Linear superposition principle of hyperbolic and trigonometric function solutions to generalized bilinear equations

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\textbf{Abstract}
Linear subspaces of hyperbolic and trigonometric function solutions to generalized bilinear equations are analyzed. Necessary and sufficient conditions are presented to apply the linear superposition principles. Applications of an algorithm using weights are made together with a few concrete examples.

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\section{Introduction}
There has been a great interest in finding exact solutions to nonlinear differential equations in mathematical physics. Exact solutions help us understand the mechanism that governs physical phenomena in plasma physics, optical fibers, biology, solid state physics, chemical physics, and others [1–7]. Integrability theory of nonlinear partial differential equations tells how and when exact solutions can be obtained [8–10]. Due to the nonlinearity of differential equations, it is often difficult to present exact solutions to nonlinear PDEs.

The Hirota bilinear technique is a powerful tool to investigate integrability of differential equations and it is applied to many integrable equations including integrable couplings by perturbation [11], for which \(N\)-soliton solutions are obtained [12–14]. The existence of \(N\)-soliton solutions often implies the integrability of differential equations by quadratures. Wronskian and Casoratian solutions [15–19] and quasi-periodic solutions [20–22] can also be presented systematically based on Hirota bilinear forms. In [23], linear superposition principles of hyperbolic and trigonometric functions solutions to Hirota bilinear equations were analyzed and specific classes of \(N\)-soliton solutions were constructed, following studies on linear superposition principles [24,25].

In this paper, we would like to find necessary and sufficient conditions to guarantee existence of linear subspaces of hyperbolic and trigonometric function solutions to generalized bilinear equations. Generalized bilinear equations were introduced by adopting a new way of assigning symbols for derivatives [26,27]. Based on an equality established in [27], we will present a condition, being sufficient and necessary, for the linear superposition principle of hyperbolic and trigonometric function solutions.

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The rest of this paper is arranged as follows. In Section 2, we study the linear superposition principle of hyperbolic and trigonometric function solutions, aiming to construct a specific class of \( N \)-wave solutions. In Section 3, a few illustrative examples will be computed, as applications of the linear superposition principle established in the previous section. Finally, some conclusions will be provided.

2. Linear superposition principle

Let us begin with a bilinear equation with generalized bilinear derivatives:

\[
P \left( D_{p,x_1}, D_{p,x_2}, \ldots, D_{p,x_M} \right) f \cdot f = 0, \quad \text{(2.1)}
\]

where \( P \) is a polynomial in the indicated variables and \( D_{p,x_i}, \ 1 \leq i \leq M, \) are generalized differential operators \([26,27]\) defined by

\[
\left( D^n_{p,x} \cdot g \right) (x) = \left( \partial_k + \alpha \partial_x \right)^n f(x)g(x')|_{x'=x} = \sum_{i=0}^{n} \binom{n}{i} \alpha^i \left( \partial^i_x f \right)(x) \left( \partial^i_x g \right)(x), \quad n \geq 1,
\]

in which the powers of \( \alpha \) are determined by

\[
\alpha^i = (-1)^{r(i)} \cdot \text{ where } i = r(i) \mod p \text{ with } 0 \leq r(i) < p, \ i \geq 0.
\]

Now introduce \( N \) wave variables:

\[
\eta_i = k_i \cdot x = k_{1,i}x_1 + k_{2,i}x_2 + \cdots + k_{M,i}x_M, \quad 1 \leq i \leq N,
\]

and exponential wave functions

\[
f_i = e^{\eta_i} = e^{k_{1,i}x_1+k_{2,i}x_2+\cdots+k_{M,i}x_M}, \quad 1 \leq i \leq N,
\]

where the \( k_{j,i}'s \) are real constants, and a wave related vector \( k_i \) and the dependent variable vector \( x \) are

\[
k_i = (k_{1,i}, k_{2,i}, \ldots, k_{M,i}), \quad 1 \leq i \leq N, \quad x = (x_1, x_2, \ldots, x_M).
\]

Take a linear combination

\[
f = \varepsilon_1 f_1 + \varepsilon_2 f_2 + \cdots + \varepsilon_N f_N = \sum_{i=1}^{N} \varepsilon_i f_i = \sum_{i=1}^{N} \varepsilon_i e^{\eta_i}, \quad \text{(2.7)}
\]

where \( \varepsilon_i, \ 1 \leq i \leq N, \) are arbitrary constants. It is known \([27]\) that a linear combination \( f \) of \( N \) exponential waves solves a generalized bilinear equation \((2.1)\) if and only if the following condition

\[
P \left( k_{1,i} + \alpha k_{1,j}, \ldots, k_{M,i} + \alpha k_{M,j} \right) + P \left( k_{1,j} + \alpha k_{1,i}, \ldots, k_{M,j} + \alpha k_{M,i} \right) = 0, \quad 1 \leq i, j \leq N,
\]

is satisfied.

2.1. Linear superposition principle of hyperbolic function solutions

We take \( f_i = c \eta_i = \frac{1}{2} \left( e^{\eta_i} + e^{-\eta_i} \right), \ 1 \leq i \leq N, \) be hyperbolic function solutions to \((2.1)\). Set

\[
f = \varepsilon_1 f_1 + \varepsilon_2 f_2 + \cdots + \varepsilon_N f_N = \sum_{i=1}^{N} \varepsilon_i c \eta_i = \sum_{i=1}^{N} \varepsilon_i \frac{1}{2} \left( e^{\eta_i} + e^{-\eta_i} \right),
\]

which is a general linear combination of hyperbolic function solutions. The following identity holds for exponential functions under generalized bilinear derivatives \([27]\):

\[
P \left( D_{p,x_1}, \ldots, D_{p,x_M} \right) e^{\eta_i} \cdot e^{\eta_j} = P \left( k_{1,i} + \alpha k_{1,j}, \ldots, k_{M,i} + \alpha k_{M,j} \right) e^{\eta_i+\eta_j}, \quad 1 \leq i, j \leq N.
\]

Based on \((2.10)\), we can compute that

\[
P \left( D_{p,x_1}, \ldots, D_{p,x_M} \right) f \cdot f = \sum_{i=1}^{N} \varepsilon_i f_i \cdot \sum_{j=1}^{N} \varepsilon_j f_j
\]

\[
= \sum_{i,j=1}^{N} \varepsilon_i \varepsilon_j P \left( D_{p,x_1}, \ldots, D_{p,x_M} \right) \frac{1}{2} \left( e^{\eta_i} + e^{-\eta_i} \right) \cdot \frac{1}{2} \left( e^{\eta_j} + e^{-\eta_j} \right)
\]
From (2.1), it is obvious that a linear combination function $f$ of the $N$ hyperbolic function solutions $f_i = \cosh \eta_i = \frac{1}{2} (e^{\eta_i} + e^{-\eta_i})$, $1 \leq i \leq N$, solves the generalized bilinear equation (2.1) if and only if the following conditions:

$$P (k_{1,i} + \alpha k_{1,j}, \ldots, k_{M,i} + \alpha k_{M,j}) + P (\alpha k_{1,i} + k_{1,j}, \ldots, \alpha k_{M,i} + k_{M,j}) = 0, \quad 1 \leq i \leq j \leq N,$$

$$P (k_{1,i} + \alpha k_{1,j}, \ldots, k_{M,i} + \alpha k_{M,j}) + P (\alpha k_{1,i} + k_{1,j}, \ldots, \alpha k_{M,i} + k_{M,j}) = 0, \quad 1 \leq i \leq j \leq N,$$

$$P (k_{1,i} - \alpha k_{1,j}, \ldots, k_{M,i} - \alpha k_{M,j}) + P (\alpha k_{1,i} - k_{1,j}, \ldots, \alpha k_{M,i} - k_{M,j}) = 0, \quad 1 \leq i \leq j \leq N,$$

$$P (k_{1,i} - \alpha k_{1,j}, \ldots, k_{M,i} - \alpha k_{M,j}) + P (\alpha k_{1,i} - k_{1,j}, \ldots, \alpha k_{M,i} - k_{M,j}) = 0, \quad 1 \leq i \leq j \leq N,$$

are satisfied.

With the aid of this result, we can obtain the following theorem.

**Theorem 1.** Let $P(x_1, x_2, \ldots, x_M)$ be a polynomial and the $N$ wave variable be defined by $\eta_i = k_i \cdot x = k_{1,i} x_1 + k_{2,i} x_2 + \cdots + k_{M,i} x_M$, $1 \leq i \leq N$, where the $k_{i,j}$’s are real constants. Then any linear combination of the hyperbolic function solutions $f_i = \cosh \eta_i = \frac{1}{2} (e^{\eta_i} + e^{-\eta_i})$, $1 \leq i \leq N$, solves the generalized bilinear equation (2.1) if and only if the system (2.12) is satisfied.

This theorem tells when a linear superposition of hyperbolic function solutions is still a solution of a given generalized bilinear equation. Furthermore, it also introduces a way to construct $N$-wave solutions to generalized bilinear equations. The system (2.12) is a key condition that a solution needs to satisfy. If we are able to solve the system (2.12), then we can present an $N$-wave solution, formed by (2.9), to a generalized bilinear equation.

### 2.2. Linear superposition principle of trigonometric function solutions

We take $f_i = \cos \eta_i = \frac{1}{2} (e^{\eta_i} + e^{-\eta_i})$, $1 \leq i \leq N$, where $\eta_i = k_i \cdot x$, $1 \leq i \leq N$, $I = \sqrt{-1}$, be trigonometric function solutions to (2.1). Set

$$f = \varepsilon_1 f_1 + \varepsilon_2 f_2 + \cdots + \varepsilon_N f_N = \sum_{i=1}^{N} \varepsilon_i \cos \eta_i = \sum_{i=1}^{N} \varepsilon_i \frac{1}{2} (e^{\eta_i} + e^{-\eta_i}),$$

(2.13)
which is a general linear combination of trigonometric function solutions. Similarly, a linear combination function $f$ of the $N$ trigonometric function solutions in $(2.13)$ solves the generalized bilinear equation $(2.1)$ if and only if the following conditions:

\[
P(ik_{1,j} + \alpha ik_{1,j}, \ldots, ik_{M,j} + \alpha ik_{M,j}) + P(\alpha ik_{1,j} + ik_{1,j}, \ldots, \alpha ik_{M,j} + ik_{M,j}) = 0, \quad 1 \leq i \leq j \leq N,
\]

\[
P(ik_{1,j} - \alpha ik_{1,j}, \ldots, ik_{M,j} - \alpha ik_{M,j}) + P(\alpha ik_{1,j} - ik_{1,j}, \ldots, \alpha ik_{M,j} - ik_{M,j}) = 0, \quad 1 \leq i \leq j \leq N,
\]

\[
P(-ik_{1,i} + \alpha ik_{1,i}, \ldots, -ik_{M,i} + \alpha ik_{M,i}) + P(-\alpha ik_{1,i} + ik_{1,i}, \ldots, -\alpha ik_{M,i} + ik_{M,i}) = 0, \quad 1 \leq i < j \leq N,
\]

\[
P(-ik_{1,i} - \alpha ik_{1,i}, \ldots, -ik_{M,i} - \alpha ik_{M,i}) + P(-\alpha ik_{1,i} - ik_{1,i}, \ldots, -\alpha ik_{M,i} - ik_{M,i}) = 0, \quad 1 \leq i \leq j \leq N,
\]

(2.14)

are satisfied. Thus we have the following theorem.

**Theorem 2.** Let $P(x_1, x_2, \ldots, x_M)$ be a polynomial and the $N$ wave variable defined by $\eta_i = k_i \cdot x = k_1, x_1 + k_2, x_2 + \cdots + k_M, x_M$, $1 \leq i \leq N$, where the $k_i$'s are real constants. Then any linear combination of the trigonometric function solutions $f_i = \cos \eta_i, \quad 1 \leq i \leq N$, solves the generalized bilinear equation $(2.1)$ if and only if the system $(2.14)$ is satisfied.

This theorem tells when a linear superposition of trigonometric function solutions is still a solution of a given generalized bilinear equation. Furthermore, it also introduces a way to construct $N$-wave solutions to generalized bilinear equations. The system $(2.14)$ is a key condition that a solution needs to satisfy. If we are able to solve the system $(2.14)$, then we can present an $N$-wave solution, formed by $(2.13)$, to a generalized bilinear equation.

3. Applications

**Example 1.** Let us introduce the weights of independent variables:

\[
(w(x), w(y), w(z), w(t)) = (1, 2, 3, 4),
\]

(3.1)

and consider a polynomial being homogeneous in weight 7:

\[
P = c_1x^3 + c_2x^2y + c_3x^2z + c_4xy^2 + c_5xyz + c_6x^2yz,
\]

(3.2)

where $c_1, c_2, c_3, c_4, c_5, c_6$ are constants. Assume that the wave variables are

\[
\eta_i = k_i x + b_1 k_i^2 y + b_2 k_i^3 z + b_3 k_i^4 t, \quad 1 \leq i \leq N,
\]

(3.3)

where $k_i, \quad 1 \leq i \leq N$, are arbitrary constants, but $b_1, b_2$ and $b_3$ are constants to be determined.

A simple direct computation shows that the corresponding generalized bilinear equation reads

\[
P(D_{3,x}, D_{3,y}, D_{3,z}, D_{3,t}) f \cdot f = 6c_1 f_{yx} f_{xy} + 2c_2 f_{xyz} + 6c_3 f_{xx} f_{xt} + 2c_4 f_{yy} f_{yt} + 2c_5 f_{xy} f_{xt} + 4c_6 f_{xyz} f_{xz} = 0
\]

(3.4)

which has the linear subspaces of $N$-wave solutions defined by

\[
f = \sum_{i=1}^{N} \varepsilon_i f_i = \sum_{i=1}^{N} \varepsilon_i c_i \eta_i = \sum_{i=1}^{N} \varepsilon_i c_i \left( k_i x + b_1 k_i^2 y + b_2 k_i^3 z + b_3 k_i^4 t \right),
\]

(3.5)

or

\[
f = \sum_{i=1}^{N} \varepsilon_i f_i = \sum_{i=1}^{N} \varepsilon_i \cos \eta_i = \sum_{i=1}^{N} \varepsilon_i \cos \left( k_i x + b_1 k_i^2 y + b_2 k_i^3 z + b_3 k_i^4 t \right),
\]

(3.6)

where $b_1, \varepsilon_i$'s and the $k_i$'s are arbitrary, and $b_2$ and $b_3$ satisfy

\[
b_2 = -\frac{3b_1^2 c_1}{2c_6}, \quad b_3 = \frac{b_1^3 c_1}{2c_3},
\]

(3.7)

when the coefficients of the polynomial $P$ satisfy

\[
9c_1 c_2 c_3 = 2c_6 \left( 3c_4 c_3 - c_5 c_6 \right).
\]

(3.8)

**Example 2.** Let us introduce the weights of independent variables:

\[
(w(x), w(y), w(z), w(t)) = (1, 3, 5, 7),
\]

(3.9)
and consider a polynomial being homogeneous in weight 8:

\[ P = c_1 x^8 + c_2 x^5 y + c_3 x^3 z + c_4 y z + c_5 x t, \]  
\[ \text{(3.10)} \]

where \( c_1, c_2, c_3, c_4, c_5 \) are constants. Assume that the wave variables are

\[ \eta_i = k_i x + b_1 k_i^3 y + b_2 k_i^5 z + b_3 k_i^3 t, \quad 1 \leq i \leq N, \]  
\[ \text{(3.11)} \]

where \( k_i, 1 \leq i \leq N, \) are arbitrary constants, but \( b_1, b_2 \) and \( b_3 \) are constants to be determined.

A simple direct computation shows that the corresponding generalized bilinear equation reads

\[ P(D_{5,x}, D_{5,y}, D_{5,z}, D_{5,t}) f \cdot f = 70c_2 f_{xxx}^2 + 20c_2 f_{xxxy} + 10c_2 f_{xxy} - 20c_2 f_{xyxy} - 2c_2 f_{xxx} - 6c_2 f_{xxx} - 6c_2 f_{xxxy} + 2c_4 f_{xz} - 2c_4 f_{xxy} + 2c_4 f_{y} - 2c_5 f_t = 0 \]
\[ \text{(3.12)} \]

which possesses the linear subspace of \( N \)-wave solutions determined by

\[ f = \sum_{i=1}^{N} \epsilon_i f_i = \sum_{i=1}^{N} \epsilon_i c h \eta_i = \sum_{i=1}^{N} \epsilon_i c h (k_i x + b_1 k_i^3 y + b_2 k_i^3 z + b_3 k_i^3 t) , \]  
\[ \text{(3.13)} \]

where the \( \epsilon_i \)'s and the \( k_i \)'s are arbitrary, and \( b_1, b_2 \) and \( b_3 \) satisfy

\[ b_1 = -\frac{7c_1}{c_2}, \quad b_2 = \frac{70c_1}{3c_3}, \quad b_3 = -\frac{70c_1}{c_5} \]  
\[ \text{(3.14)} \]

when the coefficients of the polynomial \( P \) satisfy

\[ 7c_1 c_4 = -2c_3 c_2. \]  
\[ \text{(3.15)} \]

Similarly, \( (3.12) \) has the linear subspace of \( N \)-wave solutions defined by

\[ f = \sum_{i=1}^{N} \epsilon_i f_i = \sum_{i=1}^{N} \epsilon_i \cos \eta_i = \sum_{i=1}^{N} \epsilon_i \cos (k_i x + b_1 k_i^3 y + b_2 k_i^3 z + b_3 k_i^3 t) , \]  
\[ \text{(3.16)} \]

where the \( \epsilon_i \)'s and the \( k_i \)'s are arbitrary, and \( b_1, b_2 \) and \( b_3 \) satisfy

\[ b_1 = \frac{7c_1}{c_2}, \quad b_2 = \frac{70c_1}{3c_3}, \quad b_3 = \frac{70c_1}{c_5} \]  
\[ \text{(3.17)} \]

when the coefficients of the polynomial \( P \) satisfy

\[ 7c_1 c_4 = -2c_3 c_2. \]  
\[ \text{(3.18)} \]

**Example 3.** Let us introduce the weights of independent variables:

\[ (w(x), w(y), w(z), w(t)) = (1, 3, -1, -3), \]  
\[ \text{(3.19)} \]

and consider a polynomial being homogeneous in weight 4:

\[ P = c_1 x^4 + c_2 x^2 y + c_3 x^2 z + c_4 x^2 t, \]  
\[ \text{(3.20)} \]

where \( c_1, c_2, c_3, c_4 \) are constants. Assume that the wave variables are

\[ \eta_i = k_i x + b_1 k_i^3 y + b_2 k_i^{-1} z + b_3 k_i^{-3} t, \quad 1 \leq i \leq N, \]  
\[ \text{(3.21)} \]

where \( k_i, 1 \leq i \leq N, \) are arbitrary constants, but \( b_1, b_2 \) and \( b_3 \) are constants to be determined.

A simple direct computation shows that the corresponding generalized bilinear equation reads

\[ P(D_{5,x}, D_{5,y}, D_{5,z}, D_{5,t}) f \cdot f = 2c_2 f_{xxx}^2 + 8c_2 f_{xxy} + 6c_2 f_{xx}^2 + 2c_2 f_{xy} 
- 2c_2 f_{x} f_t + 20c_2 f_{xx} f_{xx} + 10c_2 f_{x} f_{xx} 
- 20c_2 f_{xxx} f_{xx} + 70c_2 f_{xx} f_{x} = 0 \]
\[ \text{(3.22)} \]
which possesses the linear subspace of $N$-wave solutions determined by
\[
f = \sum_{i=1}^{N} \epsilon_i \phi_i = \sum_{i=1}^{N} \epsilon_i \cosh \eta_i = \sum_{i=1}^{N} \epsilon_i \left( k_i x + b_1 k_i^3 y + b_2 k_i^{-1} z + b_3 k_i^{-3} t \right),
\]
where the $\epsilon_i$'s and the $k_i$'s are arbitrary, and $b_1$, $b_2$ and $b_3$ satisfy
\[
b_1 = -\frac{c_1}{c_2}, \quad b_2 = -\frac{3c_1}{10c_3}, \quad b_3 = \frac{3c_1}{70c_4}.
\]
Similarly, (3.22) has the linear subspace of $N$-wave solutions defined by
\[
f = \sum_{i=1}^{N} \epsilon_i \phi_i = \sum_{i=1}^{N} \epsilon_i \cos \eta_i = \sum_{i=1}^{N} \epsilon_i \cos \left( k_i x + b_1 k_i^3 y + b_2 k_i^{-1} z + b_3 k_i^{-3} t \right),
\]
where the $\epsilon_i$'s and the $k_i$'s are arbitrary, and $b_1$, $b_2$ and $b_3$ satisfy
\[
b_1 = \frac{c_1}{c_2}, \quad b_2 = \frac{3c_1}{10c_3}, \quad b_3 = \frac{3c_1}{70c_4}.
\]

4. Conclusion

For generalized bilinear equations, we analyzed linear combinations of hyperbolic or trigonometric function solutions and we presented linear superposition principle of hyperbolic and trigonometric function solutions to generalized bilinear equations. A few illustrative examples were presented, by applying an algorithm using weights.

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