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Multiple-Wave Solutions to Generalized Bilinear Equations in Terms of Hyperbolic and Trigonometric Solutions

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Abstract: The linear superposition principle is applied to hyperbolic and trigonometric function solutions to generalized bilinear equations. We determine sufficient and necessary conditions for the existence of linear subspaces of hyperbolic and trigonometric function solutions to generalized bilinear equations. By using weights, three examples are given to show applicability of our theory.

Keywords: generalized bilinear equations, N-wave solution, linear superposition principle

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1 Introduction

It is important to seek exact solutions to nonlinear differential equations of mathematical physics. Exact solutions play a vital role in understanding various qualitative and quantitative features of nonlinear phenomena that nonlinear equations describe. There are many kinds of exact solutions such as periodic solutions, travelling wave solutions, soliton solutions, and complexiton solutions. The existence of those solutions helps us to explore integrability of given nonlinear differential equations [1–9].

The Hirota bilinear method is a very efficient way to investigate integrability of differential equations, and it is applied to many integrable equations to search for N-soliton solutions [10–12]. On one hand, the existence of N-soliton solutions often implies the integrability of differential equations by quadratures. On the other hand, bilinear equations are the nearest neighbors to linear equations and are expected to have features similar to those of linear equations. In [13], linear superposition principles of hyperbolic and trigonometric functions solutions to Hirota bilinear equations were analyzed and specific classes of N-soliton solutions were constructed, upon following studies on linear superposition principles [14, 15]. Recently, Hirota derivatives were generalized by adopting a new way of assigning symbols for derivatives [16, 17]. As a result, bilinear equations have been generalized with the aid of these new derivatives. After that, derivatives and bilinear equations were generalized further [18], by employing two numbers that are used to determine signs.

In this article, we analyze linear superposition principle for generalized bilinear equations, focusing on hyperbolic and trigonometric function solutions. We will introduce a criterion, being sufficient and necessary, for the linear superposition principle of hyperbolic and trigonometric function solutions to hold. This criterion is based on an equality established in [18] and also provides a way of finding N-wave solutions to the given generalized bilinear equations.

The article is organized as follows. In Section 2, we study sufficient and necessary conditions to guarantee the existence of linear subspaces of hyperbolic and trigonometric function solutions to generalized bilinear equations. In Section 3, a few illustrative examples will be given to construct N-wave solutions and to show the effectiveness of the linear superposition principle established in the previous section. Finally, some conclusions will be provided.

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2 Linear superposition principles

Let us begin with a bilinear equation with generalized bilinear derivatives:

$$P\left(D_{\bar{p}_1,x_1},\dots,D_{\bar{p}_m,x_1};\dots;D_{\bar{p}_1,x_M},\dots,D_{\bar{p}_m,x_M}\right)f\cdot f=0, \quad (1)$$

where P is a polynomial of $D_{\bar{p}_i,x_j}$, where $D_{\bar{p}_i,x_j}$, $1 \le i \le m$, $1 \le j \le M$ are generalized differential operators [18] defined by

$$\left(D^n_{\bar{p},x} f \cdot g\right)(x) = \left(D^n_{(p,p'),x} f \cdot g\right)(x) = \sum_{i=0}^{n} \binom{n}{i} \alpha_p^{n-i} \alpha_{p'}^{i} \left(\partial_x^{n-i} f\right)(x) \left(\partial_x^{i} g\right)(x), \ n \ge 1, \tag{2}$$

in which the powers of α_s are just signs determined by

$$\alpha_s^l = (-1)^{r_s(l)}, \text{ where } l = r_s(l) \bmod s \text{ with } 0 \le r_s(l) < s, \quad s \ge 1, \ l \ge 0. \tag{3}$$

These derivatives are further generalizations of the generalized derivatives in [16, 17]. Now introduce N wave variables:

$$\eta_i = \mathbf{k}_i \cdot \mathbf{x} = k_{1,i}x_1 + k_{2,i}x_2 + \dots + k_{M,i}x_M, \ 1 \le i \le N, \tag{4}$$

and exponential wave functions

$$f_i = e^{\eta_i} = e^{k_{1,i}x_1 + k_{2,i}x_2 + \dots + k_{M,i}x_M}, \ 1 \le i \le N, \tag{5}$$

where the $k_{j,i}$'s are real constants. Here, a wave-related vector $\mathbf{k}_i$ and the dependent variable vector $\mathbf{x}$ are

$$\mathbf{k}_i = (k_{1,i}, k_{2,i}, \dots, k_{M,i}), \ 1 \le i \le N, \ \mathbf{x} = (x_1, x_2, \dots, x_M). \tag{6}$$

Take a linear combination

$$f = \varepsilon_1 f_1 + \varepsilon_2 f_2 + \dots + \varepsilon_N f_N = \sum_{i=1}^{N} \varepsilon_i f_i = \sum_{i=1}^{N} \varepsilon_i e^{\eta_i}, \tag{7}$$

where ε_i, $1 \le i \le N$, are arbitrary constants. It is known [18] that a linear combination f of N exponential waves solves a generalized bilinear eq. (1) if and only if the following condition

$$P\left(\beta_{1,1}(i,j), \dots, \beta_{1,m}(i,j); \dots; \beta_{M,1}(i,j), \dots, \beta_{M,m}(i,j)\right) + P\left(\beta_{1,1}(j,i), \dots, \beta_{1,m}(j,i); \dots; \beta_{M,1}(j,i), \dots, \beta_{M,m}(j,i)\right) = 0, \quad 1 \le i \le j \le N, \tag{8}$$

is satisfied where $\beta_{r,s}(i,j)$ are defined by

$$\beta_{r,s}(i,j) = \alpha_{p_s} k_{r,i} + \alpha_{p'_s} k_{r,j}, 1 \le r \le M, 1 \le s \le m. \tag{9}$$

2.1 Linear superposition principle of hyperbolic function solutions

We take $f_i = \mathrm{ch}\eta_i = \frac{1}{2}\left(e^{\eta_i} + e^{-\eta_i}\right)$, $1 \le i \le N$, as hyperbolic function solutions to eq. (1). Consider

$$f = \varepsilon_1 f_1 + \varepsilon_2 f_2 + \dots + \varepsilon_N f_N = \sum_{i=1}^{N} \varepsilon_i \mathrm{ch}\eta_i = \sum_{i=1}^{N} \varepsilon_i \frac{1}{2}\left(e^{\eta_i} + e^{-\eta_i}\right), \tag{10}$$

which is a general linear combination of the hyperbolic function solutions. The following identity holds for exponential functions under generalized bilinear derivatives [18]:

$$P\left(D_{\bar{p}_1,x_1}, \dots, D_{\bar{p}_m,x_1}; \dots; D_{\bar{p}_1,x_M}, \dots, D_{\bar{p}_m,x_M}\right) e^{\eta_i} \cdot e^{\eta_j} = P\left(\beta_{1,1}(i,j), \dots, \beta_{1,m}(i,j); \dots; \beta_{M,1}(i,j), \dots, \beta_{M,m}(i,j)\right) e^{\eta_i + \eta_j}, \ 1 \le i, j \le N. \tag{11}$$

Based on eq. (11), we can compute that

$$\begin{aligned}
& P\left(D_{\bar{p}_1,x_1}, \dots, D_{\bar{p}_m,x_1}; \dots; D_{\bar{p}_1,x_M}, \dots, D_{\bar{p}_m,x_M}\right) f \cdot f \\
&= P\left(D_{\bar{p}_1,x_1}, \dots, D_{\bar{p}_m,x_1}; \dots; D_{\bar{p}_1,x_M}, \dots, D_{\bar{p}_m,x_M}\right) \sum_{i=1}^{N} \varepsilon_i \mathrm{ch}\eta_i \cdot \sum_{j=1}^{N} \varepsilon_j \mathrm{ch}\eta_j \\
&= \sum_{i,j=1}^{N} \varepsilon_i \varepsilon_j P\left(D_{\bar{p}_1,x_1}, \dots, D_{\bar{p}_m,x_1}; \dots; D_{\bar{p}_1,x_M}, \dots, D_{\bar{p}_m,x_M}\right) \frac{1}{2}\left(e^{\eta_i} + e^{-\eta_i}\right) \cdot \frac{1}{2}\left(e^{\eta_j} + e^{-\eta_j}\right) \\
&= \frac{1}{4} \sum_{i,j=1}^{N} \varepsilon_i \varepsilon_j P\left(D_{\bar{p}_1,x_1}, \dots, D_{\bar{p}_m,x_1}; \dots; D_{\bar{p}_1,x_M}, \dots, D_{\bar{p}_m,x_M}\right) \left(e^{\eta_i} \cdot e^{\eta_j} + e^{\eta_i} \cdot e^{-\eta_j} + e^{-\eta_i} \cdot e^{\eta_j} + e^{-\eta_i} \cdot e^{-\eta_j}\right)
\end{aligned}$$

$$\begin{aligned}
= \frac{1}{4} \sum_{1 \le i < j \le N}^{N} \varepsilon_i \varepsilon_j \Big[& \big(P\left(\beta_{1,1}(i,j), \dots, \beta_{1,m}(i,j); \dots; \beta_{M,1}(i,j), \dots, \beta_{M,m}(i,j)\right) \\
& + P\left(\beta_{1,1}(j,i), \dots, \beta_{1,m}(j,i); \dots; \beta_{M,1}(j,i), \dots, \beta_{M,m}(j,i)\right)\big) e^{\eta_i + \eta_j} \\
& + \big(P\left(\beta_{1,1}(i,-j), \dots, \beta_{1,m}(i,-j); \dots; \beta_{M,1}(i,-j), \dots, \beta_{M,m}(i,-j)\right) \\
& + P\left(\beta_{1,1}(-j,i), \dots, \beta_{1,m}(-j,i); \dots; \beta_{M,1}(-j,i), \dots, \beta_{M,m}(-j,i)\right)\big) e^{\eta_i - \eta_j} \\
& + \big(P\left(\beta_{1,1}(-i,j), \dots, \beta_{1,m}(-i,j); \dots; \beta_{M,1}(-i,j), \dots, \beta_{M,m}(-i,j)\right) \\
& + P\left(\beta_{1,1}(j,-i), \dots, \beta_{1,m}(j,-i); \dots; \beta_{M,1}(j,-i), \dots, \beta_{M,m}(j,-i)\right)\big) e^{-\eta_i + \eta_j} \\
& + \big(P\left(\beta_{1,1}(-i,-j), \dots, \beta_{1,m}(-i,-j); \dots; \beta_{M,1}(-i,-j), \dots, \beta_{M,m}(-i,-j)\right) \\
& + P\left(\beta_{1,1}(-j,-i), \dots, \beta_{1,m}(-j,-i); \dots; \beta_{M,1}(-j,-i), \dots, \beta_{M,m}(-j,-i)\right)\big) e^{-\eta_i - \eta_j} \Big]
\end{aligned}$$

$$+\frac{1}{4}\sum_{1\le i\le N}^{N}\varepsilon_i^2\left[P\left(\beta_{1,1}(i,i),\ldots,\beta_{1,m}(i,i);\ldots;\beta_{M,1}(i,i),\ldots,\beta_{M,m}\right)(i,i)e^{2\eta_i}\right.$$
$$+P\left(\beta_{1,1}(i,-i),\ldots,\beta_{1,m}(i,-i);\ldots;\beta_{M,1}(i,-i),\ldots,\beta_{M,m}(i,-i)\right)$$
$$+P\left(\beta_{1,1}(-i,i),\ldots,\beta_{1,m}(-i,i);\ldots;\beta_{M,1}(-i,i),\ldots,\beta_{M,m}(-i,i)\right)$$
$$\left.+P\left(\beta_{1,1}(-i,-i),\ldots,\beta_{1,m}(-i,-i);\ldots;\beta_{M,1}(-i,-i),\ldots,\beta_{M,m}\right)(-i,-i)e^{-2\eta_i}\right] \quad (12)$$

where

$$\begin{aligned}\beta_{r,s}(i,j) &= \alpha_{p_s}k_{r,i}+\alpha_{p_s'}k_{r,j},\\ \beta_{r,s}(i,-j) &= \alpha_{p_s}k_{r,i}-\alpha_{p_s'}k_{r,j},\\ \beta_{r,s}(-i,j) &= -\alpha_{p_s}k_{r,i}+\alpha_{p_s'}k_{r,j},\\ \beta_{r,s}(-i,-j) &= -\alpha_{p_s}k_{r,i}-\alpha_{p_s'}k_{r,j}.\end{aligned} \quad (13)$$

From eq. (12), it is obvious that a linear combination function f of the N hyperbolic function solutions $f_i = \mathrm{ch}\eta_i = \frac{1}{2}\left(e^{\eta_i}+e^{-\eta_i}\right)$, $1\le i\le N$, solves the generalized bilinear eq. (1) if and only if the following conditions:

$$P\left(\beta_{1,1}(i,j),\ldots,\beta_{1,m}(i,j);\ldots;\beta_{M,1}(i,j),\ldots,\beta_{M,m}(i,j)\right)$$
$$+P\left(\beta_{1,1}(j,i),\ldots,\beta_{1,m}(j,i);\ldots;\beta_{M,1}(j,i),\ldots,\beta_{M,m}\right)(j,i)=0,$$
$$1\le i\le j\le N,$$
$$P\left(\beta_{1,1}(i,-j),\ldots,\beta_{1,m}(i,-j);\ldots;\beta_{M,1}(i,-j),\ldots,\beta_{M,m}(i,-j)\right)$$
$$+P\left(\beta_{1,1}(-j,i),\ldots,\beta_{1,m}(-j,i);\ldots;\beta_{M,1}(-j,i),\ldots,\beta_{M,m}(-j,i)\right)$$
$$=0,\ 1\le i,j\le N,$$
$$P\left(\beta_{1,1}(-i,-j),\ldots,\beta_{1,m}(-i,-j);\ldots;\beta_{M,1}(-i,-j),\ldots,\beta_{M,m}\right)(-i,-j))$$
$$+P\left(\beta_{1,1}(-j,-i),\ldots,\beta_{1,m}(-j,-i);\ldots;\beta_{M,1}(-j,-i),\ldots,\beta_{M,m}\right)$$
$$(-j,-i))=0,\ 1\le i\le j\le N, \quad (14)$$

are satisfied.

With the aid of this result, we obtain the following theorem.

Theorem 2.1: Let $P(x_{1,1},\ldots x_{m,1};\ldots;x_{1,M},\ldots,x_{m,M})$ be a polynomial and the N wave variables be defined by $\eta_i=\mathbf{k}_i\cdot\mathbf{x}=k_{1,i}x_1+k_{2,i}x_2+\ldots+k_{M,i}x_M$, $1\le i\le N$, where $k_{j,i}$'s are real constants. Then any linear combination of the hyperbolic function solutions $f_i = \mathrm{ch}\eta_i = \frac{1}{2}\left(e^{\eta_i}+e^{-\eta_i}\right)$, $1\le i\le N$, solves the generalized bilinear eq. (1) if and only if the system eq. (14) is satisfied.

This theorem tells when a linear superposition of hyperbolic function solutions is still a solution of a given generalized bilinear equation. Besides, it also helps us to find N-wave solutions to generalized bilinear equations. Once we solve the system eq. (14), then we can present a N-wave solution, formed by eq. (10), to a generalized bilinear equation.

2.2 Linear superposition principle of trigonometric function solutions

We take $f_i=\cos\eta_i=\frac{1}{2}\left(e^{I\eta_i}+e^{-I\eta_i}\right)$, $1\le i\le N$, where $\eta_i=\mathbf{k}_i\cdot\mathbf{x}$, $1\le i\le N$, $I=\sqrt{-1}$, as trigonometric function solutions to eq. (1). Consider

$$\begin{aligned}f &= \varepsilon_1 f_1+\varepsilon_2 f_2+\ldots+\varepsilon_N f_N=\sum_{i=1}^{N}\varepsilon_i\cos\eta_i\\ &= \sum_{i=1}^{N}\varepsilon_i\frac{1}{2}\left(e^{I\eta_i}+e^{-I\eta_i}\right),\end{aligned} \quad (15)$$

which is a general linear combination of the trigonometric function solutions. Similarly, a linear combination function f of the N trigonometric function solutions in eq. (15) solves the generalized bilinear eq. (1) if and only if the following conditions:

$$P\left(\beta_{1,1}(Ii,Ij),\ldots,\beta_{1,m}(Ii,Ij);\ldots;\beta_{M,1}(Ii,Ij),\ldots,\beta_{M,m}(Ii,Ij)\right)$$
$$+P\left(\beta_{1,1}(Ij,Ii),\ldots,\beta_{1,m}(Ij,Ii);\ldots;\beta_{M,1}(Ij,Ii),\ldots,\beta_{M,m}(Ij,Ii)\right)$$
$$=0,\ 1\le i\le j\le N,$$
$$P\left(\beta_{1,1}(Ii,-Ij),\ldots,\beta_{1,m}(Ii,-Ij);\ldots;\beta_{M,1}(Ii,-Ij),\ldots,\beta_{M,m}\right)(Ii,-Ij)$$
$$+P\left(\beta_{1,1}(-Ij,Ii),\ldots,\beta_{1,m}(-Ij,Ii);\ldots;\beta_{M,1}(-Ij,Ii),\ldots,\beta_{M,m}\right)(-Ij,Ii)=0,\ 1\le i,j\le N,$$
$$P\left(\beta_{1,1}(-Ii,-Ij),\ldots,\beta_{1,m}(-Ii,-Ij);\ldots;\beta_{M,1}(-Ii,-Ij),\ldots,\beta_{M,m}\right)(-Ii,-Ij)$$
$$+P\left(\beta_{1,1}(-Ij,-Ii),\ldots,\beta_{1,m}(-Ij,-Ii);\ldots;\beta_{M,1}(-Ij,-Ii),\ldots,\right)\beta_{M,m}(-Ij,-Ii)=0,\ 1\le i\le j\le N, \quad (16)$$

are satisfied. Note that

$$\begin{aligned}\beta_{r,s}(Ii,Ij) &= \alpha_{p_s}Ik_{r,i}+\alpha_{p_s'}Ik_{r,j},\\ \beta_{r,s}(Ii,-Ij) &= \alpha_{p_s}Ik_{r,i}-\alpha_{p_s'}Ik_{r,j},\\ \beta_{r,s}(-Ii,Ij) &= -\alpha_{p_s}Ik_{r,i}+\alpha_{p_s'}Ik_{r,j},\\ \beta_{r,s}(-Ii,-Ij) &= -\alpha_{p_s}Ik_{r,i}-\alpha_{p_s'}Ik_{r,j},\ \text{and } I=\sqrt{-1}.\end{aligned} \quad (17)$$

To sum up, we have the following theorem.

Theorem 2.2: Let $P(x_{1,1}, ...x_{m,1}; ...; x_{1,M}, ..., x_{m,M})$ be a polynomial and the N wave variables defined by $\eta_i = \mathbf{k}_i \cdot \mathbf{x} = k_{1,i}x_1 + k_{2,i}x_2 + ... + k_{M,i}x_M$, $1 \le i \le N$, where $k_{j,i}$'s are real constants. Then any linear combination of the trigonometric function solutions $f_i = \cos\eta_i$, $1 \le i \le N$, solves the generalized bilinear eq. (1) if and only if the system eq. (16) is satisfied.

This theorem tells when a linear superposition of trigonometric function solutions is still a solution of a given generalized bilinear equation. Besides, it also helps us to find N-wave solutions to generalized bilinear equations. Once we solve the system eq. (16), then we can present a N-wave solution, formed by eq. (15), to a generalized bilinear equation.

3 Applications

Example 1: Let us introduce the weights of independent variables:

$$\left(w(x), w(y), w(z), w(t)\right) = (1, 3, 5, 7), \tag{18}$$

and consider a polynomial being homogeneous in weight 8 :

$$P = c_1x^8 + c_2x^5y + c_3x^3z + c_4xt + c_5yz, \tag{19}$$

where c_1, c_2, c_3, c_4, c_5 are constants. According to the chosen weights, we assume that the wave variables are

$$\eta_i = k_ix + b_1k_i^3y + b_2k_i^5z + b_3k_i^7t,\ 1 \le i \le N, \tag{20}$$

where k_i, $1 \le i \le N$, are arbitrary constants, but b_1, b_2, and b_3 are constants to be determined.

A simple and direct computation shows that the corresponding generalized bilinear equation reads

$$\begin{aligned} &P\left(D_{(2,5)x}, D_{(2,5)y}, D_{(2,5)z}, D_{(2,5)t}\right) f \cdot f \\ &= 70c_1f_{xxxx}^2 + 20c_2f_{xx}f_{xxxy} + 10c_2f_{xy}f_{xxxx} + 20c_2f_{xxx}f_{xxy} \\ &+ 2c_3ff_{xxxz} + 2c_3f_zf_{xxx} + 6c_3f_xf_{xxz} + 6c_3f_{xx}f_{xz} \\ &+ 2c_4ff_{yz} + 2c_4f_yf_z + 2c_5ff_{xt} + 2c_5f_xf_t \\ &= 0, \end{aligned} \tag{21}$$

which possesses the linear subspace of N-wave solutions determined by

$$f = \sum_{i=1}^{N} \varepsilon_if_i = \sum_{i=1}^{N} \varepsilon_i \mathrm{ch}\eta_i = \sum_{i=1}^{N} \varepsilon_i \mathrm{ch}\left(k_ix + b_1k_i^3y + b_2k_i^5z + b_3k_i^7t\right), \tag{22}$$

where ε_i's and k_i's are arbitrary, and b_1, b_2, and b_3 are given by

$$b_1 = \frac{2c_3}{c_5},\ b_2 = -\frac{20c_2}{3c_5},\ b_3 = \frac{20c_3c_2}{c_5c_4}, \tag{23}$$

when the coefficients of the polynomial P satisfy

$$7c_1c_5 = -2c_3c_2. \tag{24}$$

Similarly, eq. (21) has the linear subspace of N-wave solutions defined by

$$f = \sum_{i=1}^{N} \varepsilon_if_i = \sum_{i=1}^{N} \varepsilon_i \cos\eta_i = \sum_{i=1}^{N} \varepsilon_i \cos\left(k_ix + b_1k_i^3y + b_2k_i^5z + b_3k_i^7t\right), \tag{25}$$

where ε_i's and k_i's are arbitrary, and b_1, b_2, and b_3 are given by

$$b_1 = -\frac{2c_3}{c_5},\ b_2 = -\frac{20c_2}{3c_5},\ b_3 = -\frac{20c_3c_2}{c_5c_4} \tag{26}$$

when the coefficients of the polynomial P satisfy

$$7c_1c_5 = -2c_3c_2. \tag{27}$$

Example 2: Let us introduce the weights of independent variables:

$$\left(w(x), w(y), w(z), w(t)\right) = (1, -1, 3, 5), \tag{28}$$

and consider a polynomial being homogeneous in weight 4 :

$$P = c_1x^4 + c_2x^5y + c_3xz + c_4yt + c_5y^2z^2 + c_6x^2yz + c_7xty^2, \tag{29}$$

where $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ are constants. According to chosen weights we assume that the wave variables are

$$\eta_i = k_ix + b_1k_i^{-1}y + b_2k_i^3z + b_3k_i^5t,\ 1 \le i \le N, \tag{30}$$

where k_i, $1 \le i \le N$, are arbitrary constants, but b_1, b_2, and b_3 are constants to be determined.

A simple and direct computation shows that the corresponding generalized bilinear equation reads

$$\begin{aligned}
&P\left(D_{(3,5)x}, D_{(3,5)y}, D_{(3,5)z}, D_{(3,5)t}\right) f\cdot f \\
&= 6c_1 f_{xx}^2 - 20c_2 f_{xxy} f_{xxx} - 2c_2 f_y f_{xxxxx} - 10c_2 f_x f_{xxxy} \\
&+ 2c_3 f_{xz} f + 2c_3 f_x f_z + 2c_4 f_{yt} f + 2c_4 f_y f_t + 2c_5 f_{yy} f_{zz} \\
&+ 4c_5 f_{yz} f_{yz} + 2c_6 f_{xx} f_{yz} + 4c_6 f_{xy} f_{xz} + 2c_7 f_{yy} f_{xt} \\
&+ 4c_7 f_{xy} f_{yt} \\
&= 0,
\end{aligned} \tag{31}$$

which has the linear subspace of N-wave solutions defined by

$$f = \sum_{i=1}^{N} \varepsilon_i f_i = \sum_{i=1}^{N} \varepsilon_i \mathrm{ch}\eta_i = \sum_{i=1}^{N} \varepsilon_i \mathrm{ch}\left(k_i x + b_1 k_i^2 y + b_2 k_i^3 z + b_3 k_i^4 t\right), \tag{32}$$

where ε_i's and k_i's are arbitrary, and b_1, b_2, and b_3 are given by

$$b_1 = -\frac{8c_4c_3}{15c_6c_4 + c_7c_3}, \quad b_2 = -\frac{45(15c_6c_4 + c_7c_3)c_1}{8(2c_7c_3 - 15c_6c_4)c_3}, \quad b_3 = \frac{3c_1(15c_6c_4 + c_7c_3)^2}{64c_4^2c_3(2c_7c_3 - 15c_6c_4)}, \tag{33}$$

when the coefficients of the polynomial P satisfy

$$3c_1(15c_6c_4 + c_7c_3)^2 = 64c_2c_3c_4(2c_7c_3 - 15c_6c_4), \; 675c_1c_5c_4^2 = -c_7c_3(2c_7c_3 - 15c_6c_4). \tag{34}$$

Similarly, eq. (31) has the linear subspace of N-wave solutions defined by

$$f = \sum_{i=1}^{N} \varepsilon_i f_i = \sum_{i=1}^{N} \varepsilon_i \cos\eta_i = \sum_{i=1}^{N} \varepsilon_i \cos\left(k_i x + b_1 k_i^2 y + b_2 k_i^3 z + b_3 k_i^4 t\right), \tag{35}$$

where ε_i's and k_i's are arbitrary, and b_1, b_2, and b_3 are given by

$$b_1 = \frac{8c_4c_3}{15c_6c_4 + c_7c_3}, \quad b_2 = \frac{45(15c_6c_4 + c_7c_3)c_1}{8(2c_7c_3 - 15c_6c_4)c_3}, \quad b_3 = \frac{3c_1(15c_6c_4 + c_7c_3)^2}{64c_4^2c_3(2c_7c_3 - 15c_6c_4)}, \tag{36}$$

when the coefficients of the polynomial P satisfy

$$3c_1(15c_6c_4 + c_7c_3)^2 = 64c_2c_3c_4(2c_7c_3 - 15c_6c_4), \; 675c_1c_5c_4^2 = -c_7c_3(2c_7c_3 - 15c_6c_4). \tag{37}$$

Example 3: Let us introduce the weights of independent variables:

$$\left(w(x), w(y), w(z), w(t)\right) = (1, 3, -1, -3), \tag{38}$$

and consider a polynomial being homogeneous in weight 4 :

$$\begin{aligned}
P &= c_1 x_{\bar p_1}^4 + c_2 x_{\bar p_1} y_{\bar p_1} + c_3 x_{\bar p_1}^5 z_{\bar p_1} + c_4 x_{\bar p_1}^7 t_{\bar p_1} + c_5 y_{\bar p_1}^2 z_{\bar p_1}^2 + c_6 x_{\bar p_2}^4 \\
&+ c_7 x_{\bar p_2} y_{\bar p_1} + c_8 x_{\bar p_2}^5 z_{\bar p_1} + c_9 x_{\bar p_2}^7 t_{\bar p_1},
\end{aligned} \tag{39}$$

where $\bar p_1 = \langle 2, 5\rangle$, $\bar p_2 = \langle 3, 5\rangle$, and $c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9$ are constants. According to the chosen weights we assume that the wave variables are

$$\eta_i = k_i x + b_1 k_i^3 y + b_2 k_i^{-1} z + b_3 k_i^{-3} t, \; 1 \le i \le N, \tag{40}$$

where k_i, $1 \le i \le N$, are arbitrary constants, but b_1, b_2, and b_3 are constants to be determined.

A simple and direct computation shows that the corresponding generalized bilinear equation reads

$$\begin{aligned}
&P\left(D_{(2,5)x}, D_{(3,5)x}, D_{(2,5)y}, D_{(2,5)z}, D_{(2,5)t}\right) f\cdot f \\
&= 2c_1 f f_{xxxx} + 8c_1 f_{xxx} f_x + 6c_1 f_{xx}^2 + 2c_2 f f_{xy} \\
&+ 2c_2 f_x f_y + 20c_3 f_{xx} f_{xxxz} + 10c_3 f_{xz} f_{xxxx} \\
&+ 20c_3 f_{xxx} f_{xxz} + 70c_4 f_{xxxx} f_{xxxt} + 2c_5 f f_{yyzz} \\
&+ 4c_5 f_y f_{yzz} + 4c_5 f_z f_{zyy} + 2c_5 f_{yy} f_{zz} + 4c_5 f_{yz}^2 \\
&+ 6c_6 f_{xx}^2 + 2c_7 f f_{xy} + 2c_7 f_x f_y - 2c_8 f_{xxxxxz} f \\
&- 2c_8 f_z f_{xxxxx} - 42c_9 f_{xxt} f_{xxxxx} - 42c_9 f_{xxxxxt} f_{xx} \\
&- 70c_9 f_{xxxxt} f_{xxx} - 70c_9 f_{xxxx} f_{xxxt} \\
&= 0,
\end{aligned} \tag{41}$$

which possesses the linear subspace of N-wave solutions determined by

$$f = \sum_{i=1}^{N} \varepsilon_i f_i = \sum_{i=1}^{N} \varepsilon_i \mathrm{ch}\eta_i = \sum_{i=1}^{N} \varepsilon_i \mathrm{ch}\left(k_i x + b_1 k_i^3 y + b_2 k_i^{-1} z + b_3 k_i^{-3} t\right), \tag{42}$$

where ε_i's and k_i's are arbitrary, and b_1, b_2, and b_3 are given by

$$\begin{aligned}
b_1 &= \frac{57c_6c_9 + 17c_1c_9 - 30c_4c_6 + 10c_4c_1}{(37c_9 - 10c_4)(c_7 + c_2)}, \\
b_2 &= -\frac{3(17c_6c_9 + 20c_1c_9 - 5c_4c_6 - 5c_4c_1)}{5c_3(37c_9 - 10c_4)}, \\
b_3 &= \frac{3(-c_1 + c_6)}{7(37c_9 - 10c_4)},
\end{aligned} \tag{43}$$

when the coefficients of the polynomial P satisfy

$$15c_3c_9(-c_1+c_6) = c_8(17c_6c_9+20c_1c_9-5c_4c_6-5c_4c_1),\ c_5 = 0. \tag{44}$$

Similarly, eq. (41) has the linear subspace of N-wave solutions defined by

$$f = \sum_{i=1}^{N} \varepsilon_i f_i = \sum_{i=1}^{N} \varepsilon_i \cos \eta_i = \sum_{i=1}^{N} \varepsilon_i \cos\left(k_i x + b_1 k_i^3 y + b_2 k_i^{-1} z + b_3 k_i^{-3} t\right), \tag{45}$$

where ε_i's and k_i's are arbitrary, and b_1, b_2, and b_3 are given by

$$b_1 = -\frac{57c_6c_9 + 17c_1c_9 - 30c_4c_6 + 10c_4c_1}{(37c_9 - 10c_4)(c_7 + c_2)},$$
$$b_2 = \frac{3(17c_6c_9 + 20c_1c_9 - 5c_4c_6 - 5c_4c_1)}{5c_3(37c_9 - 10c_4)},$$
$$b_3 = \frac{3(-c_1 + c_6)}{7(37c_9 - 10c_4)}, \tag{46}$$

when the coefficients of the polynomial P satisfy

$$15c_3c_9(-c_1+c_6) = c_8(17c_6c_9+20c_1c_9-5c_4c_6-5c_4c_1),\ c_5 = 0. \tag{47}$$

4 Conclusion

The linear superposition principle does not apply to nonlinear differential equations in general. In this article, we showed that the linear superposition principle can apply to generalized bilinear equations for special kinds of wave solutions, by presenting necessary and sufficient conditions. The fact that those conditions are satisfied yields a system of nonlinear algebraic equations, and by solving these nonlinear algebraic equations, we obtained N-wave hyperbolic or trigonometric solutions to a few generalized bilinear equations. In future studies, new types of bilinear equations can be searched and linear superposition principle can be tested whether it applies to obtained bilinear equations.

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