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Linear superposition principle of hyperbolic and trigonometric function solutions to generalized bilinear equations

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ABSTRACT

Linear subspaces of hyperbolic and trigonometric function solutions to generalized bilinear equations are analyzed. Necessary and sufficient conditions are presented to apply the linear superposition principles. Applications of an algorithm using weights are made together with a few concrete examples.

1. Introduction

There has been a great interest in finding exact solutions to nonlinear differential equations in mathematical physics. Exact solutions help us understand the mechanism that governs physical phenomena in plasma physics, optical fibers, biology, solid state physics, chemical physics, and others [1–7]. Integrability theory of nonlinear partial differential equations tells how and when exact solutions can be obtained [8–10]. Due to the nonlinearity of differential equations, it is often difficult to present exact solutions to nonlinear PDEs.

The Hirota bilinear technique is a powerful tool to investigate integrability of differential equations and it is applied to many integrable equations including integrable couplings by perturbation [11], for which N-soliton solutions are obtained [12–14]. The existence of N-soliton solutions often implies the integrability of differential equations by quadratures. Wronskian and Casoratian solutions [15–19] and quasi-periodic solutions [20–22] can also be presented systematically based on Hirota bilinear forms. In [23], linear superposition principles of hyperbolic and trigonometric functions solutions to Hirota bilinear equations were analyzed and specific classes of N-soliton solutions were constructed, following studies on linear superposition principles [24,25].

In this paper, we would like to find necessary and sufficient conditions to guarantee existence of linear subspaces of hyperbolic and trigonometric function solutions to generalized bilinear equations. Generalized bilinear equations were introduced by adopting a new way of assigning symbols for derivatives [26,27]. Based on an equality established in [27], we will present a condition, being sufficient and necessary, for the linear superposition principle of hyperbolic and trigonometric function solutions.

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The rest of this paper is arranged as follows. In Section 2, we study the linear superposition principle of hyperbolic and trigonometric function solutions, aiming to construct a specific class of N-wave solutions. In Section 3, a few illustrative examples will be computed, as applications of the linear superposition principle established in the previous section. Finally, some conclusions will be provided.

2. Linear superposition principle

Let us begin with a bilinear equation with generalized bilinear derivatives:

$$P\left(D_{p,x_1}, D_{p,x_2}, \ldots, D_{p,x_M}\right) f\cdot f = 0, \tag{2.1}$$

where P is a polynomial in the indicated variables and $D_{p,x_i},\ 1 \le i \le M$, are generalized differential operators [26,27] defined by

$$\left(D_{p,x}^n f\cdot g\right)(x) = (\partial_x + \alpha\partial_{x'})^n f(x)g(x')\big|_{x'=x} = \sum_{i=0}^{n}\binom{n}{i}\alpha^i\left(\partial_x^{n-i}f\right)(x)\left(\partial_x^i g\right)(x), \quad n\ge 1, \tag{2.2}$$

in which the powers of α are determined by

$$\alpha^i = (-1)^{r(i)}, \quad \text{where } i = r(i) \bmod p \text{ with } 0\le r(i) < p,\ i\ge 0. \tag{2.3}$$

Now introduce N wave variables:

$$\eta_i = \mathbf{k}_i\cdot\mathbf{x} = k_{1,i}x_1 + k_{2,i}x_2 + \cdots + k_{M,i}x_M, \quad 1\le i\le N, \tag{2.4}$$

and exponential wave functions

$$f_i = e^{\eta_i} = e^{k_{1,i}x_1+k_{2,i}x_2+\cdots+k_{M,i}x_M}, \quad 1\le i\le N, \tag{2.5}$$

where the $k_{j,i}$'s are real constants, and a wave related vector $\mathbf{k}_i$ and the dependent variable vector $\mathbf{x}$ are

$$\mathbf{k}_i = \left(k_{1,i}, k_{2,i}, \ldots, k_{M,i}\right), \quad 1\le i\le N, \qquad \mathbf{x} = (x_1, x_2, \ldots, x_M). \tag{2.6}$$

Take a linear combination

$$f = \varepsilon_1 f_1 + \varepsilon_2 f_2 + \cdots + \varepsilon_N f_N = \sum_{i=1}^{N}\varepsilon_i f_i = \sum_{i=1}^{N}\varepsilon_i e^{\eta_i}, \tag{2.7}$$

where $\varepsilon_i,\ 1\le i\le N$, are arbitrary constants. It is known [27] that a linear combination f of N exponential wave solves a generalized bilinear equation (2.1) if and only if the following condition

$$P\left(k_{1,i}+\alpha k_{1,j}, \ldots, k_{M,i}+\alpha k_{M,j}\right) + P\left(k_{1,j}+\alpha k_{1,i}, \ldots, k_{M,j}+\alpha k_{M,i}\right) = 0, \quad 1\le i\le j\le N, \tag{2.8}$$

is satisfied.

2.1. Linear superposition principle of hyperbolic function solutions

We take $f_i = ch\eta_i = \frac{1}{2}\left(e^{\eta_i}+e^{-\eta_i}\right),\ 1\le i\le N$, be hyperbolic function solutions to (2.1). Set

$$f = \varepsilon_1 f_1 + \varepsilon_2 f_2 + \cdots + \varepsilon_N f_N = \sum_{i=1}^{N}\varepsilon_i ch\eta_i = \sum_{i=1}^{N}\varepsilon_i\frac{1}{2}\left(e^{\eta_i}+e^{-\eta_i}\right), \tag{2.9}$$

which is a general linear combination of hyperbolic function solutions. The following identity holds for exponential functions under generalized bilinear derivatives [27]:

$$P\left(D_{p,x_1}, \ldots, D_{p,x_l}\right)e^{\eta_i}\cdot e^{\eta_j} = P\left(k_{1,i}+\alpha k_{1,j}, \ldots, k_{M,i}+\alpha k_{M,j}\right)e^{\eta_i+\eta_j}, \quad 1\le i,\ j\le N. \tag{2.10}$$

Based on (2.10), we can compute that

$$\begin{aligned}
&P\left(D_{p,x_1}, \ldots, D_{p,x_M}\right) f\cdot f\\
&= P\left(D_{p,x_1}, \ldots, D_{p,x_M}\right)\sum_{i=1}^{N}\varepsilon_i ch\eta_i\cdot\sum_{j=1}^{N}\varepsilon_j ch\eta_j\\
&= \sum_{i,j=1}^{N}\varepsilon_i\varepsilon_j P\left(D_{p,x_1}, \ldots, D_{p,x_M}\right)\frac{1}{2}\left(e^{\eta_i}+e^{-\eta_i}\right)\cdot\frac{1}{2}\left(e^{\eta_j}+e^{-\eta_j}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\sum_{i,j=1}^{N}\varepsilon_i\varepsilon_j P\left(D_{p,x_1},\ldots,D_{p,x_M}\right)\left(e^{\eta_i}\cdot e^{\eta_j}+e^{\eta_i}\cdot e^{-\eta_j}+e^{-\eta_i}\cdot e^{\eta_j}+e^{-\eta_i}\cdot e^{-\eta_j}\right)\\
&= \frac{1}{4}\sum_{1\le i<j\le N}^{N}\varepsilon_i\varepsilon_j\Big[P\left(k_{1,i}+\alpha k_{1,j},\ldots,k_{M,i}+\alpha k_{M,j}\right)e^{\eta_i+\eta_j}+P\left(k_{1,i}-\alpha k_{1,j},\ldots,k_{M,i}-\alpha k_{M,j}\right)e^{\eta_i-\eta_j}\\
&\quad +P\left(-k_{1,i}+\alpha k_{1,j},\ldots,-k_{M,i}+\alpha k_{M,j}\right)e^{-\eta_i+\eta_j}+P\left(-k_{1,i}-\alpha k_{1,j},\ldots,-k_{M,i}-\alpha k_{M,j}\right)e^{-\eta_i-\eta_j}\Big]\\
&\quad +\frac{1}{4}\sum_{1\le i>j\le N}^{N}\varepsilon_i\varepsilon_j\Big[P\left(k_{1,i}+\alpha k_{1,j},\ldots,k_{M,i}+\alpha k_{M,j}\right)e^{\eta_i+\eta_j}+P\left(k_{1,i}-\alpha k_{1,j},\ldots,k_{M,i}-\alpha k_{M,j}\right)e^{\eta_i-\eta_j}\\
&\quad +P\left(-k_{1,i}+\alpha k_{1,j},\ldots,-k_{M,i}+\alpha k_{M,j}\right)e^{-\eta_i+\eta_j}+P\left(-k_{1,i}-\alpha k_{1,j},\ldots,-k_{M,i}-\alpha k_{M,j}\right)e^{-\eta_i-\eta_j}\Big]\\
&\quad +\frac{1}{4}\sum_{1\le i=j\le N}^{N}\varepsilon_i\varepsilon_j\Big[P\left(k_{1,i}+\alpha k_{1,j},\ldots,k_{M,i}+\alpha k_{M,j}\right)e^{\eta_i+\eta_j}+P\left(k_{1,i}-\alpha k_{1,j},\ldots,k_{M,i}-\alpha k_{M,j}\right)e^{\eta_i-\eta_j}\\
&\quad +P\left(-k_{1,i}+\alpha k_{1,j},\ldots,-k_{M,i}+\alpha k_{M,j}\right)e^{-\eta_i+\eta_j}+P\left(-k_{1,i}-\alpha k_{1,j},\ldots,-k_{M,i}-\alpha k_{M,j}\right)e^{-\eta_i-\eta_j}\Big]\\
&= \frac{1}{4}\sum_{1\le i<j\le N}^{N}\varepsilon_i\varepsilon_j\Big[\Big(P\left(k_{1,i}+\alpha k_{1,j},\ldots,k_{M,i}+\alpha k_{M,j}\right)+P\left(\alpha k_{1,i}+k_{1,j},\ldots,\alpha k_{M,i}+k_{M,j}\right)\Big)e^{\eta_i+\eta_j}\\
&\quad +\Big(P\left(k_{1,i}-\alpha k_{1,j},\ldots,k_{M,i}-\alpha k_{M,j}\right)+P\left(\alpha k_{1,i}-k_{1,j},\ldots,\alpha k_{M,i}-k_{M,j}\right)\Big)e^{\eta_i-\eta_j}\\
&\quad +\Big(P\left(-k_{1,i}+\alpha k_{1,j},\ldots,-k_{M,i}+\alpha k_{M,j}\right)+P\left(-\alpha k_{1,i}+k_{1,j},\ldots,-\alpha k_{M,i}+k_{M,j}\right)\Big)e^{-\eta_i+\eta_j}\\
&\quad +\Big(P\left(-k_{1,i}-\alpha k_{1,j},\ldots,-k_{M,i}-\alpha k_{M,j}\right)+P\left(-\alpha k_{1,i}-k_{1,j},\ldots,-\alpha k_{M,i}-k_{M,j}\right)\Big)e^{-\eta_i-\eta_j}\Big]\\
&\quad +\frac{1}{4}\sum_{1\le i\le N}^{N}\varepsilon_i^2\Big[P\left(k_{1,i}+\alpha k_{1,i},\ldots,k_{M,i}+\alpha k_{M,i}\right)e^{2\eta_i}+P\left(k_{1,i}-\alpha k_{1,i},\ldots,k_{M,i}-\alpha k_{M,i}\right)\\
&\quad +P\left(-k_{1,i}+\alpha k_{1,i},\ldots,-k_{M,i}+\alpha k_{M,i}\right)+P\left(-k_{1,i}-\alpha k_{1,i},\ldots,-k_{M,i}-\alpha k_{M,i}\right)e^{-2\eta_i}\Big].
\end{aligned}
\tag{2.11}
$$

From (2.11), it is obvious that a linear combination function f of the N hyperbolic function solutions $f_i = ch\eta_i = \frac{1}{2}\left(e^{\eta_i}+e^{-\eta_i}\right),\ 1\le i\le N$, solves the generalized bilinear equation (2.1) if and only if the following conditions:

$$
\begin{aligned}
P\left(k_{1,i}+\alpha k_{1,j},\ldots,k_{M,i}+\alpha k_{M,j}\right)+P\left(\alpha k_{1,i}+k_{1,j},\ldots,\alpha k_{M,i}+k_{M,j}\right)&=0,\quad 1\le i\le j\le N,\\
P\left(k_{1,i}-\alpha k_{1,j},\ldots,k_{M,i}-\alpha k_{M,j}\right)+P\left(\alpha k_{1,i}-k_{1,j},\ldots,\alpha k_{M,i}-k_{M,j}\right)&=0,\quad 1\le i\le j\le N,\\
P\left(-k_{1,i}+\alpha k_{1,j},\ldots,-k_{M,i}+\alpha k_{M,j}\right)+P\left(-\alpha k_{1,i}+k_{1,j},\ldots,-\alpha k_{M,i}+k_{M,j}\right)&=0,\quad 1\le i<j\le N,\\
P\left(-k_{1,i}-\alpha k_{1,j},\ldots,-k_{M,i}-\alpha k_{M,j}\right)+P\left(-\alpha k_{1,i}-k_{1,j},\ldots,-\alpha k_{M,i}-k_{M,j}\right)&=0,\quad 1\le i\le j\le N,
\end{aligned}
\tag{2.12}
$$

are satisfied.

With the aid of this result, we can obtain the following theorem.

Theorem 1. *Let $P(x_1,x_2,\ldots,x_M)$ be a polynomial and the N wave variable be defined by $\eta_i = \mathbf{k}_i\cdot\mathbf{x} = k_{1,i}x_1+k_{2,i}x_2+\cdots+k_{M,i}x_M,\ 1\le i\le N$, where the $k_{j,i}$'s are real constants. Then any linear combination of the hyperbolic function solutions $f_i = ch\eta_i = \frac{1}{2}\left(e^{\eta_i}+e^{-\eta_i}\right),\ 1\le i\le N$, solves the generalized bilinear equation (2.1) if and only if the system (2.12) is satisfied.*

This theorem tells when a linear superposition of hyperbolic function solutions is still a solution of a given generalized bilinear equation. Furthermore, it also introduces a way to construct N-wave solutions to generalized bilinear equations. The system (2.12) is a key condition that a solution needs to satisfy. If we are able to solve the system (2.12), then we can present an N-wave solution, formed by (2.9), to a generalized bilinear equation.

2.2. Linear superposition principle of trigonometric function solutions

We take $f_i = \cos\eta_i = \frac{1}{2}\left(e^{I\eta_i}+e^{-I\eta_i}\right),\ 1\le i\le N$, where $\eta_i=\mathbf{k}_i\cdot\mathbf{x},\ 1\le i\le N,\ I=\sqrt{-1}$, be trigonometric function solutions to (2.1). Set

$$
f=\varepsilon_1 f_1+\varepsilon_2 f_2+\cdots+\varepsilon_N f_N=\sum_{i=1}^{N}\varepsilon_i\cos\eta_i=\sum_{i=1}^{N}\varepsilon_i\frac{1}{2}\left(e^{I\eta_i}+e^{-I\eta_i}\right),
\tag{2.13}
$$

which is a general linear combination of trigonometric function solutions. Similarly, a linear combination function f of the N trigonometric function solutions in (2.13) solves the generalized bilinear equation (2.1) if and only if the following conditions:

$$\begin{aligned}
P\left(Ik_{1,i}+\alpha Ik_{1,j},\ldots,Ik_{M,i}+\alpha Ik_{M,j}\right)+P\left(\alpha Ik_{1,i}+Ik_{1,j},\ldots,\alpha Ik_{M,i}+Ik_{M,j}\right)&=0, \quad 1\le i\le j\le N,\\
P\left(Ik_{1,i}-\alpha Ik_{1,j},\ldots,Ik_{M,i}-\alpha Ik_{M,j}\right)+P\left(\alpha Ik_{1,i}-Ik_{1,j},\ldots,\alpha Ik_{M,i}-Ik_{M,j}\right)&=0, \quad 1\le i\le j\le N,\\
P\left(-Ik_{1,i}+\alpha Ik_{1,j},\ldots,-Ik_{M,i}+\alpha Ik_{M,j}\right)+P\left(-\alpha Ik_{1,i}+Ik_{1,j},\ldots,-\alpha Ik_{M,i}+Ik_{M,j}\right)&=0, \quad 1\le i< j\le N,\\
P\left(-Ik_{1,i}-\alpha Ik_{1,j},\ldots,-Ik_{M,i}-\alpha Ik_{M,j}\right)+P\left(-\alpha Ik_{1,i}-Ik_{1,j},\ldots,-\alpha Ik_{M,i}-Ik_{M,j}\right)&=0, \quad 1\le i\le j\le N,
\end{aligned} \tag{2.14}$$

are satisfied. Thus we have the following theorem.

Theorem 2. *Let $P(x_1,x_2,\ldots,x_M)$ be a polynomial and the N wave variable defined by $\eta_i=\mathbf{k}_i\cdot\mathbf{x}=k_{1,i}x_1+k_{2,i}x_2+\cdots+k_{M,i}x_M,\ 1\le i\le N$, where the $k_{j,i}$'s are real constants. Then any linear combination of the trigonometric function solutions $f_i=\cos\eta_i,\ 1\le i\le N$, solves the generalized bilinear equation (2.1) if and only if the system (2.14) is satisfied.*

This theorem tells when a linear superposition of trigonometric function solutions is still a solution of a given generalized bilinear equation. Furthermore, it also introduces a way to construct N-wave solutions to generalized bilinear equations. The system (2.14) is a key condition that a solution needs to satisfy. If we are able to solve the system (2.14), then we can present an N-wave solution, formed by (2.13), to a generalized bilinear equation.

3. Applications

Example 1. Let us introduce the weights of independent variables:

$$(w(x),w(y),w(z),w(t))=(1,2,3,4), \tag{3.1}$$

and consider a polynomial being homogeneous in weight 7:

$$P=c_1xy^3+c_2xz^2+c_3x^3t+c_4y^2z+c_5xyt+c_6x^2yz, \tag{3.2}$$

where c_1,c_2,c_3,c_4,c_5,c_6 are constants. Assume that the wave variables are

$$\eta_i=k_ix+b_1k_i^2y+b_2k_i^3z+b_3k_i^4t, \quad 1\le i\le N, \tag{3.3}$$

where $k_i,\ 1\le i\le N$, are arbitrary constants, but b_1, b_2 and b_3 are constants to be determined.

A simple direct computation shows that the corresponding generalized bilinear equation reads

$$\begin{aligned}
P\left(D_{3,x},D_{3,y},D_{3,z},D_{3,t}\right)f\cdot f&=6c_1f_{yy}f_{yx}+2c_2f_{xzz}f+6c_3f_{xx}f_{xt}+2c_4f_{yyz}f\\
&\quad+2c_5f_{xyt}f+2c_6f_{xx}f_{yz}+4c_6f_{xy}f_{xz}\\
&=0
\end{aligned} \tag{3.4}$$

which has the linear subspaces of N-wave solutions defined by

$$f=\sum_{i=1}^N\varepsilon_if_i=\sum_{i=1}^N\varepsilon_i ch\eta_i=\sum_{i=1}^N\varepsilon_i ch\left(k_ix+b_1k_i^2y+b_2k_i^3z+b_3k_i^4t\right), \tag{3.5}$$

or

$$f=\sum_{i=1}^N\varepsilon_if_i=\sum_{i=1}^N\varepsilon_i\cos\eta_i=\sum_{i=1}^N\varepsilon_i\cos\left(k_ix+b_1k_i^2y+b_2k_i^3z+b_3k_i^4t\right), \tag{3.6}$$

where b_1, the ε_i's and the k_i's are arbitrary, and b_2 and b_3 satisfy

$$b_2=-\frac{3b_1^2c_1}{2c_6}, \qquad b_3=\frac{b_1^3c_1}{2c_3}, \tag{3.7}$$

when the coefficients of the polynomial P satisfy

$$9c_1c_2c_3=2c_6\left(3c_4c_3-c_5c_6\right). \tag{3.8}$$

Example 2. Let us introduce the weights of independent variables:

$$(w(x),w(y),w(z),w(t))=(1,3,5,7), \tag{3.9}$$

and consider a polynomial being homogeneous in weight 8:

$$P = c_1x^8 + c_2x^5y + c_3x^3z + c_4yz + c_5xt, \tag{3.10}$$

where c_1, c_2, c_3, c_4, c_5 are constants. Assume that the wave variables are

$$\eta_i = k_ix + b_1k_i^3y + b_2k_i^5z + b_3k_i^7t, \quad 1 \le i \le N, \tag{3.11}$$

where $k_i,\ 1 \le i \le N$, are arbitrary constants, but b_1, b_2 and b_3 are constants to be determined.

A simple direct computation shows that the corresponding generalized bilinear equation reads

$$\begin{aligned} P\left(D_{5,x}, D_{5,y}, D_{5,z}, D_{5,t}\right) f\cdot f &= 70c_1f_{xxxx}^2 + 20c_2f_{xx}f_{xxxy} + 10c_2f_{xy}f_{xxxx} - 20c_2f_{xxx}f_{xxy} \\ &\quad + 2c_3ff_{xxxz} - 2c_3f_zf_{xxx} - 6c_3f_xf_{xxz} + 6c_3f_{xx}f_{xz} \\ &\quad + 2c_4ff_{yz} - 2c_4f_yf_z + 2c_5ff_{xt} - 2c_5f_xf_t \\ &= 0 \end{aligned} \tag{3.12}$$

which possesses the linear subspace of N-wave solutions determined by

$$f = \sum_{i=1}^{N} \varepsilon_i f_i = \sum_{i=1}^{N} \varepsilon_i ch\eta_i = \sum_{i=1}^{N} \varepsilon_i ch\left(k_ix + b_1k_i^3y + b_2k_i^5z + b_3k_i^7t\right), \tag{3.13}$$

where the ε_i's and the k_i's are arbitrary, and b_1, b_2 and b_3 satisfy

$$b_1 = -\frac{7c_1}{c_2}, \qquad b_2 = \frac{70c_1}{3c_3}, \qquad b_3 = -\frac{70c_1}{c_5} \tag{3.14}$$

when the coefficients of the polynomial P satisfy

$$7c_1c_4 = -2c_3c_2. \tag{3.15}$$

Similarly, (3.12) has the linear subspace of N-wave solutions defined by

$$f = \sum_{i=1}^{N} \varepsilon_i f_i = \sum_{i=1}^{N} \varepsilon_i \cos\eta_i = \sum_{i=1}^{N} \varepsilon_i \cos\left(k_ix + b_1k_i^3y + b_2k_i^5z + b_3k_i^7t\right), \tag{3.16}$$

where the ε_i's and the k_i's are arbitrary, and b_1, b_2 and b_3 satisfy

$$b_1 = \frac{7c_1}{c_2}, \qquad b_2 = \frac{70c_1}{3c_3}, \qquad b_3 = \frac{70c_1}{c_5} \tag{3.17}$$

when the coefficients of the polynomial P satisfy

$$7c_1c_4 = -2c_3c_2. \tag{3.18}$$

Example 3. Let us introduce the weights of independent variables:

$$(w(x), w(y), w(z), w(t)) = (1, 3, -1, -3), \tag{3.19}$$

and consider a polynomial being homogeneous in weight 4:

$$P = c_1x^4 + c_2xy + c_3x^5z + c_4x^7t, \tag{3.20}$$

where c_1, c_2, c_3, c_4 are constants. Assume that the wave variables are

$$\eta_i = k_ix + b_1k_i^3y + b_2k_i^{-1}z + b_3k_i^{-3}t, \quad 1 \le i \le N, \tag{3.21}$$

where $k_i,\ 1 \le i \le N$, are arbitrary constants, but b_1, b_2 and b_3 are constants to be determined.

A simple direct computation shows that the corresponding generalized bilinear equation reads

$$\begin{aligned} P\left(D_{5,x}, D_{5,y}, D_{5,z}, D_{5,t}\right) f\cdot f &= 2c_1ff_{xxxx} - 8c_1f_{xxx}f_x + 6c_1f_{xx}^2 + 2c_2ff_{xy} \\ &\quad - 2c_2f_xf_y + 20c_3f_{xx}f_{xxxz} + 10c_3f_{xz}f_{xxxx} \\ &\quad - 20c_3f_{xxx}f_{xxz} + 70c_4f_{xxxx}f_{xxxt} \\ &= 0 \end{aligned} \tag{3.22}$$

which possesses the linear subspace of N-wave solutions determined by

$$f = \sum_{i=1}^{N} \varepsilon_i f_i = \sum_{i=1}^{N} \varepsilon_i ch\eta_i = \sum_{i=1}^{N} \varepsilon_i ch\left(k_i x + b_1 k_i^3 y + b_2 k_i^{-1} z + b_3 k_i^{-3} t\right), \tag{3.23}$$

where the ε_i's and the k_i's are arbitrary, and b_1, b_2 and b_3 satisfy

$$b_1 = -\frac{c_1}{c_2}, \qquad b_2 = -\frac{3c_1}{10c_3}, \qquad b_3 = \frac{3c_1}{70c_4}. \tag{3.24}$$

Similarly, (3.22) has the linear subspace of N-wave solutions defined by

$$f = \sum_{i=1}^{N} \varepsilon_i f_i = \sum_{i=1}^{N} \varepsilon_i \cos\eta_i = \sum_{i=1}^{N} \varepsilon_i \cos\left(k_i x + b_1 k_i^3 y + b_2 k_i^{-1} z + b_3 k_i^{-3} t\right), \tag{3.25}$$

where the ε_i's and the k_i's are arbitrary, and b_1, b_2 and b_3 satisfy

$$b_1 = \frac{c_1}{c_2}, \qquad b_2 = \frac{3c_1}{10c_3}, \qquad b_3 = \frac{3c_1}{70c_4}. \tag{3.26}$$

4. Conclusion

For generalized bilinear equations, we analyzed linear combinations of hyperbolic or trigonometric function solutions and we presented linear superposition principle of hyperbolic and trigonometric function solutions to generalized bilinear equations. A few illustrative examples were presented, by applying an algorithm using weights.

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