \frac{6}{\beta}(\ln f)_x, \quad (2)$$

Eq. (1) has been converted into the following bilinear form

$$(D_yD_t + \alpha_1(t)D_x^3D_y + \gamma\alpha_1(t)D_x^2 + \alpha_2(t)D_y^2 + \alpha_3(t)D_xD_y + \alpha_4(t)D_yD_z)f\cdot f = 0, \quad (3)$$

with the D-operator defined as,

$$D_x^{n_1}D_t^{n_2}D_y^{n_3}D_z^{n_4}(f\cdot g) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^{n_1}\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^{n_2} \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^{n_3}$$

$$\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial wz'}\right)^{n_4} f(x,t,y,z)g(x',t',y',z')\Big|_{x'=x,t'=t,y'=y,z'=z},$$

where $f(x,y,z,t)$ and $g(x',t',y',z')$ are differentiable functions, and n_1, n_2, n_3 and n_4 are nonnegative integers.

We will pay attention to the integrability of Eq. (1) via applying the Painlevé analysis, Hirota bilinear method and Bell polynomials approach. The paper is organized as follows. In Sect. 2, the Painlevé analysis will prove that Eq. (1) possesses the Painlevé property. In Sect. 3, we will derive the soliton solutions through the Hirota bilinear method including one- and two-soliton ones. In Sect. 4, the Lax pair and two kinds of BT, including the bilinear BT and Bell-polynomial-typed BT, will be constructed via the Hirota bilinear method and Bell polynomials. In Sect. 5, the infinite conservation laws will be derived with the aid of the Lax pair. In Sect. 6, the lump-kink solution and periodic lump solutions for Eq. (1) will be constructed and analyzed, respectively. Finally, Sect. 7 wil give some discussions.

2 Painlevé analysis

A NLEE has the Painlevé property when its general solution is single-valued about all the movable singularity manifold [41,42]. We assume that Eq. (1) has the solutions as the Laurent expansions about a singular manifold ϕ,

$$u=\sum_{j=0}^{\infty} u_j\phi^{j+\alpha}, \tag{4}$$

where α is a negative integer, $\phi(x,y,z,t)$ and $u_j(x,y,z,t)$ are analytic functions near a singularity manifold $\phi(x,y,z,t)=0$. Using the leading order analysis, we assume the leading terms in the form of

$$u\sim u_0\phi^{\alpha}, \tag{5}$$

and obtain $\alpha=-1$ and $u_0=\frac{6}{\beta}\phi_x$.

Substituting

$$u\sim u_0\phi^{-1}+u_j\phi^{j-1} \tag{6}$$

into Eq. (1) and make the coefficient of ϕ^{j-5} to zero, leading to

$$(j+1)(j-1)(j-4)(j-6)\,\alpha_1(t)u_j\phi_y\phi_x^3=0. \tag{7}$$

It can obtain that $j=-1,1,4,6$ are the resonance points from Eq. (7). The resonance at $j=-1$ corresponds to the arbitrariness of ϕ, which describes the singular hypersurface. By computation, u_2, u_3 and u_5 can be expressed explicitly, u_1, u_4 and u_6 are arbitrary functions, which shows that Eq. (1) passes the Painlevé test and hence possess the Painlevé integrability.

3 Soliton solutions

There are many effective methods for solving soliton solutions. In Ref. [43], a series of optical soliton solutions were obtained by The generalized Kudryashov method. the Exp$(-\phi(\xi))$-expansion method in Ref. [44] is practical for the determination of soliton solutions of high-dimensional equations with conformable time fractional derivative. By applying the generalized auxiliary equation method [45], a series of soliton solutions are generated for the derived equations. New soliton solutions can also be obtained from the auto-Bäcklund transformation derived from the homogenous balance technique [46]. We will focus on the soliton solutions including one- and two-soliton ones with the Hirota bilinear method. For obtaining the one-soliton solution, f is given by

$$f=1+e^{\eta_1}, \tag{8}$$

where $\eta_1=p_1x+q_1y+r_1z+\Omega_1(t)+\eta_1^0$, p_1, q_1, r_1 and η_1^0 are constants, and the dispersion relation is

$$\Omega_1(t)+p_1\int\left(p_1(p_1+\frac{\gamma}{q_1})\alpha_1(t)+\alpha_3(t)\right)dt+q_1\int\alpha_2(t)\,dt+r_1\int\alpha_4(t)\,dt=0. \tag{9}$$

Based on the transformation (2), the one-soliton solution to Eq. (1) is shown as

$$u=\frac{6p_1}{\beta}\frac{e^{p_1x+q_1y+r_1z+\Omega_1(t)+\eta_1^0}}{1+e^{p_1x+q_1y+r_1z+\Omega_1(t)+\eta_1^0}}. \tag{10}$$

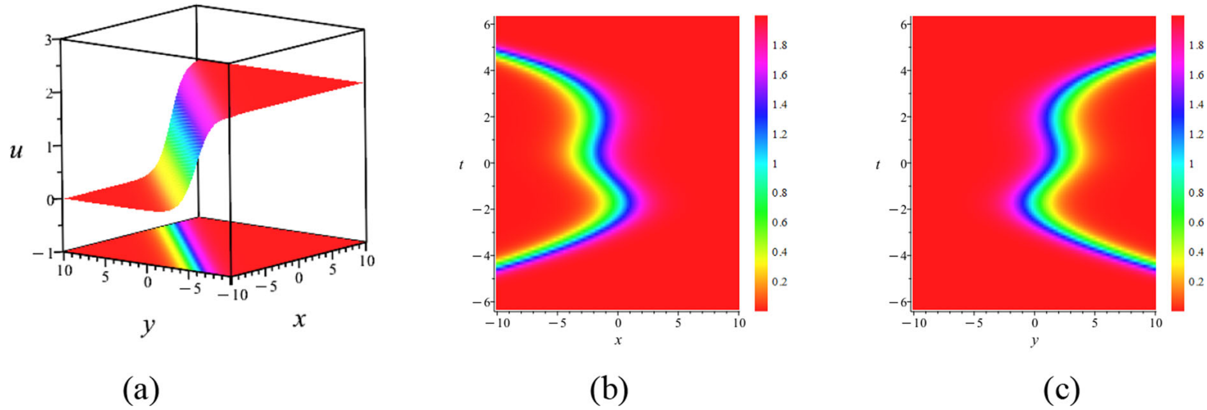

Fig. 1 One-soliton solution (10) with $\beta = 3, \gamma = 2, \alpha_1(t) = \cos(t), \alpha_2(t) = t, \alpha_3(t) = \cos(t)\sin(t), \alpha_4(t) = \sin(t), t = \pi$, $p_1 = 1, 1_1 = -1, r_1 = 2$: **a** three-dimensional plots and density plots in the (x, y)-plane; **b** in the (x, t)-plane; **c** in the (y, t)-plane

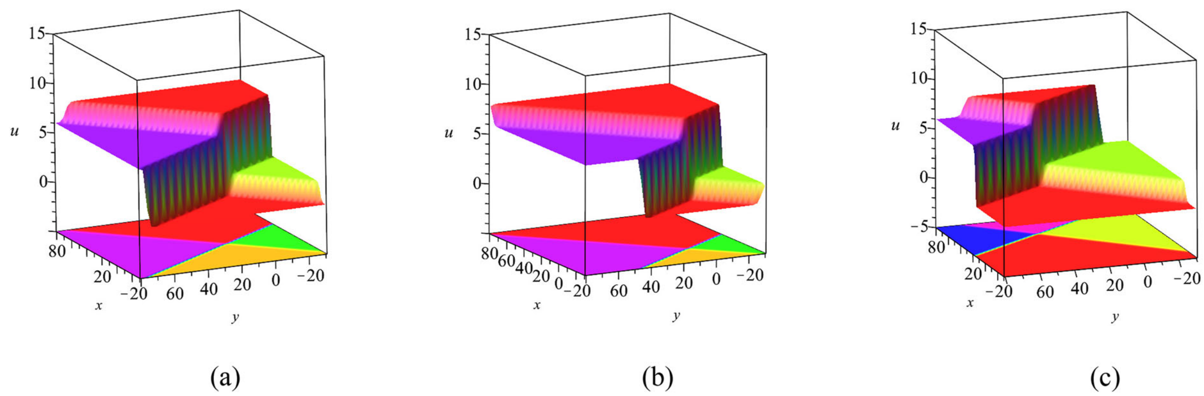

Fig. 2 Two-soliton solutions (12) with $\beta = 3, \gamma = 2, \alpha_1(t) = t, \alpha_2(t) = t^2, \alpha_3(t) = t^2 + t, \alpha_4(t) = 2t$, $p_1 = 1, p_2 = 3, q_1 = r_2 = -1, r_1 = q_2 = 2$: **a** $t = -2$; **b** $t = 0$; **c** $t = 3$

Assuming that

$$f = 1 + \mathrm{e}^{\eta_1} + \mathrm{e}^{\eta_2} + a_{12}\mathrm{e}^{\eta_1+\eta_2},$$
$$\eta_i = p_i x + q_i y + r_i z + \Omega_i(t) + \eta_i^0 \ (i = 1, 2), \quad (11)$$

the two-soliton solution to Eq. (1) is derived as

$$u = \frac{6}{\beta}\frac{p_1\mathrm{e}^{\eta_1} + p_2\mathrm{e}^{\eta_1} + a_{12}(p_1 + p_2)\mathrm{e}^{\eta_1+\eta_2}}{1 + \mathrm{e}^{\eta_1} + \mathrm{e}^{\eta_1} + a_{12}\mathrm{e}^{\eta_1+\eta_2}},$$
$$a_{12} = \frac{-3\,p_1p_2q_1q_2(q_1 - q_2)(p_1 - p_2) + \gamma\,(p_1q_2 - q_1p_2)^2}{-3\,p_1p_2q_1q_2(q_1 + q_2)(p_1 + p_2) + \gamma\,(p_1q_2 - q_1p_2)^2}, \quad (12)$$

where $\Omega_i(t) = -p_i \int \left(p_i(p_i + \frac{\gamma}{q_i})\alpha_1(t) + \alpha_3(t) \right) dt - q_i \int \alpha_2(t)\, dt - r_i \int \alpha_4(t)\, dt$, p_i, q_i, r_i and η_i^0 $(i = 1, 2)$ are constants.

Figures 1 and 2 present a visual analysis of the one-soliton (10) and two-soliton (12) solutions, respectively, through three-dimensional plots and corresponding density plots. Figures 1a reveals spatial structure and amplitude, which is maintained at 2. The plot highlights the kink soliton's distinct shape and the distribution of its density across the (x, y)-plane. The velocity of one-soliton solution is given by $(v_x = -t + \sin(t)(\cos(t) + 2) - \cos(t),\ v_y = -v_x)$. Figures 1b and c show the plots in the (x, t)- and (y, t)-planes, respectively, offering a temporal perspective on the soliton's propagation. For the two-soliton solution, the velocity of the two solitary waves is $(v_x = 4t, v_y = -4t)$ and $(v_x = \frac{5t^2+37t}{3}, v_y = \frac{5t^2+37t}{2})$. The interaction generates platforms resembling steps with distinct heights, characteristic of kink solitons interactions.

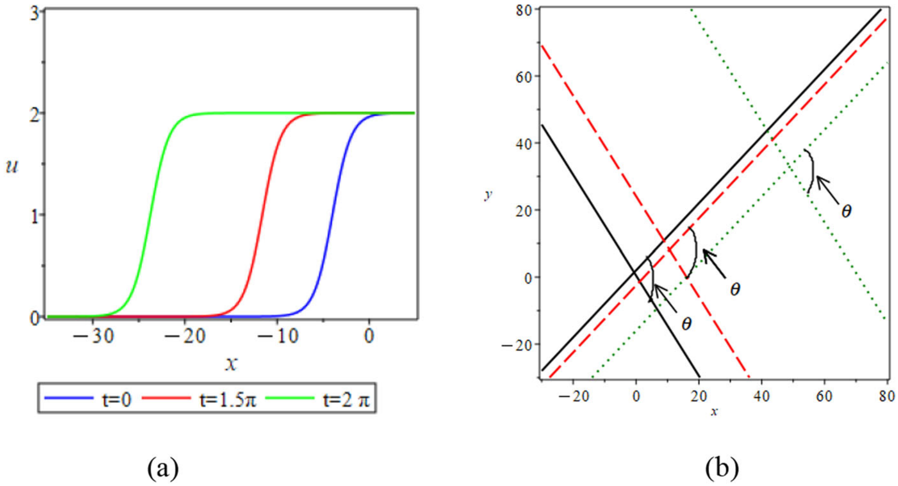

Fig. 3 The propagation of the one-soliton solution (10) and the two-soliton solutions (12) **a** Time evolution plot of the one-soliton solution (10); **b** Characteristic lines of the two-soliton solutions (12) at corresponding time

Figures 3 show the propagation of the one-soliton (10) and the two-soliton (12). It can be seen from the time evolution plot that the single kink wave always maintains its amplitude. The Angle of the corresponding characteristic lines of the two kink waves does not change before and after the collision, which means that the shape of the kink waves is unchanged. Thus, interactions between solitons result in elastic collisions.

4 Bäcklund transformation and Lax pair

By establishing a relationship between binary Bell polynomials and Hirota bilinear operators, Bell-polynomial-typed BTs can be effectively converted into bilinear forms. This transformation process is pivotal for advancing the study and application of nonlinear dynamical systems. [47]. In this section, we will construct BTs and Lax pair for Eq. (1) via the Hirota bilinear method and Bell polynomials.

4.1 Bilinear BT and Lax pair

In this subsection, we construct the bilinear BT for Eq. (1). First of all, we list several bilinear identities as follows:

$$\begin{aligned}
&[D_yD_tf\cdot f]f'^2-f^2[D_yD_tf'\cdot f']\\
&\quad=2D_y(D_tf\cdot f')\cdot ff',\\
&[D_x^2f\cdot f]f'^2-f^2[D_x^2f'\cdot f']\\
&\quad=2D_x(D_tf\cdot f')\cdot ff',\\
&[D_xD_yf\cdot f]f'^2-f^2[D_x^2f'\cdot f']\\
&\quad=2D_x(D_yf\cdot f')\cdot ff',\\
&[D_x^3D_yf\cdot f]f'^2-f^2[D_x^3D_yf'\cdot f']\\
&\quad=2D_y(D_x^3f\cdot f')\cdot ff'\\
&\quad-6D_x(D_xD_yf\cdot f')\cdot(D_xf\cdot f'),\\
&D_x(D_yf\cdot f')\cdot ff'=D_y(D_xf\cdot f')\cdot ff',\\
&D_x(D_xf\cdot f')\cdot ff'=-D_x(ff')(D_xf\cdot f').
\end{aligned}\tag{13}$$

Theorem 1 *If f is a solution of Eq. (3) and satisfies the following relationships,*

$$\begin{aligned}
B_1f\cdot f'&=(D_t+\alpha_1(t)D_x^3+\alpha_2(t)D_y\\
&\qquad+\alpha_3(t)D_x+\alpha_4(t)D_z+\mu_1(t))f\cdot f',\\
B_2f\cdot f'&=(3D_xD_y+\gamma+\mu_2D_x)f\cdot f',
\end{aligned}$$

where $\mu_1(t)$ is undetermined function and μ_2 is an arbitrary constant, f and f' are real functions of variables x, y, z and t. Then, f' is also a solution of Eq. (3) and $B_j(j=1,2)$ are the bilinear Bäcklund transformation of Eq. (1).

Proof Considering the equation

$$\begin{aligned}
P&=[(D_yD_t+\alpha_1(t)D_x^3D_y+\gamma\alpha_1(t)D_x^2\\
&\quad+\alpha_2(t)D_y^2+\alpha_3(t)D_xD_y+\alpha_4(t)D_yD_z)f\cdot f]f'^2\\
&\quad-f^2[(D_yD_t+\alpha_1(t)D_x^3D_y+\gamma\alpha_1(t)D_x^2\\
&\quad+\alpha_2(t)D_y^2+\alpha_3(t)D_xD_y+\alpha_4(t)D_yD_z)f'\cdot f'],
\end{aligned}\tag{14}$$

and setting $P=0$. Using identities (13), we have

$$
\begin{aligned}
P &= 2D_y[(D_t+\alpha_1(t)D_x^3+\alpha_2(t)D_y+\alpha_3(t)D_x \\
&\quad +\alpha_4(t)D_z)f\cdot f']\cdot ff' \\
&\quad -2\alpha_1(t)D_x[(3D_xD_y+\gamma)f\cdot f']\cdot(D_xf\cdot f'), \\
&= 2D_y[(D_t+\alpha_1(t)D_x^3+\alpha_2(t)D_y+\alpha_3(t)D_x \\
&\quad +\alpha_4(t)D_z+\mu_1(t))f\cdot f']\cdot ff' \\
&\quad -2\alpha_1(t)D_x[(3D_xD_y+\gamma \\
&\quad +\mu_2D_x)f\cdot f']\cdot(D_xf\cdot f'), \\
&= 2D_y[B_1f\cdot f']\cdot ff' \\
&\quad -2\alpha_1(t)D_x[B_2f\cdot f']\cdot(D_xf\cdot f').
\end{aligned} \tag{15}
$$

So the bilinear BT for Eq. (1) is

$$
\begin{cases}
(D_t+\alpha_1(t)D_x^3+\alpha_2(t)D_y+\alpha_3(t)D_x \\
\quad +\alpha_4(t)D_z+\mu_1(t))f\cdot f'=0, \\
(3D_xD_y+\gamma+\mu_2D_x)f\cdot f'=0.
\end{cases} \tag{16}
$$

This completes the proof. □

According to the rational transformation $\psi=\frac{f'}{f}$ and $u=\frac{6}{\beta}(\ln f)_x$, Eq. (16) is equivalent to

$$
\begin{cases}
\psi_t+\alpha_1(t)\psi_{xxx}+\beta\alpha_1u_x\psi_x+\alpha_2(t)\psi_y \\
\quad +\alpha_3(t)\psi_x+\alpha_4(t)\psi_z+\mu_1(t)=0, \\
3\psi_{xy}+\beta u_y\psi+\gamma\psi+\mu_2\psi_x=0.
\end{cases} \tag{17}
$$

Introducing two linear differential operators L_1 and L_2, we can cast Eq. (17) into

$$
\begin{cases}
L_1(\psi)=\partial_t+\alpha_1(t)\partial_x{}^3+\beta\alpha_1u_x\partial_x+\alpha_2(t)\partial_y \\
\qquad\qquad +\alpha_3(t)\partial_x+\alpha_4(t)\partial_z+\mu_1(t)=0, \\
L_2(\psi)=3\partial_{xy}+\beta u_y+\gamma+\mu_2\partial_x=0,
\end{cases} \tag{18}
$$

Actually, the compatibility condition $[L_1,L_2]=L_1L_2-L_2L_1=0$, leads to

$$
\begin{aligned}
&\big(u_{yt}-\alpha_1(t)u_{xxxy}+\beta\alpha_1(t)(u_{xy})_x \\
&+\gamma\alpha_1(t)u_{xx}+\alpha_2(t)u_{yy} \\
&+\alpha_3(t)u_{xy}+\alpha_4(t)u_{yz}\big)\beta\psi=0,
\end{aligned} \tag{19}
$$

and the linear operators form the Lax pair of Eq. (1).

4.2 Bell-polynomial-typed BT

Considering $f=f(x_1,x_2,\ldots,x_s)$ is a C^∞ multi-variable function and the multi-dimensional generalization with the variables $f_{r_1x_1,r_2x_2,\ldots,r_sx_s}=\partial_{x_1}^{r_1}\partial_{x_2}^{r_2}\cdots\partial_{x_s}^{r_s}f$, the multi-dimensional Bell polynomials is defined as

$$
\begin{aligned}
&Y_{k_1x_1,k_2x_2,\ldots,k_sx_s}(f)\equiv Y_{k_1,k_2,\ldots,k_s}(f_{r_1x_1,r_2x_2,\ldots,r_sx_s}) \\
&\quad =e^{-f}\partial_{x_1}^{k_1}\partial_{x_2}^{k_2}\cdots\partial_{x_s}^{k_s}e^f,
\end{aligned} \tag{20}
$$

where k_i are arbitrary integers and $r_i=0,\ldots,k_i$ $(1\le i\le s)$. With the new C^∞ functions $v=v(x_1,x_2,\ldots,x_s)$ and $w=w(x_1,x_2,\ldots,x_s)$, the multi-dimensional binary Bell polynomials will be shown as

$$
\begin{aligned}
&\mathcal{Y}_{k_1x_1,k_2x_2,\ldots,k_sx_s}(v,w) \\
&\quad \equiv Y_{k_1,k_2,\ldots,k_s}(f)\Bigg|_{f_{r_1x_1,r_2x_2,\ldots,r_sx_s}=\begin{cases} v_{r_1x_1,r_2x_2,\ldots,r_sx_s}, & \text{if } r_1+\cdots+r_s \text{ is odd,} \\ w_{r_1x_1,r_2x_2,\ldots,r_sx_s}, & \text{if } r_1+\cdots+r_s \text{ is even,}\end{cases}}
\end{aligned} \tag{21}
$$

When $\sum_{i=1}^{s}k_i$ is even, in the special case of $v=0$ and $w=q$, even-order $\mathcal{Y}$-polynomials are defined as the $\mathcal{P}$-polynomials

$$
\begin{aligned}
&\mathcal{P}_{k_1x_1,k_2x_2,\ldots,k_sx_s}(q) \\
&\quad =\mathcal{Y}_{k_1x_1,k_2x_2,\ldots,k_sx_s}(0,q).
\end{aligned} \tag{22}
$$

when $q=q(x,y)$, the $\mathcal{P}$-polynomials are

$$
\begin{aligned}
&\mathcal{P}_{2x}(q)=q_{xx},\quad \mathcal{P}_{x,y}(q)=q_{xy}, \\
&\mathcal{P}_{3x,y}(q)=q_{3x,y}+3q_{2x}q_{xy},\ \ldots
\end{aligned} \tag{23}
$$

Setting $v=\ln(\frac{F}{G})$ and $w=\ln(FG)$, the bilinear term $(FG)^{-1}D_{x_1}^{k_1}D_{x_2}^{k_2}\cdots D_{x_s}^{k_s}F\cdot G$ can be written as the Bell polynomial

$$
\begin{aligned}
&\mathcal{Y}_{k_1x_1,k_2x_2,\ldots,k_sx_s}(v,w) \\
&\quad =(FG)^{-1}D_{x_1}^{k_1}D_{x_2}^{k_2}\cdots D_{x_s}^{k_s}F\cdot G,
\end{aligned} \tag{24}
$$

where $\sum_{i=1}^{s}k_i\ge 1$. Specifically, by substituting $F=G$ into formula (24), we obtain

$$
\begin{aligned}
&\mathcal{P}_{k_1x_1,k_2x_2,\ldots,k_sx_s}(q=2\ln(F)) \\
&\quad =(F)^{-2}D_{x_1}^{k_1}D_{x_2}^{k_2}\cdots D_{x_s}^{k_s}F\cdot F.
\end{aligned} \tag{25}
$$

Setting $u=cq_x$, substituting it into Eq. (1) and integrating twice with respect to x, Eq. (1) is converted into the $\mathcal{P}$-polynomials form

$$
\begin{aligned}
E(q) &= q_{yt} + \alpha_1(t)q_{xxxy} + 3\alpha_1 q_{xx}q_{xy} + \gamma\alpha_1 q_{xx} \\
&\quad + \alpha_2(t)q_{yy} + \alpha_3(t)q_{xy} + \alpha_4(t)q_{yz} \\
&= \mathcal{P}_{y,t}(q) + \alpha_1(t)\mathcal{P}_{3x,y}(q) + \gamma\alpha_1\mathcal{P}_{2x}(q) \\
&\quad + \alpha_2(t)\mathcal{P}_{2y}(q) + \alpha_3(t)\mathcal{P}_{x,y}(q) + \alpha_4(t)\mathcal{P}_{y,z}(q) \\
&= 0, \qquad (26)
\end{aligned}
$$

with the assumption $c = \frac{3}{\beta}$. In the case of $q = 2\ln f$, Eq. (26) is equivalent to the bilinear equation (3). Setting two new variables $v = \frac{q'-q}{2} = \ln\frac{g}{f}$, $w = \frac{q'+q}{2} = \ln(gf)$, where $q' = 2\ln g$ is another solution to Eq. (26).

The so-called two-field condition between q and q' can be derived as

$$
\begin{aligned}
E(q') - E(q) &= (q'-q)_{yt} + \alpha_1(t)(q'-q)_{xxxy} \\
&\quad + \frac{3\alpha_1(t)}{2}\Big[(q'-q)_{xx}(q'+q)_{xy} \\
&\quad + (q'-q)_{xy}(q'+q)_{xx}\Big] \\
&\quad + \gamma\alpha_1(t)(q'-q)_{xx} + \alpha_2(t)(q'-q)_{yy} \\
&\quad + \alpha_3(t)(q'-q)_{xy} + \alpha_4(t)(q'-q)_{yz} \\
&= 2v_{yt} + 2\alpha_1(t)v_{3x,y} + 6\alpha_1(t)(v_{xx}w_{xy} \\
&\quad + v_{xy}w_{xx}) + \gamma\alpha_1(t)v_{xx} + \alpha_2(t)v_{yy} \\
&\quad + \alpha_3(t)v_{xy} + \alpha_4(t)v_{yz} \\
&= 2\partial_y\big(\mathcal{Y}_t(v,w) + \alpha_1(t)\mathcal{Y}_{3x}(v,w) \\
&\quad + \alpha_2(t)\mathcal{Y}_y(v,w) + \alpha_3(t)\mathcal{Y}_x(v,w) \\
&\quad + \alpha_4(t)\mathcal{Y}_z(v,w) + \mu_1(t)\big) \\
&\quad - 6\alpha_1(t)\text{Wronskian}[\mathcal{Y}_{x,y}(v,w) \\
&\quad + \frac{\gamma}{3}, \mathcal{Y}_x(v,w)] \\
&= 0, \qquad (27)
\end{aligned}
$$

where $\mu_1(t)$is a function of t.

Taking $3\mathcal{Y}_{x,y}(v,w) + \gamma = -\mu_2\mathcal{Y}_x(v,w)$, where μ_2 is an constant. Thus, the Bell-polynomial-typed BT for Eq. (1) is

$$
\begin{cases}
\mathcal{Y}_t(v,w) + \alpha_1(t)\mathcal{Y}_{3x}(v,w) + \alpha_2(t)\mathcal{Y}_y(v,w) \\
+\alpha_3(t)\mathcal{Y}_x(v,w) + \alpha_4(t)\mathcal{Y}_z(v,w) + \mu_1(t) = 0, & (28a) \\
3\mathcal{Y}_{x,y}(v,w) + \gamma + \mu_2\mathcal{Y}_x(v,w) = 0, & (28b)
\end{cases}
$$

which can lead to the bilinear BT (16).

5 Infinite conservation laws

The conservation laws are basic laws which make sure given physical quantities do not change in time. Further, conservation laws gives a way to prove the integrability of NLEEs and conduct some important analysis of solutions [48]. In this section, we will use the binary Bell polynomials to derive infinite conservation laws of Eq. (1).

By introducing potential function $\eta = v_x$, the $\mathcal{Y}$-polynomial (28a) and (28b) become

$$
\begin{aligned}
&\eta_t + \alpha_2(t)\eta_y + \alpha_4(t)\eta_z \\
&\quad + (\alpha_1(t)\eta_{xx} + 3\alpha_1(t)\eta\eta_x + \beta\alpha_1(t)\eta u_x \\
&\quad + \alpha_1(t)\eta^3 + \alpha_3(t)\eta)_x = 0, \qquad (29) \\
&3\eta_y + \beta u_y + 3\partial_x^{-1}\eta_y + \gamma + \mu_2\eta = 0. \qquad (30)
\end{aligned}
$$

Substituting

$$
\eta = \varepsilon + \sum_{n=1}^{\infty} I_n(u, u_x, \ldots)\varepsilon^{-n}, \ \mu_2 = -\gamma\varepsilon^{-1} \qquad (31)
$$

into Eq. (30) and setting the coefficients of the each power of ε be zero, the recurrence relationships of conservation laws density I_n are obtained

$$
\begin{aligned}
I_1 &= -\frac{\beta}{3}u_x, \\
I_2 &= \frac{\beta}{3}u_{xx}, \\
I_3 &= -\frac{\beta}{3}u_{xxx} - \frac{\beta^2}{9}\partial_y^{-1}(u_x u_y)_x - \frac{\beta\gamma}{9}\partial_y^{-1}u_{xx}, \\
&\ldots, \\
I_{n+1} &= -I_n - \sum_{i=1}^{n}\partial_y^{-1}(I_i\partial_x^{-1}I_{n-i,y})_x \\
&\quad + \frac{\gamma}{3}\partial_y^{-1}I_{n-1,x}, \ (n = 2, 3, \ldots). \qquad (32)
\end{aligned}
$$

According to Eq. (29), the infinite conservation laws are obtained

$$
L_{n,t} + M_{n,x} + N_{n,y} + K_{n,z} = 0, \quad n = 1, 2, \ldots, \qquad (33)
$$

where the conversed densities L_n $(n = 1, 2, \ldots)$ are represented by I_n

$$L_n = I_n,\ n = 1, 2, \ldots. \tag{34}$$

The first fluxes M_n $(n = 1, 2, \ldots)$ are

$$\begin{aligned}
M_1 &= -\alpha_1(t)\frac{\beta}{3}u_{xxx} - \alpha_3(t)\frac{\beta}{3}u_x \\
&\quad - \alpha_1(t)\frac{\beta^2}{3}\partial_y^{-1}(u_x u_y)_x \\
&\quad - \alpha_1(t)\frac{\beta\gamma}{3}\partial_y^{-1}u_{xx}, \\
M_2 &= \alpha_1(t)\frac{\beta}{3}u_{xxxx} + \alpha_3(t)\frac{\beta}{3}u_{xx} \\
&\quad + \alpha_1(t)\frac{\beta^2}{3}\partial_y^{-1}(u_x u_y)_{xx} \\
&\quad + \alpha_1(t)\frac{\beta\gamma}{3}\partial_y^{-1}u_{xxx}, \\
&\ldots, \\
M_n &= \alpha_1(t)I_{n,xx} + 3\alpha_1(t)I_{n+1,x} \\
&\quad + 3\alpha_1(t)\sum_{i=1}^{n} I_i I_{n-i,x} \\
&\quad + (\beta\alpha_1(t)u_x + \alpha_3(t))I_n + 3\alpha_1(t)I_{n+2} \\
&\quad + 3\alpha_1(t)\sum_{j=1}^{n+1} I_j I_{n+1,j} \\
&\quad + \alpha_1(t)\sum_{k=1}^{n}\left(\sum_{l=1}^{n-k} I_k I_l I_{n-k-l}\right),\quad n = 3, 4, \ldots,
\end{aligned} \tag{35}$$

the second fluxes N_n $(n = 1, 2, \ldots)$ are

$$N_n = \alpha_2(t)I_n,\ n = 1, 2, \ldots, \tag{36}$$

and the third fluxes K_n $(n = 1, 2, \ldots)$ are

$$K_n = \alpha_4(t)I_n,\ n = 1, 2, \ldots. \tag{37}$$

We present recursion formulas (34)–(37) for generating an infinite sequence of conservation laws. And the relations between the first two conservation laws and Eq. (1) are given by

$$\begin{aligned}
&L_{1,t} + M_{1,x} + N_{1,y} + K_{1,z} \\
&= -\frac{\beta}{3}\partial_x\partial_y^{-1}(u_{yt} + \alpha_1(t)u_{xxxy} \\
&\quad + \beta\alpha_1(t)(u_{xy})_x + \gamma\alpha_1(t)u_{xx} + \alpha_2(t)u_{yy} \\
&\quad + \alpha_3(t)u_{xy} + \alpha_4(t)u_{yz}) \\
&= 0,
\end{aligned} \tag{38}$$

$$\begin{aligned}
&L_{2,t} + M_{2,x} + N_{2,y} + K_{2,z} \\
&= \frac{\beta}{3}\partial_x^2\partial_y^{-1}(u_{yt} + \alpha_1(t)u_{xxxy} \\
&\quad + \beta\alpha_1(t)(u_{xy})_x + \gamma\alpha_1(t)u_{xx} + \alpha_2(t)u_{yy} \\
&\quad + \alpha_3(t)u_{xy} + \alpha_4(t)u_{yz}) \\
&= 0.
\end{aligned} \tag{39}$$

Refs. [49–51] utilize Noether's theorem and the Euler-Lagrange equations to derive conservation laws for several types of equations with constant coefficients, in which the densities and fluxes of the conservation laws are related to the constant coefficients. For the variable coefficient shallow water wave equation (1) presented in this paper, we first obtained the Bell-polynomial-typed BT and used binary Bell polynomials to derive the infinite conservation laws. The results indicate that the three types of fluxes M_n, N_n, K_n vary with the changes in the variable coefficient functions $\alpha_i(t)$ $(i = 1, 2, 3, 4)$, and this dependency highlights the complex integrability characteristics of the equation, which is conducive to gaining a further understanding of its structural properties. In conclusion, Eq. (1) is completely integrable in the sense that it admits bilinear BT, Bell-polynomial-typed BT, Lax pair and infinite conservation laws.

6 Interaction solutions

In the process of studying nonlinear evolution equations, the combination of quadratic functions with exponential, trigonometric, or hyperbolic functions is used to explain the nature of the collision between kink, lump, rogue, and periodic waves. This approach helps produce solutions such as lump-kink, rogue-kink, periodic solitons, periodic lump, and periodic rogue-kink interactions [52,53]. In this section, we will use test functions to construct the interaction solutions, including lump-kink and periodic lump solutions for Eq. (1), which will be computed explicitly through symbolic computations using Maple.

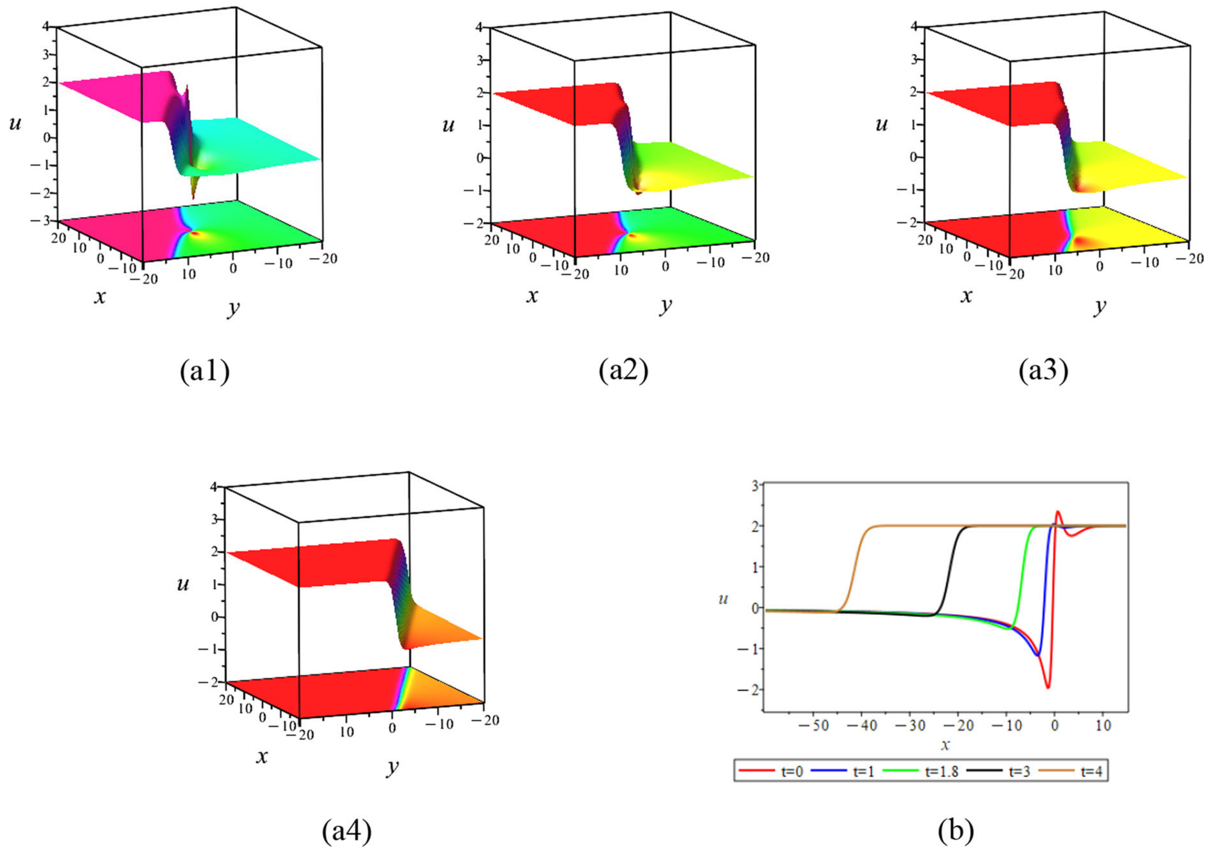

Fig. 4 The plots of the lump-kink solutions (42) for $z = 1$ at times **a1** $t = 0$; **a2** $t = 1$; **a3** $t = 1.8$; **a4** $t = 3$; **b** Time evolution plot

6.1 Lump-kink solution

We will investigate the interaction between a lump solution and a kink soliton for Eq. (1), by taking the function f in the following form

$$f = k^2 + h^2 + a_{13}(t)\mathrm{e}^{\eta} + a_{14}(t), \tag{40}$$

with $k = a_1 x + a_2 y + a_3 z + a_4(t)$, $h = a_5 x + a_6 y + a_7 z + a_8(t)$ and $\eta = a_9 x + a_{10} y + a_{11} z + a_{12}(t)$, where a_i $(i = 1, 2, 3, 5, 6, 7, 9, 10, 11)$ are real parameters to be determined, $a_i(t)$ $(i = 4, 8, 12, 13, 14)$ are functions and $a_{14}(t) > 0$. Substituting Eq. (40) into Eq. (3), we obtain a set of constraining equations for parameters

$$\Big\{ a_1 = 0, a_i = a_i \ (i = 3, 5, 7, 9, 10, 11), a_{12}(t)$$
$$= a_{12}(t), a_2 = \frac{a_5\, a_{10}\, Q}{2\, a_9\, \gamma}, a_6 = \frac{a_{10}\, a_5 W}{2\, a_9\, \gamma},$$
$$a_4(t) = -\int_0^t \left(\frac{-Q\alpha_1(t)\, a_5\, a_9{}^2\gamma + Q\alpha_2(t)\, a_5\, a_{10}{}^2}{2\, a_{10}\, a_9\, \gamma} + a_3\alpha_4(t) \right) \mathrm{d}t + \delta_1,$$
$$a_8(t) = -\int_0^t \left(\frac{a_9{}^2 a_5 \gamma\, \alpha_1(t) W + a_5 a_{10}^2 \alpha_2(t) W}{2\, a_{10}\, a_9 \gamma} + \alpha_4(t) a_7 + \alpha_3(t) a_5 \right) \mathrm{d}t + \delta_2,$$
$$a_{13}(t) = \mathrm{e}^{-\int_0^t (\alpha_1(t) a_9{}^2 (a_9 + \frac{\gamma}{a_{10}}) + a_9\, \alpha_3(t) + a_{11}\, \alpha_4(t) + a_{10}\, \alpha_2(t) + a_{12}'(t))\mathrm{d}t + \delta_3},$$
$$a_{14}(t) = \frac{2 a_5{}^2 W}{a_9{}^2 (W + 2\gamma)} \Big\},$$
$$Q = \sqrt{-3\, a_9\, a_{10}\, (3\, a_9\, a_{10} + 4\, \gamma)},$$
$$W = 3\, a_9\, a_{10} + 2\, \gamma. \tag{41}$$

where a_i $(i = 3, 5, 7, 9, 10, 11)$ and δ_i $(i = 1, 2, 3)$ are constants, while

$$a_{10}\, a_9 \neq 0, a_9\, a_{10}\, (3\, a_9\, a_{10} + 4\, \gamma)$$

$$< 0, 1 - \frac{2\gamma}{(3\,a_9\,a_{10} + 4\,\gamma)} > 0.$$

Selecting the parameters as $a_3 = a_9 = \frac{a_{10}}{2} = a_{11} = 1, a_5 = a_7 = -1, \beta = 3, \gamma = -4, a_{12}(t) = t^2,\ \delta_1 = \delta_2 = \delta_3 = 0,\ \alpha_1(t) = -\alpha_2(t) = 2t,\ \alpha_3(t) = -\alpha_4(t) = -t$, we obtain the corresponding lump-kink solution to Eq. (1) as

$$u = \frac{4t^2 + 4x + 2y + 4z + 2e^{3t^2+x+2y+z}}{\left(\frac{\sqrt{15}\,y}{2} + z - \frac{t^2}{2}\right)^2 + \left(-x - \frac{y}{2} - z - t^2\right)^2 + e^{3t^2+x+2y+z} + \frac{2}{5}}. \tag{42}$$

Figures 4 show the spatial structure of lump-kink solution (42) in the (x, y)-plane at different times, which include three-dimensional plots and the corresponding density plots. Combining the time evolution plot, we can see that one lump soliton and one kink soliton gradually fuse into one kink soliton over time. These visualizations collectively provide a detailed account of the lump-kink interaction process. It can be seen from the shape and amplitude of the final kink soliton that lump waves do not return to the state before the collision, which reflects the inelastic characteristics of the interaction.

6.2 Periodic lump solution

Then, we will investigate the interaction between a lump solution and trigonometric function sin for Eq. (1), by taking the function f in the following form

$$f = k^2 + h^2 + a_{13}\sin(\eta) + a_{14}(t), \tag{43}$$

with $k = a_1x + a_2y + a_3z + a_4(t), h = a_5x + a_6y + a_7z + a_8(t)$ and $\eta = a_9x + a_{10}y + a_{11}z + a_{12}(t)$, where a_i $(i = 1, 2, 3, 5, 6, 7, 9, 10, 11, 13)$ are real parameters to be determined, $a_i(t)$ $(i = 4, 8, 12, 14)$ are functions and $a_{14}(t) > 0$. Substituting Eq. (43) into Eq. (3), we obtain the constraints between the parameters, which can be classified into the following cases.

Case I

$$\begin{aligned}
&\Big\{ a_i = a_i\ (i = 2, 3, 7, 9, 10, 11, 13),\\
&a_1 = -\frac{a_2 a_9\ (3\,a_9\,a_{10} - 2\,\gamma)}{2\,a_{10}\,\gamma},\\
&a_5 = \frac{M a_9\,a_2}{2\,a_{10}\,\gamma}, a_6 = 0,\\
&a_4(t) = -\int_0^t \Bigg(\frac{a_2 {a_9}^2 \alpha_1(t) N + a_{10} a_2\,a_9 \alpha_3(t)(6\,\gamma - 3 a_9\,a_{10})}{2\,\gamma\,{a_{10}}^2}\\
&\quad + a_3\alpha_4(t) + a_2\alpha_2(t)\Bigg)\mathrm{d}t + \sigma_1,\\
&a_8(t) = -\int_0^t \frac{-M a_9\,a_2\,\left(3\,\alpha_1\,(t)\,a_{10}\,{a_9}^2 - 2\,\alpha_1\,(t)\,a_9\,\gamma - a_{10}\,\alpha_3\,(t)\right) + 2\gamma\alpha_4\,(t)\,a_7\,{a_{10}}^2}{2\,\gamma\,{a_{10}}^2}\\
&\mathrm{d}t + \sigma_2, a_{12}(t) = -\int_0^t \frac{-\alpha_1\,(t)\,a_{10}\,{a_9}^3 + \lambda\,\alpha_1\,(t)\,{a_9}^2 + {a_{10}}^2\alpha_2\,(t) + \alpha_4\,(t)\,a_{11}\,a_{10} + a_9\,a_{10}\,\alpha_3\,(t)}{a_{10}}\\
&\mathrm{d}t + \sigma_3, a_{14}(t) = \frac{\gamma\,({a_{10}}^4{a_{13}}^2 + 4\,{a_2}^4) - 6\,{a_2}^4 a_{10}\,a_9}{{a_{10}}^2{a_2}^2\,(3\,a_9\,a_{10} - 4\,\gamma)}\Big\},\\
&M = \sqrt{3a_9\,a_{10}\,(-3\,a_9\,a_{10} + 4\,\gamma)}, N = {a_{10}}^2{a_9}^2 - 12\,a_9\,a_{10}\,\gamma + 2\gamma^2.
\end{aligned} \tag{44}$$

where a_i $(i = 2, 3, 7, 9, 10, 11, 13)$ and σ_i $(i = 1, 2, 3)$ are constants, while

$$a_{10} \neq 0, a_9\, a_{10}\,(-3\, a_9\, a_{10} + 4\,\gamma) > 0,$$
$$\frac{\gamma\,({a_{10}}^4 {a_{13}}^2 + 4\, {a_2}^4) - 6\, {a_2}^4 a_{10}\, a_9}{(3\, a_9\, a_{10} - 4\,\gamma)} > 0.$$

Selecting the parameters as $a_2 = 1, a_3 = a_7 = \frac{1}{3}, a_9 = 2, a_{10} = a_{11} = -1, a_{13} = -\frac{1}{2}, \beta = 3, \gamma = -2, a_{12}(t) = t + 2,\ \sigma_1 = \sigma_2 = \sigma_3 = 0,\ \alpha_1(t) = t,\ \alpha_2(t) = 2t,\ \alpha_3(t) = t - 1,\ \alpha_4(t) = t + 2$, we obtain the corresponding periodic lump solution to Eq. (1) as

$$u = \frac{-2\,\cos\left(2\,x - y - z + \frac{t^2}{2} + 4\,t\right) + \frac{(-2\,t^2 - 8\,t + 4\,z)\sqrt{3}}{3} + \frac{10\,t^2}{3} + \frac{40\,t}{3} + 16\,x + 4\,y + \frac{4\,z}{3}}{\left(x + y + \frac{z}{3} - \frac{11\,t^2}{3} + \frac{t}{3}\right)^2 + \left(\sqrt{3}x + \frac{z}{3} + \frac{\left(3\sqrt{3} - \frac{1}{3}\right)t^2}{2} + \sqrt{3}t - \frac{2\,t}{3}\right)^2 - \frac{\sin\left(2x - y - z + \frac{t^2}{2} + 4t\right)}{2} + \frac{7}{4}}. \tag{45}$$

Case II

$$\Big\{a_i = a_i\ (i = 2, 3, 9, 10, 11, 13), a_1 = \frac{a_2 a_9 \gamma}{3 a_{10}(-a_{10} a_9 + \gamma)},$$
$$a_5 = \frac{\sqrt{3} a_2 a_9 \gamma\, Q}{9 a_{10}\, P\,(a_{10} a_9 - \gamma)(3 a_{10} a_9 - \gamma)},$$
$$a_6 = \frac{\sqrt{3} a_2\, P\,(3 a_{10} a_9 - \gamma)}{9 a_9 a_{10}(a_{10} a_9 - \gamma)}, a_7 = \frac{\sqrt{3} a_3\, Q}{3 P(-3 a_{10} a_9 + \gamma)},$$
$$a_4(t) = -\int_0^t \left(\frac{a_2 a_9 \gamma\,(a_9 \gamma \alpha_1(t) - a_{10}\alpha_3(t))}{3 {a_{10}}^2 (a_{10} a_9 - \gamma)} + a_3 \alpha_4(t) + a_2 \alpha_2(t)\right)\mathrm{d}t + \delta_1,$$
$$a_8(t) = \sqrt{3} P \int_0^t \left(\frac{(3 a_{10} a_9 - \gamma) a_2 \alpha_2(t)}{9 a_9 a_{10}(a_{10} a_9 - \gamma)} + \frac{a_3 \alpha_4(t)}{3 a_9 a_{10}} - \frac{a_2 \gamma\left(a_9 \gamma \alpha_1(t) + a_{10}\alpha_3(t)\right)}{9 {a_{10}}^3 (a_{10} a_9 - \gamma)}\right)\mathrm{d}t + \delta_2,$$
$$a_{12}(t) = \int_0^t \frac{\alpha_1(t) a_{10} {a_9}^3 - \gamma \alpha_1(t) {a_9}^2 - {a_{10}}^2 \alpha_2(t) - \alpha_4(t) a_{11} a_{10} - a_9 a_{10} \alpha_3(t)}{a_{10}} + \delta_3,$$
$$a_{14}(t) = \frac{729 {a_{10}}^6 {a_9}^2 {a_{13}}^2 (a_{10} a_9 - \gamma)^4 - 32 \gamma^5 {a_2}^4 (3 a_{10} a_9 - 2\gamma)}{108 a_9 {a_{10}}^3 \gamma^2 {a_2}^2 (a_{10} a_9 - \gamma)^2 (3 a_{10} a_9 - 4\gamma)}\Big\},$$
$$P = \sqrt{a_9\, a_{10}\,(-3\, a_9\, a_{10} + 4\,\gamma)},$$
$$Q = 9 {a_{10}}^2 {a_9}^2 - 15 a_{10} a_9 \gamma + 4\gamma^2. \tag{46}$$

where a_i $(i = 2, 3, 9, 10, 11, 13)$ and δ_i $(i = 1, 2, 3)$ are constants, while

$$a_9,\ a_{10},\ (a_{10} a_9 - \gamma),\ (3 a_{10} a_9 - 4\gamma)$$
$$\neq 0,\ a_9\, a_{10}\,(-3\, a_9\, a_{10} + 4\,\gamma) > 0,$$
$$32\gamma^5 {a_2}^4 (3 a_{10} a_9 - 2\gamma)$$
$$> 729 {a_{10}}^6 {a_9}^2 {a_{13}}^2 (a_{10} a_9 - \gamma)^4.$$

Selecting the parameters as $a_2 = a_3 = a_{11} = 1, a_9 = 2, a_{10} = \beta = 3, a_{13} = \frac{1}{3}, \gamma = 5, a_{12}(t) = t + 2,\ \delta_1 = \delta_2 = \delta_3 = 0,\ \alpha_1(t) = t,\ \alpha_2(t) = 2t,\ \alpha_3(t) = t - 1,\ \alpha_4(t) = -t$, we obtain the corresponding periodic lump solution to Eq. (1) as

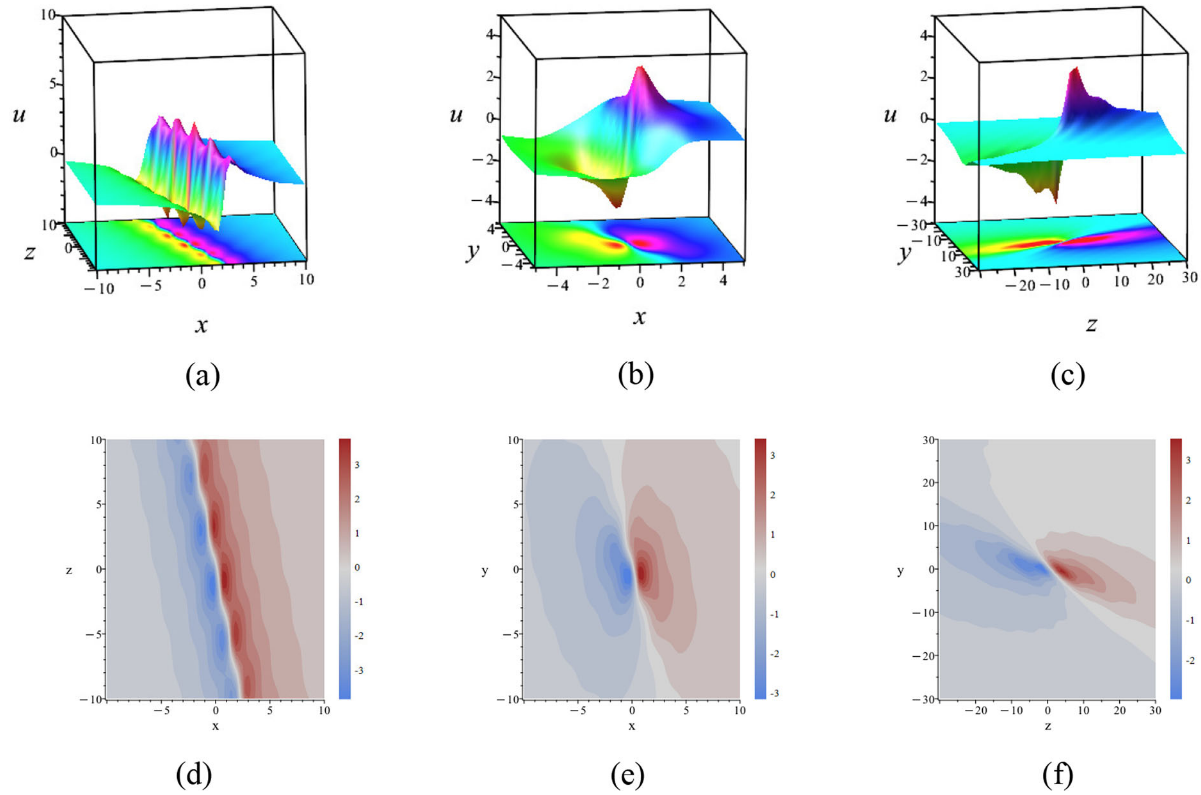

Fig. 5 Periodic lump solution (45) at time $t = 0$: **a** 3-D in (x, z)-plane; **b** 3-D in (x, y)-plane; **c** 3-D in (y, z)-plane; **d** Contour in (x, z)-plane; **e** Contour in (x, y)-plane; **f** Contour in (y, z)-plane

$$u = \frac{120000\left(-47.5t^2 - 7.29\cos(-2x - 3y - z + \frac{17}{6}t^2 - 2t) - 30t - 30x + 36y + 27z\right)}{\left(\frac{-10x}{9} + y + z - \frac{97t^2}{54} - \frac{-10t}{9}\right)^2 + \left(\frac{-10x}{27} + \frac{13y}{9} + \frac{z}{3} - \frac{77t^2}{162} - \frac{10t}{27}\right)^2 - \frac{\sin\left(-2x - 3y - z + \frac{17t^2}{6} - 2t\right)}{3} + \frac{140951}{72900}}. \tag{47}$$

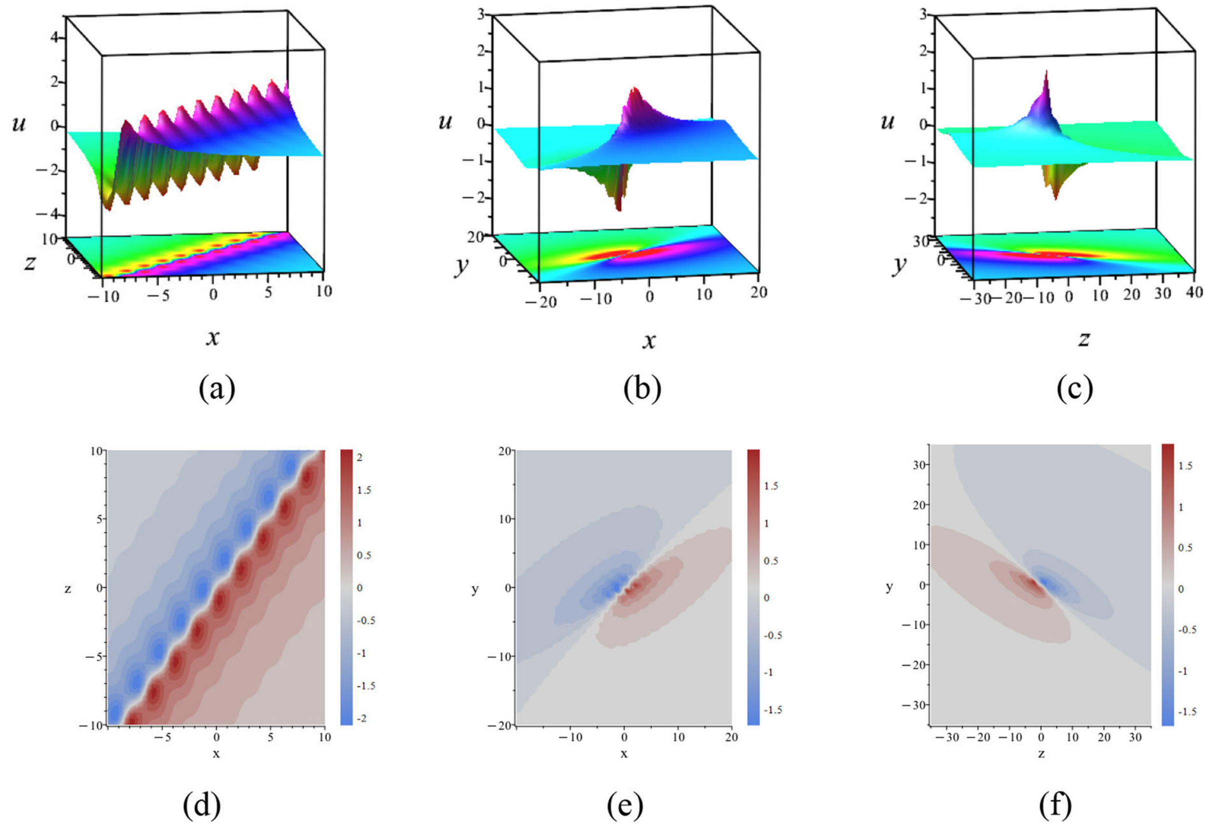

Fig. 6 Periodic lump solution (47) at time $t = 0$: **a** 3-D in (x, z)-plane; **b** 3-D in (x, y)-plane; **c** 3-D in (y, z)-plane; **d** Contour in (x, z)-plane; **e** Contour in (x, y)-plane; **f** Contour in (y, z)-plane

Case III

$$
\Big\{ a_i = a_i \ (i = 2, 3, 7, 9, 10, 11, 13), a_1 = 0, a_5 = -\frac{2\gamma a_2 a_9}{\sqrt{3}Pa_{10}} a_6 = \frac{Sa_2}{\sqrt{3}P},
$$

$$
a_4(t) = \int_0^t \frac{a_2 {a_9}^2 \gamma \alpha_1(t) - {a_{10}}^2 \alpha_4(t) a_3 - a_2 {a_{10}}^2 \alpha_2(t)}{{a_{10}}^2} \mathrm{d}t + \tau_1,
$$

$$
a_8(t) = \int_0^t a_2 \left(\frac{\Big(-S\alpha_1(t) a_9^2 \gamma - 2\gamma a_{10} a_9 \alpha_3(t) \Big)}{3a_{10}^2 P} - \frac{\alpha_2(t)(3a_{10}a_9 + 2\gamma)}{3P} - \frac{a_7 \alpha_4(t)}{3} \right) \mathrm{d}t + \tau_2,
$$

$$
a_{12}(t) = \int_0^t \frac{\alpha_1(t) a_{10} {a_9}^3 - \gamma \alpha_1(t) {a_9}^2 - {a_{10}}^2 \alpha_2(t) - \alpha_4(t) a_{11} a_{10} - a_9 a_{10} \alpha_3(t)}{a_{10}} + \tau_3,
$$

$$
a_{14}(t) = \frac{-9{a_{13}}^2 {a_{10}}^6 {a_9}^2 (-3a_9 a_{10} + 4\gamma)^2 - 32\gamma^3 {a_2}^4 S}{12 a_9 {a_{10}}^3 \gamma {a_2}^2 (-3a_9 a_{10} + 4\gamma)^2} \Big\},
$$

$$
S = 3a_9 a_{10} - 2\gamma. \tag{48}
$$

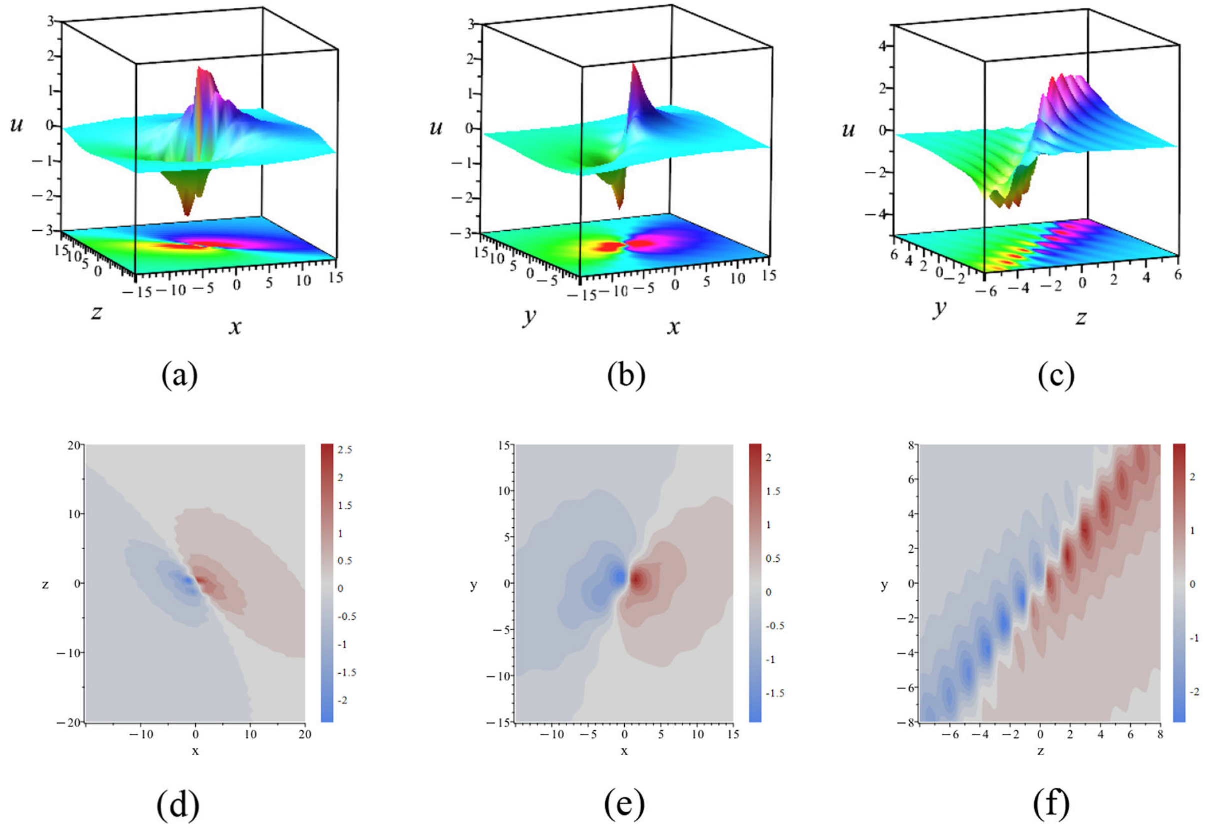

Fig. 7 Periodic lump solution (49) at time $t = 0$: **a** 3-D in (x, z)-plane; **b** 3-D in (x, y)-plane; **c** 3-D in (y, z)-plane; **d** Contour in (x, z)-plane; **e** Contour in (x, y)-plane; **f** Contour in (y, z)-plane

where a_i $(i = 2, 3, 7, 9, 10, 11, 13)$ and σ_i $(i = 1, 2, 3)$ are constants, while

$$
\begin{aligned}
& a_9,\ a_{10},\ (3a_{10}a_9 - 4\gamma) \neq 0,\\
& a_9\, a_{10}\, (-3\, a_9\, a_{10} + 4\, \gamma) > 0,\\
& \begin{cases} 32\gamma^3 {a_2}^4 S < -9{a_{13}}^2{a_{10}}^6{a_9}^2(-3a_9a_{10} + 4\gamma)^2, \\ \quad a_9a_{10}\gamma > 0 \\ 32\gamma^3 {a_2}^4 S > -9{a_{13}}^2{a_{10}}^6{a_9}^2(-3a_9a_{10} + 4\gamma)^2, \\ \quad a_9a_{10}\gamma < 0. \end{cases}
\end{aligned}
$$

Selecting the parameters as $a_2 = a_{13} = -1, a_3 = a_7 = a_9 = a_{10} = \gamma = 1, a_{11} = 4, \beta = 3, a_{12} = t + 2,\ \tau_1 = \tau_2 = \tau_3 = 0,\ \alpha_1(t) = t,\ \alpha_2(t) = 2t,\ \alpha_3(t) = t - 1,\ \alpha_4(t) = -t$, we obtain the corresponding periodic lump solution to Eq. (1) as

$$
u = \frac{-48\sqrt{3}t^2 - 48\,t^2 - 192\,t - 192\,x + 96\,y - 96\,\sqrt{3}z + 72\,\cos(x + y + 4\,z + \frac{1}{2}t^2 + t)}{\left(t^2 - y + z\right)^2 + \left(\frac{2\sqrt{3}x}{3} - \frac{\sqrt{3}y}{3} + z - \frac{(-\sqrt{3}-1)\sqrt{3}t^2}{6} + \frac{2\sqrt{3}t}{3}\right)^2 - \sin\left(x + y + 4z + \frac{t^2}{2} + t\right) + \frac{23}{12}}. \tag{49}
$$

Figures 5, 6 and 7 illustrate the spatial structures of the periodic lump solutions (45), (47) and (49) at time

$t = 0$, respectively. Under the influence of trigonometric function, periodic lump solution takes the form of periodic wave in a certain plane. If trigonometric functions are not included in f, the periodic lump solution degenerates to a single lump wave in all three planes.

Specifically, Figs. 5, 6 and 7(a) and (d) display the three-dimensional plost, density plots and contour plots in the (x, z)-plane with $y = 0$. Similarly, (b) and (e) present the three-dimensional plost, density plots and contour plots in the (x, y)-plane with $z = 0$. Finally, (c) and (f) depict the three-dimensional plost, density plots and contour plots in the (y, z)-plane with $x = 0$. These visualizations provide a comprehensive view of the solution's behavior in the dimensions, revealing the structural complexity and density variations. Each set of plots-3-D, density, and contour-complements the others, providing a multifaceted view of the periodic lump solutions in different planes. This comprehensive visualization aids in analyzing the spatial dynamics and structural properties of the solutions.

7 Conclusions

In this work, we have delved into the integrability of the extended (3 + 1)-dimensional variable-coefficient shallow water equation (1), confirming its integrability through the Painlevé test. Using the hirota bilinear method, we successfully derived both one-soliton and two-soliton solutions, which are fundamental in understanding the dynamic behaviors of the system. Furthermore, we reported on bilinear BT, Bell-polynomial-typed BT, Lax pair, and infinite conservation laws, with the conserved densities and fluxes explicitly depending on the variable coefficient functions. This dependency highlights the equation's complex integrability characteristics and offers a deeper insight into its structural properties.

Additionally, based on test functions, the lump-kink solution and periodic lump solutions have been constructed. In more detail, one lump soliton and one kink soliton fuse to one kink soliton, and the final kink soliton preserves its shape and amplitude in Fig. 4. The periodic lump solutions, as shown in Figs. 5, 6 and 7. The combination of 3-D plots and contour plots offers a comprehensive analysis of the solutions' spatial dynamics and structural properties, enhancing our understanding of the integrability characteristics of the equation (1). These solutions not only demonstrate the equation's capability to support complex wave interactions but also provide a basis for predicting and managing wave dynamics in practical applications.

Overall, our findings enhance the understanding of the integrability characteristics of this model and contribute to the broader study of nonlinear wave interactions in various physical systems. The results obtained from this research could have significant implications for the study of soliton theory and its applications in fluid dynamics, nonlinear optics, and other fields where soliton solutions play a crucial role.

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Declarations

Conflict of interest The authors declare that they have no Conflict of interest.

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