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On some soliton structures to the Schamel–Korteweg-de Vries model via two analytical approaches

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The Schamel–Korteweg-de Vries (S-KdV) model is used to predict the influence of surface for deep water in the presence of solitary waves. The aim of the study is to study the governing model analytically by employing the extended modified auxiliary equation mapping approach and the extended FAN sub-equation method. The 3D, 2D and contour plots are drawn to demonstrate the physical nature of the nonlinear model for a set of parameters. As a result, dark solitons, light solitons, solitary waves, periodic solitary waves, rational functions, and elliptic function solutions are established. Furthermore, the the developed results are verified with the aid of latest computing tool such as Mathematica or Maple. The applied strategy appears to be a more powerful and

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efficient scheme for achieving exact solutions to a number of diversified contemporary models of recent eras.

Keywords: Schamel–Korteweg-de Vries equation; analytical solutions; the extended modified auxiliary equation mapping method; extended FAN sub-equation method.

1. Introduction

Nonlinear evolution equations (NLEEs) are of key importance due to their significant role in diverse disciplines of science and technology. The nonlinear wave structures have fascinated many researchers in recent decades due to their diverse properties observed in various disciplines of contemporary sciences. In the presence of solitary waves, the nonlinear evolution models are utilized to simulate the effect of surface for deep water and weakly nonlinear dispersive long waves. Therefore, the exact solutions of such models play a vital role of study of dynamical structures and further properties of physical phenomenon occurring several fields to name a few, electromagnetism, physical chemistry, geophysics, ionized physics, elastic medium, fluid motion, fluid mechanics, elastic medium, nuclear physics, electrochemistry, optical fibers, energy physics, chemical mechanics, gravity, biostatistics, statistical and natural physics.[1–10]

With the recent developments in various contemporary analytical methodologies, solitons play a key role to understand the nonlinear phenomenon of many crucial structures in an exceptional way. The major feature of solitons is that they have nearly the same forms and speeds after colliding; also, the production of optical solitons is linked to optical frequency. Kink solutions are asymptotic waves that ascend or descend from one asymptotic state to the next, and they also approach a constant at infinity. Kink solutions, like classical particles, have a constant shape; nevertheless, their widths shrink, which can change. Solitons are transmitted as dark ones in the normal dispersion domain, but as bright ones in the anomalous dispersion domain. With the rapid advancement of information technology and telecommunications, the optical solitons play an important rule in understanding the dynamics of nonlinear wave propagation through a variety of wave-guides. The polarization of pulse propagation over trans-oceanic and trans-continental distances is an inherent problem with the dynamics of pulse propagation.[11–20]

To analyze various properties of electrostatic waves is especially significant because of their potential applications in the improvement of new theories of chemical physics, nuclear physics, astrophysics, dusty plasma, fluid dynamics, optical physics, fluid mechanics, geophysics and distinctive other fields of applied physics. For the better understanding of nonlinear models also further uses in everyday life, it is crucial to get their exact traveling wave solution. To attain the exact solutions of nonlinear models there are powerful and effective techniques such as the Hirota bilinear method,[21–23] the Sine-Cosine method,[24] Adomian decomposition methods,[25] generalized Riccati equation expansion method,[26,27] homotopy analysis method,[28] extended Kurdyashov method,[29] residual power series method,[30] modified simple equation method,[31] $(\frac{G'}{G}, \frac{1}{G})$-expansion method,[32] extended simple equation methods,[33,34] the Hirota method,[35] modified Kudryashov method,[36] extended Sinh–Gordon equation expansion method,[37] extended generalized Riccati equation mapping method,[38] homotopy perturbation method,[39] multiple exp-function method[40,41] and many more.[42–51]

The Gardner equation originated by an American mathematician Clifford Gardner in 1968 to disseminate KdV equation and modified KdV equation. The Gardner equation has applications in quantum field theory, plasma physics and hydrodynamics. The generalized Gardner equations[52] is read as

$$u_t + (qu^n + ru^{2n} + p)u_x + u_{xxx} = 0, \quad n \geq 0. \tag{1}$$

The S-Kdv equation is a special case of the generalized Gardner equations. For $p = 0$ and $n = \frac{1}{2}$ in Eq. (1), we have the S-Kdv equation.

In 1973, a German Mathematician Hans Schamel first derived Schamel–Korteweg-de Vries equation[53] to express the outcomes of electron entrapment in plasma physics to study the ion acoustic solitons and it also describes the electrostatic potential for a specific electron scattering in velocity space.[54] The Schamel-KdV equation is also used to study the wave properties in dusty space plasma containing positively and negatively charged particles as well as non-isothermal electrons. The most common type of plasma in our Solar System is dusty plasma. In fact, it is hard to find a plasma environment free of dust particles anywhere in the Solar System.

In the recent past, the Schamel–Korteweg-de Vries equation (S-KdV) as been studied by various renowned scholars by applying diverse analytical approaches such as the Exp-function method,[60] the extended (G'/G)-expansion method,[61] the simplest equation method and the Kudryashov method,[62] classical Khater and the modified Khater approaches,[63] the modified Kudrayshov scheme[64] and more. The main focus of our work is to extract some new traveling wave solutions to the S-KdV model[55] by utilizing the extended modified auxiliary equation mapping (AEM) method[56,57] and the extended FAN sub-equation method, which reads

$$u_t + (\alpha u^{\frac{1}{2}} + \beta u)u_x + \delta u_{xxx} = 0, \tag{2}$$

where the variables β, α and δ correspond to the activation convection, trapping, and dispersion coefficients, respectively. Furthermore, Eq. (2) reduces to the classical KdV equation[58] and the Schamel equation[59] for $\alpha = 0$ and $\beta = 0$, respectively.

In this work, Sec. 1 includes the brief introduction of S-Kdv model, while in Sec. 2, the extended modified AEM and the extended FAN sub-equation techniques are successfully implemented to demonstrate different solitary wave structures. In Sec. 3, the physical interpretation of the obtained solutions is represented by 3D, 2D and contour graphs. At the end, the concluding remarks are given.

2. Mathematical Analysis

Consider the wave transformation

$$u(x,t) = \Omega(x,t)^2, \tag{3}$$

where

$$\Omega(x,t) = \Omega(\xi), \quad \xi = \kappa x - t\omega. \tag{4}$$

Equation (2) transforms to nonlinear ode of the form

$$-\omega\Omega\Omega' + \kappa\Omega'(\beta\Omega^3 + \alpha\Omega^2) + \delta\kappa^3(\Omega^{'''}\Omega + 3\Omega'\Omega'') = 0. \tag{5}$$

2.1. *Applications of the modified AEM method*

By the homogenous balance principle from Eq. (5), we find the positive integer $n = 1$, the solution is of the form[65]

$$\Omega(\xi) = a_0 + a_1\psi(\xi) + \frac{b_1}{\psi(\xi)} + \frac{d_1\psi'(\xi)}{\psi(\xi)}, \tag{6}$$

where a_0, a_1, b_1 and d_1 are to be determined and ψ satisfies

$$\psi'(\xi)^2 = \chi_1\psi(\xi)^2 + \chi_2\psi(\xi)^3 + \chi_3\psi(\xi)^4, \tag{7}$$

χ_i $(i = 0, 1, 2, 3, 4)$ are real constants.

Putting Eq. (20) along with its desired derivatives into Eq. (5), by collecting the coefficients of $\psi^{'}(\eta)\ \psi(\eta)$ and equating them to zero we obtain the following algebraic system:

$$3a_0\beta b_1^2 d_1\kappa\chi_1 + \alpha b_1^2 d_1\kappa\chi_1 + \frac{3}{2}\beta b_1^3 d_1\kappa\chi_2 = 0,$$

$$\begin{aligned}&2\alpha a_0 b_1 d_1\kappa\chi_1 + 3a_1\beta b_1^2 d_1\kappa\chi_1 + 3a_0^2\beta b_1 d_1\kappa\chi_1 + \frac{9}{2}a_0\beta b_1^2 d_1\kappa\chi_2\\&\quad + \frac{3}{2}\alpha b_1^2 d_1\kappa\chi_2 + 2\beta b_1^3 d_1\kappa\chi_3 - 5b_1\delta d_1\kappa^3\chi_1^2 - b_1 d_1\chi_1\omega = 0,\end{aligned}$$

$$\begin{aligned}&2\alpha a_1 b_1 d_1\kappa\chi_1 + 3\alpha a_0 b_1 d_1\kappa\chi_2 + 6a_0 a_1\beta b_1 d_1\kappa\chi_1\\&\quad + \frac{9}{2}a_1\beta b_1^2 d_1\kappa\chi_2 + \frac{9}{2}a_0^2\beta b_1 d_1\kappa\chi_2 + 6a_0\beta b_1^2 d_1\kappa\chi_3\\&\quad + \alpha a_0^2 d_1\kappa\chi_1 + a_0^3\beta d_1\kappa\chi_1 - 2a_0\delta d_1\kappa^3\chi_1^2 - a_0 d_1\chi_1\omega\\&\quad + 2\alpha b_1^2 d_1\kappa\chi_3 - \frac{27}{2}b_1\delta d_1\kappa^3\chi_1\chi_2 - \frac{3}{2}b_1 d_1\chi_2\omega = 0,\end{aligned}$$

$$\begin{aligned}&3\alpha a_1 b_1 d_1\kappa\chi_2 + 4\alpha a_0 b_1 d_1\kappa\chi_3 + 3a_1^2\beta b_1 d_1\kappa\chi_1 + 9a_0 a_1\beta b_1 d_1\kappa\chi_2\\&\quad + 6a_1\beta b_1^2 d_1\kappa\chi_3 + 6a_0^2\beta b_1 d_1\kappa\chi_3 + 2\alpha a_0 a_1 d_1\kappa\chi_1 + \frac{3}{2}\alpha a_0^2 d_1\kappa\chi_2\\&\quad + 3a_0^2 a_1\beta d_1\kappa\chi_1 + \frac{3}{2}a_0^3\beta d_1\kappa\chi_2 + a_1\delta d_1\kappa^3\chi_1^2 - \frac{9}{2}a_0\delta d_1\kappa^3\chi_1\chi_2\end{aligned}$$

$$
\begin{aligned}
&- a_1 d_1 \chi_1 \omega - \frac{3}{2} a_0 d_1 \chi_2 \omega - 9 b_1 \delta d_1 \kappa^3 \chi_2^2 - 16 b_1 \delta d_1 \kappa^3 \chi_1 \chi_3 - 2 b_1 d_1 \chi_3 \omega = 0,\\
&4\alpha a_1 b_1 d_1 \kappa \chi_3 + \frac{9}{2} a_1^2 \beta b_1 d_1 \kappa \chi_2 + 12 a_0 a_1 \beta b_1 d_1 \kappa \chi_3 + \alpha a_1^2 d_1 \kappa \chi_1\\
&\quad + 3\alpha a_0 a_1 d_1 \kappa \chi_2 + 2\alpha a_0^2 d_1 \kappa \chi_3 + 3 a_0 a_1^2 \beta d_1 \kappa \chi_1 + \frac{9}{2} a_0^2 a_1 \beta d_1 \kappa \chi_2\\
&\quad + 2 a_0^3 \beta d_1 \kappa \chi_3 - \frac{1}{4} 9 a_0 \delta d_1 \kappa^3 \chi_2^2 + \frac{9}{2} a_1 \delta d_1 \kappa^3 \chi_1 \chi_2\\
&\quad - 4 a_0 \delta d_1 \kappa^3 \chi_1 \chi_3 - \frac{3}{2} a_1 d_1 \chi_2 \omega - 2 a_0 d_1 \chi_3 \omega - 21 b_1 \delta d_1 \kappa^3 \chi_2 \chi_3 = 0,\\
&6 a_1^2 \beta b_1 d_1 \kappa \chi_3 + \frac{3}{2} \alpha a_1^2 d_1 \kappa \chi_2 + 4\alpha a_0 a_1 d_1 \kappa \chi_3\\
&\quad + a_1^3 \beta d_1 \kappa \chi_1 + \frac{9}{2} a_0 a_1^2 \beta d_1 \kappa \chi_2 + 6 a_0^2 a_1 \beta d_1 \kappa \chi_3\\
&\quad + \frac{9}{2} a_1 \delta d_1 \kappa^3 \chi_2^2 + 8 a_1 \delta d_1 \kappa^3 \chi_1 \chi_3 - 3 a_0 \delta d_1 \kappa^3 \chi_2 \chi_3\\
&\quad - 2 a_1 d_1 \chi_3 \omega - 12 b_1 \delta d_1 \kappa^3 \chi_3^2 = 0,\\
&2\alpha a_1^2 d_1 \kappa \chi_3 + \frac{3}{2} a_1^3 \beta d_1 \kappa \chi_2 + 6 a_0 a_1^2 \beta d_1 \kappa \chi_3 + 15 a_1 \delta d_1 \kappa^3 \chi_2 \chi_3 = 0,\\
&\alpha a_0^2 a_1 \kappa + \alpha a_1^2 b_1 \kappa + 3 a_0 a_1^2 \beta b_1 \kappa\\
&\quad + 9 a_1 \beta b_1 d_1^2 \kappa \chi_2 + 12 a_0 \beta b_1 d_1^2 \kappa \chi_3 + 6 a_0 b_1 \delta \kappa^3 \chi_3 + a_0^3 a_1 \beta \kappa\\
&\quad + 2\alpha a_1 d_1^2 \kappa \chi_1 + 3\alpha a_0 d_1^2 \kappa \chi_2 + 6 a_0 a_1 \beta d_1^2 \kappa \chi_1 + \frac{9}{2} a_0^2 \beta d_1^2 \kappa \chi_2\\
&\quad + a_0 a_1 \delta \kappa^3 \chi_1 - a_0 a_1 \omega + 4\alpha b_1 d_1^2 \kappa \chi_3 - 18 \delta d_1^2 \kappa^3 \chi_1 \chi_2 - \frac{3}{2} d_1^2 \chi_2 \omega = 0,\\
&-3 a_0 \beta b_1^3 \kappa - \alpha b_1^3 \kappa = 0,\\
&-2\alpha a_0 b_1^2 \kappa - 2 a_1 \beta b_1^3 \kappa - 3 a_0^2 \beta b_1^2 \kappa + 3\beta b_1^2 d_1^2 \kappa \chi_1 + 8 b_1^2 \delta \kappa^3 \chi_1 + b_1^2 \omega = 0,\\
&-\alpha a_1 b_1^2 \kappa - \alpha a_0^2 b_1 \kappa - 3 a_0 a_1 \beta b_1^2 \kappa - a_0^3 \beta b_1 \kappa + 6 a_0 \beta b_1 d_1^2 \kappa \chi_1\\
&\quad + 5 a_0 b_1 \delta \kappa^3 \chi_1 + a_0 b_1 \omega + 2\alpha b_1 d_1^2 \kappa \chi_1 + \frac{9}{2} \beta b_1^2 d_1^2 \kappa \chi_2 + \frac{21}{2} b_1^2 \delta \kappa^3 \chi_2 = 0,\\
&6 a_1 \beta b_1 d_1^2 \kappa \chi_1 + 9 a_0 \beta b_1 d_1^2 \kappa \chi_2 + 6 a_0 b_1 \delta \kappa^3 \chi_2 + 2\alpha a_0 d_1^2 \kappa \chi_1 + 3 a_0^2 \beta d_1^2 \kappa \chi_1\\
&\quad + 3\alpha b_1 d_1^2 \kappa \chi_2 + 6\beta b_1^2 d_1^2 \kappa \chi_3 + 12 b_1^2 \delta \kappa^3 \chi_3 - 8 \delta d_1^2 \kappa^3 \chi_1^2 - d_1^2 \chi_1 \omega = 0,\\
&2\alpha a_0 a_1^2 \kappa + 2 a_1^3 \beta b_1 \kappa + 12 a_1 \beta b_1 d_1^2 \kappa \chi_3 + 3 a_0^2 a_1^2 \beta \kappa + 3\alpha a_1 d_1^2 \kappa \chi_2\\
&\quad + 4\alpha a_0 d_1^2 \kappa \chi_3 + 3 a_1^2 \beta d_1^2 \kappa \chi_1 + 9 a_0 a_1 \beta d_1^2 \kappa \chi_2 + 6 a_0^2 \beta d_1^2 \kappa \chi_3 + 4 a_1^2 \delta \kappa^3 \chi_1\\
&\quad + 3 a_0 a_1 \delta \kappa^3 \chi_2 - a_1^2 \omega - 9 \delta d_1^2 \kappa^3 \chi_2^2 - 16 \delta d_1^2 \kappa^3 \chi_1 \chi_3 - 2 d_1^2 \chi_3 \omega = 0,
\end{aligned}
$$

$$\alpha a_1^3\kappa + 3a_0a_1^3\beta\kappa + 4\alpha a_1 d_1^2\kappa\chi_3 + \frac{9}{2}a_1^2\beta d_1^2\kappa\chi_2$$
$$+ 12a_0a_1\beta d_1^2\kappa\chi_3 + \frac{15}{2}a_1^2\delta\kappa^3\chi_2 + 6a_0a_1\delta\kappa^3\chi_3 - 12\delta d_1^2\kappa^3\chi_2\chi_3 = 0,$$
$$a_1^4\beta\kappa + 6a_1^2\beta d_1^2\kappa\chi_3 + 12a_1^2\delta\kappa^3\chi_3 = 0,$$
$$\alpha a_1^2 d_1\kappa + \frac{9}{2}a_1\beta d_1^3\kappa\chi_2 + 6a_0\beta d_1^3\kappa\chi_3 + 3a_0a_1^2\beta d_1\kappa$$
$$+ 3a_1\delta d_1\kappa^3\chi_2 + 12a_0\delta d_1\kappa^3\chi_3 + 2\alpha d_1^3\kappa\chi_3 = 0,$$
$$-9a_0\beta b_1^2 d_1\kappa - 3\alpha b_1^2 d_1\kappa = 0,$$
$$-4\alpha a_0 b_1 d_1\kappa - 6a_1\beta b_1^2 d_1\kappa - 6a_0^2\beta b_1 d_1\kappa$$
$$+ 3\beta b_1 d_1^3\kappa\chi_1 + 28b_1\delta d_1\kappa^3\chi_1 + 2b_1 d_1\omega = 0,$$
$$-2\alpha a_1 b_1 d_1\kappa - 6a_0a_1\beta b_1 d_1\kappa - \alpha a_0^2 d_1\kappa + 3a_0\beta d_1^3\kappa\chi_1 - a_0^3\beta d_1\kappa$$
$$+ 8a_0\delta d_1\kappa^3\chi_1 + a_0 d_1\omega + \frac{9}{2}\beta b_1 d_1^3\kappa\chi_2 + 33b_1\delta d_1\kappa^3\chi_2 + \alpha d_1^3\kappa\chi_1 = 0,$$
$$3a_1\beta d_1^3\kappa\chi_1 + \frac{9}{2}a_0\beta d_1^3\kappa\chi_2 + 9a_0\delta d_1\kappa^3\chi_2 + 6\beta b_1 d_1^3\kappa\chi_3$$
$$+ 36b_1\delta d_1\kappa^3\chi_3 + \frac{3}{2}\alpha d_1^3\kappa\chi_2 = 0,$$
$$6a_1\beta d_1^3\kappa\chi_3 + 2a_1^3\beta d_1\kappa + 12a_1\delta d_1\kappa^3\chi_3 = 0,$$
$$-6\beta b_1^2 d_1^2\kappa - 12b_1^2\delta\kappa^3 = 0,$$
$$-9a_0\beta b_1 d_1^2\kappa - 6a_0 b_1\delta\kappa^3 - 3\alpha b_1 d_1^2\kappa = 0,$$
$$-6a_1\beta b_1 d_1^2\kappa - 2\alpha a_0 d_1^2\kappa - 3a_0^2\beta d_1^2\kappa + \beta d_1^4\kappa\chi_1 + 20\delta d_1^2\kappa^3\chi_1 + d_1^2\omega = 0,$$
$$-\alpha a_1 d_1^2\kappa - 3a_0a_1\beta d_1^2\kappa + \frac{3}{2}\beta d_1^4\kappa\chi_2 + \frac{45}{2}\delta d_1^2\kappa^3\chi_2 = 0,$$
$$2\beta d_1^4\kappa\chi_3 + 24\delta d_1^2\kappa^3\chi_3 = 0,$$
$$-4\beta b_1 d_1^3\kappa - 24b_1\delta d_1\kappa^3 = 0,$$
$$-3a_0\beta d_1^3\kappa - 6a_0\delta d_1\kappa^3 - \alpha d_1^3\kappa = 0,$$
$$-\beta d_1^4\kappa - 12\delta d_1^2\kappa^3 = 0.$$

The following cases arise by solving the above system:

Family I:

$$\psi(\xi) = -\frac{4\chi_1 e^{\xi\sqrt{\chi_1}}}{\chi_2^2(-e^{2\xi\sqrt{\chi_1}}) + 2\chi_2 e^{\xi\sqrt{\chi_1}} + 4\chi_1\chi_3 e^{2\xi\sqrt{\chi_1}} - 1}.$$

Case I: $a_0 = 0, a_1 = \lambda, b_1 = 0, d_1 = 0, \chi_1 = \frac{\omega}{4\delta\kappa^3}, \chi_2 = -\frac{2\alpha a_1}{15\delta\kappa^2}, \chi_3 = -\frac{a_1^2\beta}{12\delta\kappa^2}$.

$$u_1(x,t) = \frac{\lambda^2\omega^2 e^{\xi\sqrt{\frac{\omega}{\delta\kappa^3}}}}{\delta^2\kappa^6\left(-\frac{4\alpha^2\lambda^2 e^{\xi\sqrt{\frac{\omega}{\delta\kappa^3}}}}{225\delta^2\kappa^4} - \frac{4\alpha\lambda e^{\frac{1}{2}\xi\sqrt{\frac{\omega}{\delta\kappa^3}}}}{15\delta\kappa^2} - \frac{\beta\lambda^2\omega e^{\xi\sqrt{\frac{\omega}{\delta\kappa^3}}}}{12\delta^2\kappa^5} - 1\right)^2}. \tag{8}$$

Case II: $a_0 = \frac{-\sqrt{6}\sqrt{6\alpha^2\kappa^2+25\beta\kappa\omega}-6\alpha\kappa}{10\beta\kappa}, a_1 = \frac{30\delta\delta_2\kappa^3}{\sqrt{6}\sqrt{\kappa(6\alpha^2\kappa+25\beta\omega)}+2\alpha\kappa}, b_1 = 0, d_1 = 0, \chi_1 = \frac{-\frac{\sqrt{6}\alpha\sqrt{\kappa(6\alpha^2\kappa+25\beta\omega)}}{\beta}-\frac{6\alpha^2\kappa}{\beta}-25\omega}{50\delta\kappa^3}, \chi_2 = \lambda, \chi_3 = -\frac{a_1^2\beta}{12\delta\kappa^2}$.

$$u_2(x,t) = \left(\frac{30\delta\kappa^3\lambda\psi(\xi)}{\sqrt{6}\sqrt{\kappa(6\alpha^2\kappa+25\beta\omega)}+2\alpha\kappa} + \frac{-\sqrt{6}\sqrt{6\alpha^2\kappa^2+25\beta\kappa\omega}-6\alpha\kappa}{10\beta\kappa}\right)^2. \tag{9}$$

Case III: $a_0 = \frac{\sqrt{6}\sqrt{6\alpha^2\kappa^2+25\beta\kappa\omega}-6\alpha\kappa}{10\beta\kappa}, a_1 = -\frac{30\delta\chi_2\kappa^3}{\sqrt{6}\sqrt{\kappa(6\alpha^2\kappa+25\beta\omega)}-2\alpha\kappa}, b_1 = 0, d_1 = 0, \chi_1 = \frac{\frac{\sqrt{6}\alpha\sqrt{\kappa(6\alpha^2\kappa+25\beta\omega)}}{\beta}-\frac{6\alpha^2\kappa}{\beta}-25\omega}{50\delta\kappa^3}, \chi_2 = \lambda, \chi_3 = -\frac{a_1^2\beta}{12\delta\kappa^2}$.

$$u_3(x,t) = \left(\frac{\sqrt{6}\sqrt{6\alpha^2\kappa^2+25\beta\kappa\omega}-6\alpha\kappa}{10\beta\kappa} - \frac{30\delta\kappa^3\lambda\psi(\xi)}{\sqrt{6}\sqrt{\kappa(6\alpha^2\kappa+25\beta\omega)}-2\alpha\kappa}\right)^2. \tag{10}$$

Case IV: $a_0 = \frac{-\sqrt{6}\sqrt{6\alpha^2\kappa^2+25\beta\kappa\omega}-6\alpha\kappa}{10\beta\kappa}, a_1 = \lambda, b_1 = 0, d_1 = 0, \chi_3 = -\frac{a_1^2\beta}{12\delta\kappa^2}, \chi_1 = \frac{\frac{\sqrt{6}\alpha\sqrt{\kappa(6\alpha^2\kappa+25\beta\omega)}}{\beta}-\frac{6\alpha^2\kappa}{\beta}-25\omega}{50\delta\kappa^3}, \chi_2 = \frac{\frac{\sqrt{6}a_1\sqrt{\kappa(6\alpha^2\kappa+25\beta\omega)}}{\kappa}+2\alpha a_1}{30\delta\kappa^2}$.

$$u_4(x,t) = \left(\frac{-\sqrt{6}\sqrt{6\alpha^2\kappa^2+25\beta\kappa\omega}-6\alpha\kappa}{10\beta\kappa} + \lambda\psi(\xi)\right)^2. \tag{11}$$

Case V: $a_0 = \frac{\sqrt{6}\sqrt{6\alpha^2\kappa^2+25\beta\kappa\omega}-6\alpha\kappa}{10\beta\kappa}, a_1 = \lambda, b_1 = 0, d_1 = 0, \chi_3 = -\frac{a_1^2\beta}{12\delta\kappa^2}, \chi_1 = \frac{\frac{\sqrt{6}\alpha\sqrt{\kappa(6\alpha^2\kappa+25\beta\omega)}}{\beta}-\frac{6\alpha^2\kappa}{\beta}-25\omega}{50\delta\kappa^3}, \chi_2 = \frac{2\alpha a_1-\frac{\sqrt{6}a_1\sqrt{\kappa(6\alpha^2\kappa+25\beta\omega)}}{\kappa}}{30\delta\kappa^2}$.

$$u_5(x,t) = \left(\frac{\sqrt{6}\sqrt{6\alpha^2\kappa^2+25\beta\kappa\omega}-6\alpha\kappa}{10\beta\kappa} - \frac{30\delta\kappa^3\lambda\psi(\xi)}{\sqrt{6}\sqrt{\kappa(6\alpha^2\kappa+25\beta\omega)}-2\alpha\kappa}\right)^2. \tag{12}$$

Family II:

$$\psi(\xi) = \frac{4\chi_1 e^{\xi\sqrt{\chi_1}}}{-2\chi_2 e^{\xi\sqrt{\chi_1}} + e^{2\xi\sqrt{\chi_1}} + \chi_2^2 - 4\chi_1\chi_3}$$

Case I: $a_0 = 0, a_1 = \lambda, b_1 = 0, d_1 = 0, \chi_1 = \frac{\omega}{4\delta\kappa^3}, \chi_2 = -\frac{2\alpha a_1}{15\delta\kappa^2}, \chi_3 = -\frac{a_1^2\beta}{12\delta\kappa^2}$.

$$u_6(x,t) = \frac{225\lambda^2\omega^2 e^{\xi\sqrt{\frac{\omega}{\delta\kappa^3}}}}{4\alpha^2\kappa^2\left(\frac{75\beta\lambda^2\omega}{16\alpha^2\kappa} - 2\lambda e^{\frac{1}{2}\xi\sqrt{\frac{\omega}{\delta\kappa^3}}} + e^{\xi\sqrt{\frac{\omega}{\delta\kappa^3}}} + \lambda^2\right)^2}. \tag{13}$$

Case II: $a_0 = \frac{-\sqrt{6}\sqrt{6\alpha^2\kappa^2+25\beta\kappa\omega}-6\alpha\kappa}{10\beta\kappa}, a_1 = \frac{30\delta\delta_2\kappa^3}{\sqrt{6}\sqrt{\kappa(6\alpha^2\kappa+25\beta\omega)}+2\alpha\kappa}, b_1 = 0, d_1 = 0, \chi_1 = \frac{-\frac{\sqrt{6}\alpha\sqrt{\kappa(6\alpha^2\kappa+25\beta\omega)}}{\beta}-\frac{6\alpha^2\kappa}{\beta}-25\omega}{50\delta\kappa^3}, \chi_2 = \lambda, \chi_3 = -\frac{a_1^2\beta}{12\delta\kappa^2}$.

$$u_7(x,t) = \left(\frac{30\delta\kappa^3\lambda\psi(\xi)}{\sqrt{6}\sqrt{\kappa(6\alpha^2\kappa+25\beta\omega)}+2\alpha\kappa} + \frac{-\sqrt{6}\sqrt{6\alpha^2\kappa^2+25\beta\kappa\omega}-6\alpha\kappa}{10\beta\kappa}\right)^2. \tag{14}$$

Case III: $a_0 = \frac{\sqrt{6}\sqrt{6\alpha^2\kappa^2+25\beta\kappa\omega}-6\alpha\kappa}{10\beta\kappa}, a_1 = -\frac{30\delta\chi_2\kappa^3}{\sqrt{6}\sqrt{\kappa(6\alpha^2\kappa+25\beta\omega)}-2\alpha\kappa}, b_1 = 0, d_1 = 0, \chi_1 = \frac{\frac{\sqrt{6}\alpha\sqrt{\kappa(6\alpha^2\kappa+25\beta\omega)}}{\beta}-\frac{6\alpha^2\kappa}{\beta}-25\omega}{50\delta\kappa^3}, \chi_2 = \lambda, \chi_3 = -\frac{a_1^2\beta}{12\delta\kappa^2}$.

$$u_8(x,t) = \left(\frac{\sqrt{6}\sqrt{6\alpha^2\kappa^2+25\beta\kappa\omega}-6\alpha\kappa}{10\beta\kappa} - \frac{30\delta\kappa^3\lambda\psi(\xi)}{\sqrt{6}\sqrt{\kappa(6\alpha^2\kappa+25\beta\omega)}-2\alpha\kappa}\right)^2. \tag{15}$$

Case IV: $a_0 = \frac{-\sqrt{6}\sqrt{6\alpha^2\kappa^2+25\beta\kappa\omega}-6\alpha\kappa}{10\beta\kappa}, a_1 = \lambda, b_1 = 0, d_1 = 0, \chi_3 = -\frac{a_1^2\beta}{12\delta\kappa^2}, \chi_1 = \frac{\frac{\sqrt{6}\alpha\sqrt{\kappa(6\alpha^2\kappa+25\beta\omega)}}{\beta}-\frac{6\alpha^2\kappa}{\beta}-25\omega}{50\delta\kappa^3}, \chi_2 = \frac{\frac{\sqrt{6}a_1\sqrt{\kappa(6\alpha^2\kappa+25\beta\omega)}}{\kappa}+2\alpha a_1}{30\delta\kappa^2}$.

$$u_9(x,t) = \left(\frac{-\sqrt{6}\sqrt{6\alpha^2\kappa^2+25\beta\kappa\omega}-6\alpha\kappa}{10\beta\kappa} + \lambda\psi(\xi)\right)^2. \tag{16}$$

Case V: $a_0 = \frac{\sqrt{6}\sqrt{6\alpha^2\kappa^2+25\beta\kappa\omega}-6\alpha\kappa}{10\beta\kappa}, a_1 = \lambda, b_1 = 0, d_1 = 0, \chi_3 = -\frac{a_1^2\beta}{12\delta\kappa^2}, \chi_1 = \frac{\frac{\sqrt{6}\alpha\sqrt{\kappa(6\alpha^2\kappa+25\beta\omega)}}{\beta}-\frac{6\alpha^2\kappa}{\beta}-25\omega}{50\delta\kappa^3}, \chi_2 = \frac{2\alpha a_1-\frac{\sqrt{6}a_1\sqrt{\kappa(6\alpha^2\kappa+25\beta\omega)}}{\kappa}}{30\delta\kappa^2}$.

$$u_{10}(x,t) = \left(\frac{\sqrt{6}\sqrt{6\alpha^2\kappa^2+25\beta\kappa\omega}-6\alpha\kappa}{10\beta\kappa} + \lambda\psi(\xi)\right)^2. \tag{17}$$

2.2. *Applications of the extended FAN sub-equation method*

By the homogenous balance principle from Eq. (5) we find the positive integer $n = 1$, the solution is of the form[66]

$$\Omega = \zeta_0 + \zeta_1\psi(\xi), \tag{18}$$

where a_0, a_1 are to be determined and ψ satisfies

$$\left(\frac{d\psi(\xi)}{d\xi}\right)^2 = \ell_0 + \ell_1\psi(\xi) + \ell_2\psi^2(\xi) + \ell_3\psi^3(\xi) + \ell_4\psi^4(\xi), \tag{19}$$

ℓ_i $(i = 0, 1, 2, 3, 4)$ are real constants.

Inserting Eq. (18) along Eq. (19) in Eq. (5) and picking the coefficients of $\phi^j\phi^{(k)}$,

$$\begin{aligned}
2\alpha\zeta_0^2\kappa + 2\zeta_0^3\beta\kappa + 3\zeta_1\delta\ell_1\kappa^3 + 2\zeta_0\zeta_1\delta\ell_2\kappa^3 - 2\zeta_0\omega &= 0,\\
4\alpha\zeta_0\zeta_1\kappa + 6\zeta_0^2\zeta_1\beta\kappa + 2\zeta_1(\zeta_1+3)\delta\ell_2\kappa^3 + 6\zeta_0\zeta_1\delta\ell_3\kappa^3 - 2\zeta_1\omega &= 0,\\
6\zeta_0\zeta_1^2\beta\kappa + \zeta_1\kappa(2\alpha\zeta_1 + 3(2\zeta_1+3)\delta\ell_3\kappa^2) + 12\zeta_0\zeta_1\delta\ell_4\kappa^3 &= 0,\\
2\zeta_1\kappa(\zeta_1^2\beta + 6(\zeta_1+1)\delta\ell_4\kappa^2) &= 0.
\end{aligned}$$

We select variables suitably which gives

$$\begin{aligned}
\zeta_0 = \pm&\frac{1}{12\beta\ell_4(3\beta - 4\delta\ell_4\kappa^2)}\\
&\times\Bigg[\ell_4\Big(4\alpha\Big(\sqrt{3}\sqrt{\delta\ell_4\kappa^2(3\delta\ell_4\kappa^2-2\beta)} - 3\beta\Big)\\
&+ 3\delta\ell_3\kappa^2\big(4\sqrt{3}\sqrt{\delta\ell_4\kappa^2(3\delta\ell_4\kappa^2-2\beta)} - 9\beta\big)\Big)\\
&+ 12\delta\ell_4^2\kappa^2(\alpha + 3\delta\ell_3\kappa^2) - 9\sqrt{3}\beta\ell_3\sqrt{\delta\ell_4\kappa^2(3\delta\ell_4\kappa^2-2\beta)}\Bigg],
\end{aligned} \tag{20}$$

$$\zeta_1 = \pm\frac{\sqrt{3}\sqrt{3\delta^2\ell_4^2\kappa^4 - 2\beta\delta\ell_4\kappa^2} - 3\delta\ell_4\kappa^2}{\beta}. \tag{21}$$

Thus, we have following collection of soliton solutions for the given model:

Case I. If $\ell_0 = \vartheta_3^2, \ell_1 = 2\vartheta_1\vartheta_3, \ell_2 = 2\vartheta_2\vartheta_3 + \vartheta_1^2, \ell_3 = 2\vartheta_1\vartheta_2, \ell_4 = \vartheta_2^2$, there may exist parameters ϑ_1, ϑ_2, satisfy ϑ_3, the solutions of (2) are ϕ_η^{I}, $(\eta = 1, 2, \ldots, 24)$. Some of important solitons are listed below.

Type I: when $\gamma_1^2 - 4\gamma_2\gamma_3 > 0$, $\gamma_1\gamma_2 \neq 0$, $\gamma_2\gamma_3 \neq 0$. We obtain the dark optical solitons in the form as follows

$$u_1^{\mathrm{I}}(x,t) = \left[\zeta_0 - \zeta_1\left\{\frac{\sqrt{\gamma_1^2 - 4\gamma_2\gamma_3}\tanh\left(\frac{1}{2}\xi\sqrt{\gamma_1^2-4\gamma_2\gamma_3}\right) + \gamma_1}{2\gamma_2}\right\}\right]^2. \tag{22}$$

We obtain the combined bright-dark optical soliton in the form as follows

$$\begin{aligned}
u_3^{\mathrm{I}}(x,t) = \Bigg[\zeta_0 - \zeta_1\Bigg\{&\sqrt{\gamma_1^2-4\gamma_2\gamma_3}\Big(i\ \mathrm{sech}\Big(\xi\sqrt{\gamma_1^2-4\gamma_2\gamma_3}\Big)\\
&+ \tanh\Big(\xi\sqrt{\gamma_1^2-4\gamma_2\gamma_3}\Big)\Big) + \gamma_1\Bigg\}\Bigg]^2.
\end{aligned} \tag{23}$$

We obtain the combined dark-singular optical solitons in the form as follows:

$$u_5^{\mathrm{I}}(x,t)=\Bigg[\zeta_0-\zeta_1\Bigg\{\sqrt{\gamma_1^2-4\gamma_2\gamma_3}\Bigg(\tanh\left(\frac{1}{4}\xi\sqrt{\gamma_1^2-4\gamma_2\gamma_3}\right)$$
$$+\coth\left(\frac{1}{4}\xi\sqrt{\gamma_1^2-4\gamma_2\gamma_3}\right)\Bigg)+\gamma_1\Bigg\}\Bigg]^2. \tag{24}$$

The family of solitons is obtained as

$$u_{10}^{\mathrm{I}}(x,t)=\Bigg[\zeta_0-\zeta_1\Bigg\{\left(2\cosh\left(\xi\sqrt{\gamma_1^2-4\gamma_2\gamma_3}\right)\right)$$
$$\times\left(\sqrt{\gamma_1^2-4\gamma_2\gamma_3}\sinh\left(\xi\sqrt{\gamma_1^2-4\gamma_2\gamma_3}\right)\right.$$
$$\left.-\left(\gamma_1\cosh\left(\xi\sqrt{\gamma_1^2-4\gamma_2\gamma_3}\right)\pm i\sqrt{\gamma_1^2-4\gamma_2\gamma_3}\right)\right)^{-1}\Bigg\}\Bigg]^2. \tag{25}$$

Type II: when $\gamma_1^2-4\gamma_2\gamma_3<0$, $\gamma_1\gamma_2\neq 0$, $\gamma_2\gamma_3\neq 0$. The following families of periodic solitons are obtained

$$u_{13}^{\mathrm{I}}(x,t)=\left[\zeta_0-\zeta_1\left\{\frac{\sqrt{4\gamma_2\gamma_3-\gamma_1^2}\tan\left(\frac{1}{2}\xi\sqrt{4\gamma_2\gamma_3-\gamma_1^2}\right)-\gamma_1}{2\gamma_2}\right\}\right]^2. \tag{26}$$

$$u_{20}^{\mathrm{I}}(x,t)=\left[\zeta_0-\zeta_1\left\{\frac{2\gamma_3\cos\left(\frac{1}{2}\xi\sqrt{4\gamma_2\gamma_3-\gamma_1^2}\right)}{\sqrt{4\gamma_2\gamma_3-\gamma_1^2}\sin\left(\frac{1}{2}\xi\sqrt{4\gamma_2\gamma_3-\gamma_1^2}\right)+\gamma_1\cos\left(\frac{1}{2}\xi\sqrt{4\gamma_2\gamma_3-\gamma_1^2}\right)}\right\}\right]^2. \tag{27}$$

$$\Phi_{24}^{\mathrm{I}}(x,t)=\Bigg[\zeta_0-\zeta_1\Bigg\{4r\sin\left(\frac{1}{4}\xi\sqrt{4\gamma_2\gamma_3-\gamma_1^2}\right)\cos\left(\frac{1}{4}\xi\sqrt{4\gamma_2\gamma_3-\gamma_1^2}\right)\Bigg)$$
$$\times\Bigg(2\sqrt{4\gamma_2\gamma_3-\gamma_1^2}\cos^2\left(\frac{1}{4}\xi\sqrt{4\gamma_2\gamma_3-\gamma_1^2}\right)-2\gamma_1\sin\left(\frac{1}{4}\xi\sqrt{4\gamma_2\gamma_3-\gamma_1^2}\right)$$
$$\times\cos\left(\frac{1}{4}\xi\sqrt{4\gamma_2\gamma_3-\gamma_1^2}\right)-\sqrt{4\gamma_2\gamma_3-\gamma_1^2}\Bigg)\Bigg)^{-1}\Bigg\}\Bigg]^2. \tag{28}$$

Case II. If $\ell_0=\gamma_3^2,\ell_1=2\gamma_1\gamma_3,\ell_2=0,\ell_3=2\gamma_1\gamma_2,\ell_4=\gamma_2^2$, u is one of the u_η^{II}, $(\eta=1,2,\ldots,12)$. A family of dark optical soliton is obtained

$$u_1^{\mathrm{II}}(x,t)=\left[\zeta_0-\zeta_1\left\{\frac{\sqrt{-6\gamma_2\gamma_3}\tanh\left(\frac{1}{2}\xi\sqrt{-6\gamma_2\gamma_3}\right)+\sqrt{-2\gamma_2\gamma_3}}{2\gamma_2}\right\}\right]^2, \tag{29}$$

another form of dark-singular optical soliton is obtained

$$u_5^{\mathrm{II}}(x,t) = \left[\zeta_0 - \zeta_1\left\{\sqrt{-6\gamma_2\gamma_3}\left(\tanh\left(\frac{1}{4}\xi\sqrt{-6\gamma_2\gamma_3}\right)\right.\right.\right.$$
$$\left.\left.\left.+ \coth\left(\frac{1}{4}\xi\sqrt{-6\gamma_2\gamma_3}\right)\right) + 2\sqrt{-2\gamma_2\gamma_3}\right\}\right]^2. \tag{30}$$

Case III. If $\ell_0 = \ell_1 = 0$, we have the following solution of (2) in the form u_η^{III}, $(\eta = 1, 2, \ldots, 10)$:

Type I: $\ell_2 = 1, \ell_3 = \frac{-2\varkappa_3}{\varkappa_1}, \ell_4 = \frac{\varkappa_3^2 - \varkappa_2^2}{\varkappa_1^2}$, where $\varkappa_1, \varkappa_2, \varkappa_3$ are arbitrary constants.

$$u_1^{\mathrm{III}}(x,t) = \left[\zeta_0 - \zeta_1\left(\frac{\varkappa_1 \operatorname{sech}(\xi)}{\varkappa_2 \operatorname{sech}(\xi) + \varkappa_3}\right)\right]^2. \tag{31}$$

Type II: $\ell_2 = 1, \ell_3 = \frac{-2\varkappa_3}{\varkappa_1}, \ell_4 = \frac{\varkappa_3^2 + \varkappa_2^2}{\varkappa_1^2}$, where $\varkappa_1, \varkappa_2, \varkappa_3$ are arbitrary constants.

$$u_2^{\mathrm{III}}(x,t) = \left[\zeta_0 - \zeta_1\left(\frac{\varkappa_1 \operatorname{csch}(\xi)}{\varkappa_2 \operatorname{csch}(\xi) + \varkappa_3}\right)\right]^2. \tag{32}$$

In particular, if we take $\varkappa_2 = 0$ in Eqs. (31) and (32). We obtain the families of bright and singular optical solitons as follows:

$$u_1^{\mathrm{III}}(x,t) = \left[\zeta_0 - \zeta_1\left(\frac{\varkappa_1 \operatorname{sech}(\xi)}{\varkappa_3}\right)\right]^2, \tag{33}$$

$$u_2^{\mathrm{III}}(x,t) = \left[\zeta_0 - \zeta_1\left(\frac{\varkappa_1 \operatorname{csch}(\xi)}{\varkappa_3}\right)\right]^2. \tag{34}$$

Type III: $\ell_2 = 4, \ell_3 = -\frac{4(2\varkappa_2 + \varkappa_4)}{\varkappa_1}, \ell_4 = \frac{4\varkappa_2^2 + 4\varkappa_4\varkappa_2 + \varkappa_3^2}{\varkappa_1^2}$, where $\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4$ are arbitrary constants.

$$u_3^{\mathrm{III}}(x,t) = \left[\zeta_0 - \zeta_1\left(\frac{\varkappa_1 \operatorname{sech}^2(\xi)}{\varkappa_2 \tanh(\xi) + \varkappa_3 + \varkappa_4 \operatorname{sech}^2(\xi)}\right)\right]^2. \tag{35}$$

Type IV: $\ell_2 = 4, \ell_3 = \frac{4(\varkappa_4 - 2\varkappa_2)}{\varkappa_1}, \ell_4 = \frac{4\varkappa_2^2 - 4\varkappa_4\varkappa_2 + \varkappa_3^2}{\varkappa_1^2}$, where $\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4$ are arbitrary constants.

$$u_4^{\mathrm{III}}(x,t) = \left[\zeta_0 - \zeta_1\left(\frac{\varkappa_1 \operatorname{csch}^2(\xi)}{\varkappa_2 \coth(\xi) + \varkappa_3 + \varkappa_4 \operatorname{csch}^2(\xi)}\right)\right]^2. \tag{36}$$

In particular, if we consider $\varkappa_2 = \varkappa_4$; another family of dark and singular optical solitons are obtained as follows:

$$u_4^{\mathrm{III}}(x,t) = \left[\zeta_0 - \zeta_1\left(\frac{\varkappa_1 \operatorname{csch}^2(\xi)}{\varkappa_2 \coth(\xi) + \varkappa_3 + \varkappa_2 \operatorname{csch}^2(\xi)}\right)\right]^2. \tag{37}$$

Type V: $\ell_2 = -1, \ell_3 = \frac{2\varkappa_3}{\varkappa_1}, \ell_4 = \frac{\varkappa_3^2 - \varkappa_2^2}{\varkappa_1^2}$, where $\varkappa_1, \varkappa_2, \varkappa_3$ are arbitrary constants.

$$u_6^{\mathrm{III}}(x,t) = \left[\zeta_0 + \frac{\zeta_1}{\varkappa_3}(\varkappa_1(\sinh(\varkappa_1\xi) + \cosh(\varkappa_1\xi))(\sinh(\varkappa_1\xi) + \cosh(\varkappa_1\xi) + \varkappa_2))\right]^2. \tag{38}$$

Type VI: $\ell_2 = 4, \ell_3 = \frac{-2\varkappa_3}{\varkappa_1}, \ell_4 = \frac{\varkappa_3^2 - \varkappa_2^2}{\varkappa_1^2}$, where $\varkappa_1, \varkappa_2, \varkappa_3$ are arbitrary constants.

$$u_8^{\mathrm{III}}(x,t) = \left[\zeta_0 - \zeta_1\left(\frac{\varkappa_1 \csc(\xi)}{\varkappa_2 \csc(\xi) + \varkappa_3}\right)\right]^2. \tag{39}$$

Type VII: $\ell_2 = -4, \ell_3 = \frac{4(2\varkappa_2 + \varkappa_4)}{\varkappa_1}, \ell_4 = -\frac{4\varkappa_2^2 + 4\varkappa_4\varkappa_2 - \varkappa_3^2}{\varkappa_1^2}$, where $\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4$ are arbitrary constants.

$$u_9^{\mathrm{III}}(x,t) = \left[\zeta_0 - \zeta_1\left(\frac{\varkappa_1 \sec^2(\xi)}{\varkappa_2 \tan(\xi) + \varkappa_3 + \varkappa_4 \sec^2(\xi)}\right)\right]^2. \tag{40}$$

Case IV. If $\ell_1 = \ell_3 = 0$, we have the following solution of (2) in the form u_η^{IV}, $(\eta = 1, 2, \ldots, 16)$.

For $\ell_0 = \frac{1}{4}, \ell_2 = \frac{1-2m^2}{2}, \ell_4 = \frac{1}{4}$, the solutions of (2) is of the form

$$u_3^{\mathrm{IV}}(\xi) = (\zeta_0 - \zeta_1(cn\xi))^2, \tag{41}$$

gives the bright optical soliton for $m \to 1$,

$$u_3^{\mathrm{IV}}(\xi) = (\zeta_0 - \zeta_1 \operatorname{sech}(\xi))^2, \tag{42}$$

and the periodic singular solutions for $m \to 0$,

$$u_3^{\mathrm{IV}}(\xi) = (\zeta_0 - \zeta_1 \cos(\xi))^2, \tag{43}$$

for $\ell_0 = \frac{1}{4}, \ell_2 = \frac{1-2m^2}{2}, \ell_4 = \frac{1}{4}$, the solutions of (2) is of the form

$$u_{13}^{\mathrm{IV}}(\xi) = (\zeta_0 - \zeta_1(ns\xi \pm cs\xi))^2, \tag{44}$$

gives the combined dark-singular wave solutions for $m \to 1$,

$$u_{13}^{\mathrm{IV}}(\xi) = (\zeta_0 - \zeta_1(\coth(\xi) + \operatorname{csch}(\xi)))^2, \tag{45}$$

and the periodic singular solutions for $m \to 0$,

$$u_{13}^{\mathrm{IV}}(\xi) = (\zeta_0 - \zeta_1(\cot(\xi) + \csc(\xi)))^2. \tag{46}$$

3. Discussion and Results

In this section, the graphical visualization of S-Kdv model have been illustrated with the assist of latest scientific tools. A variety of solitary wave solutions are observed with the aid of the extended modified AEM method and the extended FAN sub-equation method for a set of parameters. The 3D, contour and 2D graphs visualize the physical behavior of nonlinear waves constructed from Eq. (2) for various paymasters viz. the activation convection β, the trapping value α, the dispersion coefficients δ. When the results from Ref. 64 are compared, it can be seen that this study showed a variety of new wave patterns while some of the solutions showed the same behaviors, such as dark solitons as seen in Figs. 7 and 12.

Figures 1 and 9 represent periodic solitary wave $|u_2(x,t)|$ and $|u_{13}^{\mathrm{I}}(x,t)|$ when $\alpha = 5, \beta = 3, \omega = -0.5, \lambda = -2, \delta = 3, \kappa = 1$ while Fig. 2 μ-shaped periodic soliton for $|u_4(x,t)|$ when $\alpha = 5, \beta = 3, \omega = -0.5, \lambda = -2, \delta = 3, \kappa = 1$ whereas Figs. 3, 4 and 13 depict the bell-shaped bright soliton for $|u_5(x,t)|, |u_6(x,t)|$ and $|u_3^{\mathrm{IV}}(x,t)|$ when $\alpha = 5, \beta = 3, \omega = -0.5, \lambda = -2, \delta = 3, \kappa = 1$, respectively.

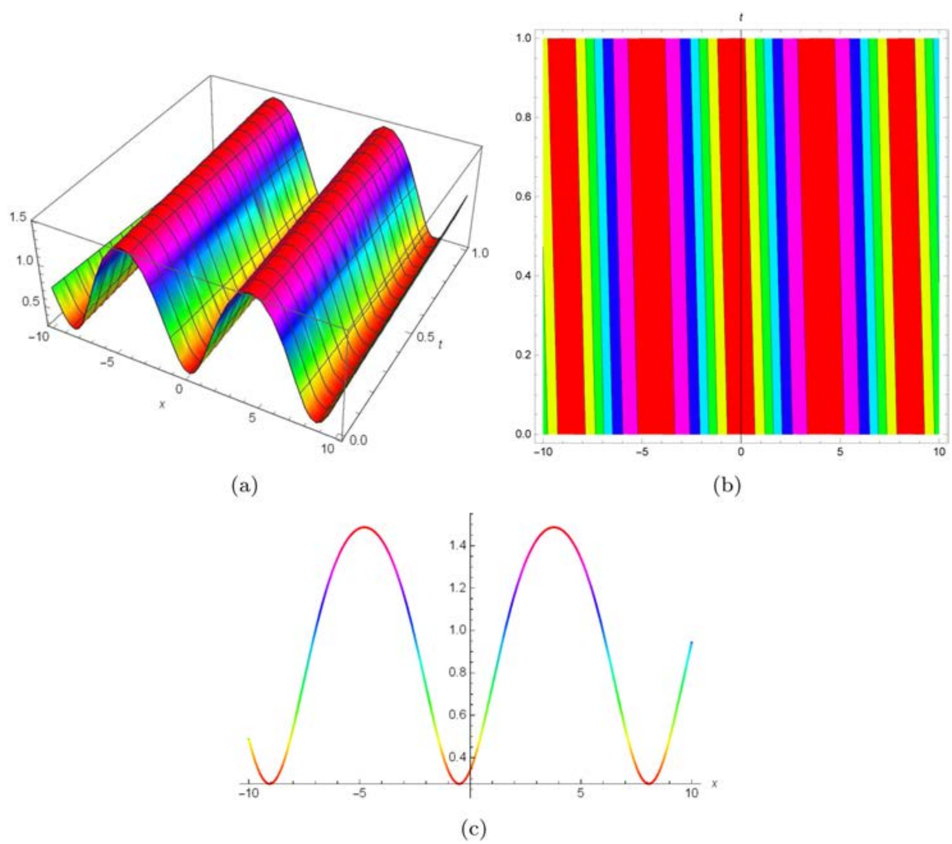


Fig. 1. (Color online) 3D, Contour and 2D plots for $|u_2(x,t)|$ when $\alpha = 5, \beta = 3, \omega = -0.5, \lambda = -2, \delta = 3, \kappa = 1$.

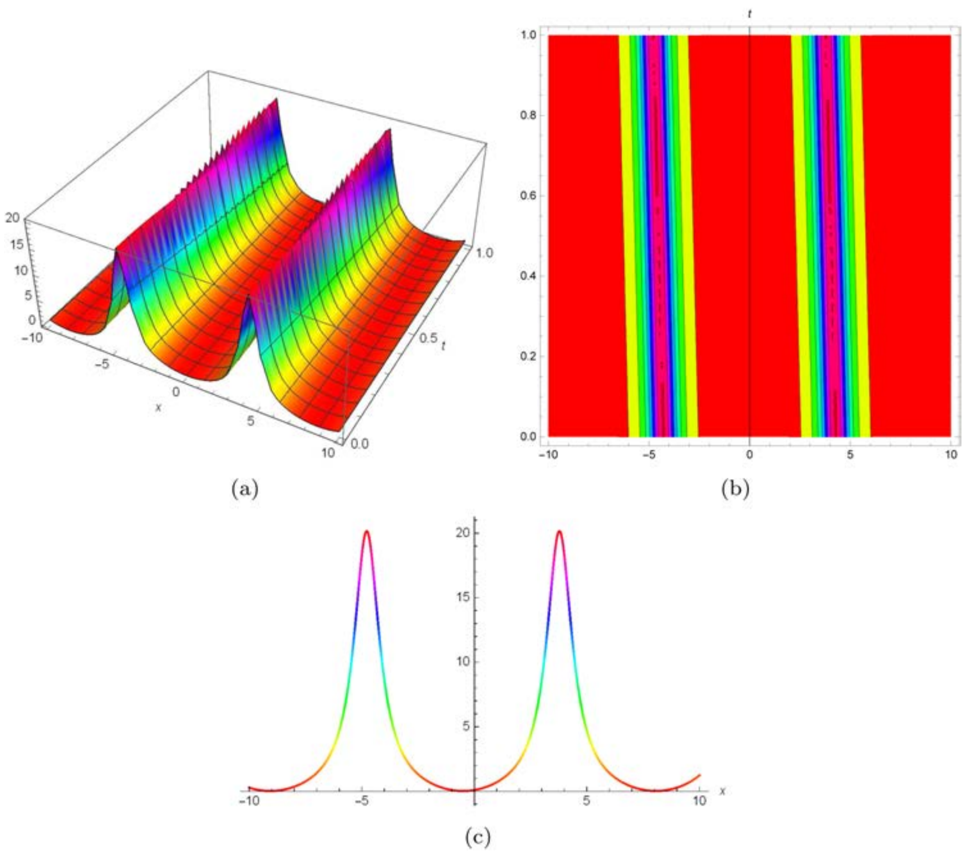


Fig. 2. (Color online) 3D, Contour and 2D plots for $|u_4(x,t)|$ when $\alpha = 5, \beta = 3, \omega = -0.5, \lambda = -2, \delta = 3, \kappa = 1$.

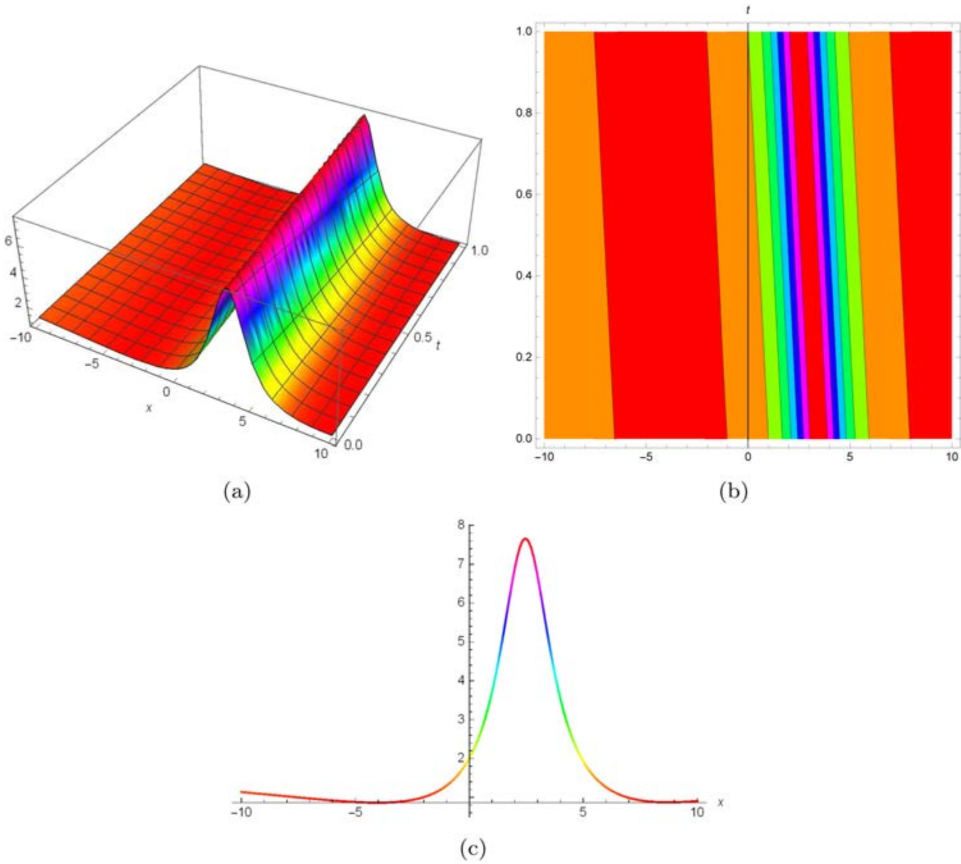


Fig. 3. (Color online) 3D, Contour and 2D plots for $|u_5(x,t)|$ when $\alpha = 5, \beta = 3, \omega = -0.5, \lambda = -2, \delta = 3, \kappa = 1$.

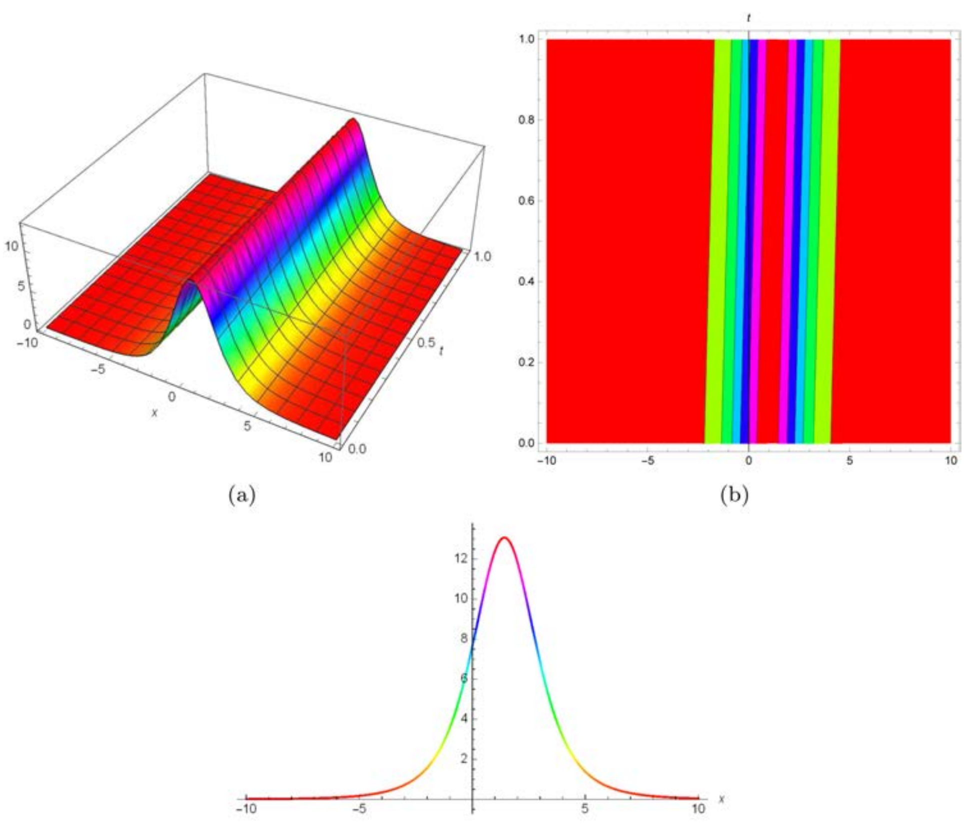


Fig. 4. (Color online) 3D, Contour and 2D plots for $|u_6(x,t)|$ when $\alpha = 5, \beta = 3, \omega = -0.5, \lambda = -2, \delta = 3, \kappa = 1$.

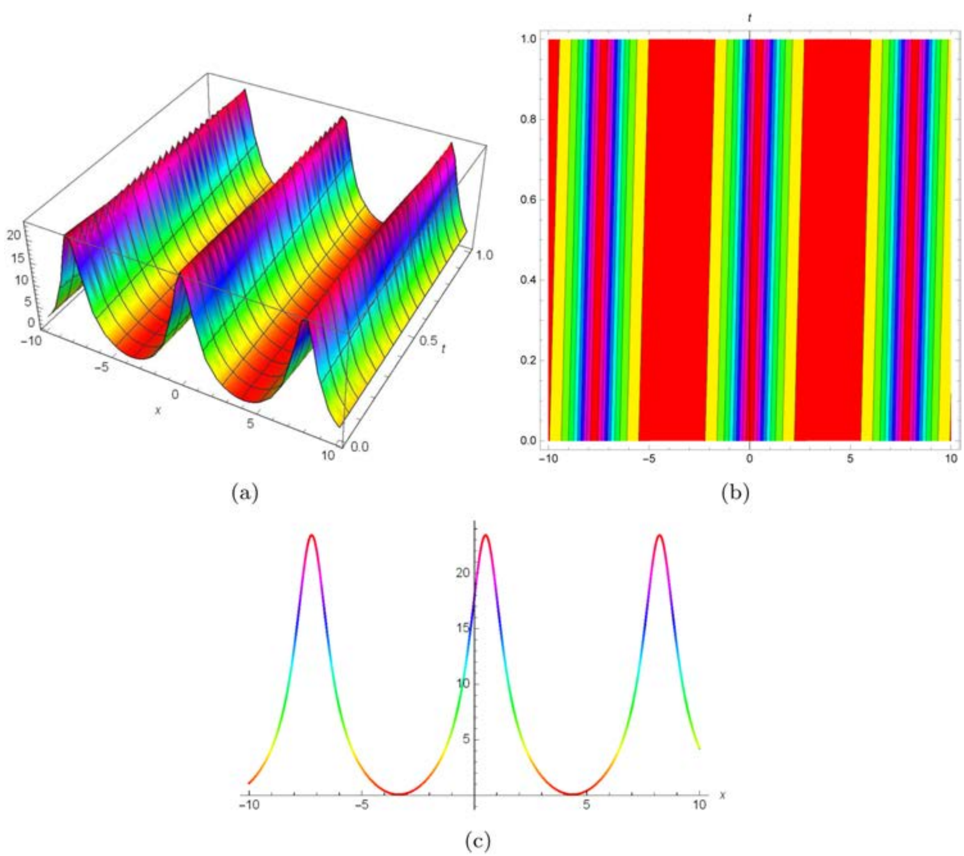


Fig. 5. (Color online) 3D, Contour and 2D plots for $|u_7(x,t)|$ when $\alpha = 5, \beta = 3, \omega = -0.5, \lambda = -2, \delta = 3, \kappa = 1$.

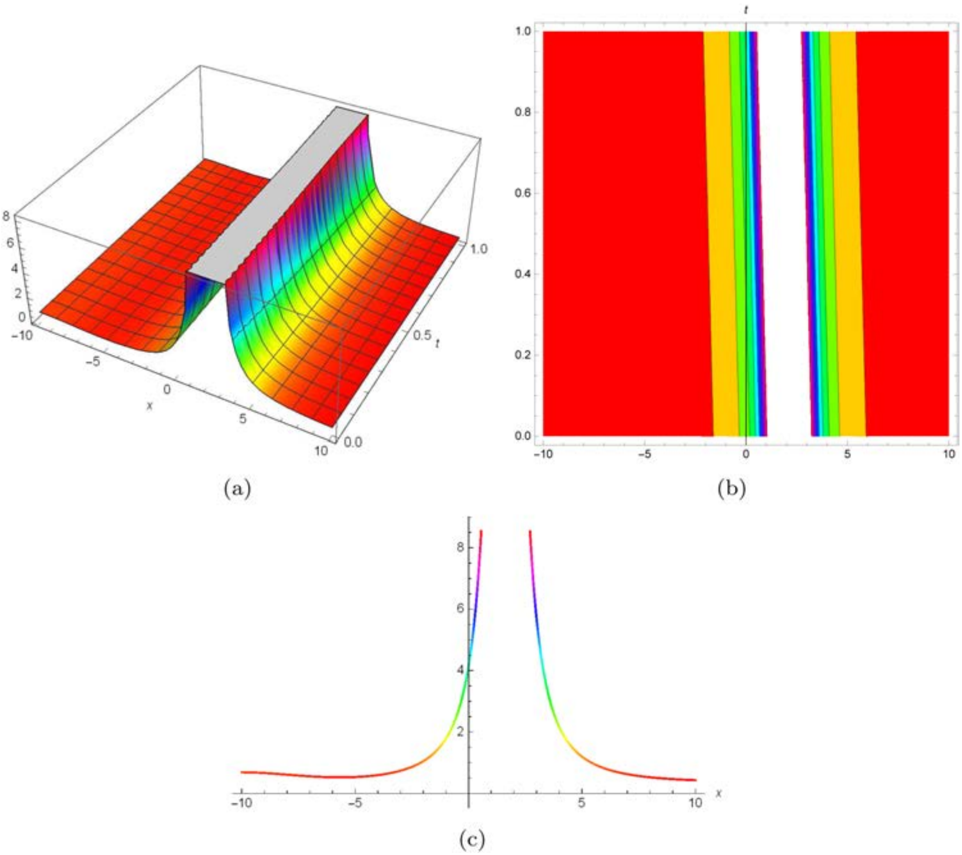


Fig. 6. (Color online) 3D, Contour and 2D plots for $|u_8(x,t)|$ when $\alpha = 5, \beta = 3, \omega = -0.5, \lambda = -2, \delta = 3, \kappa = 1$.

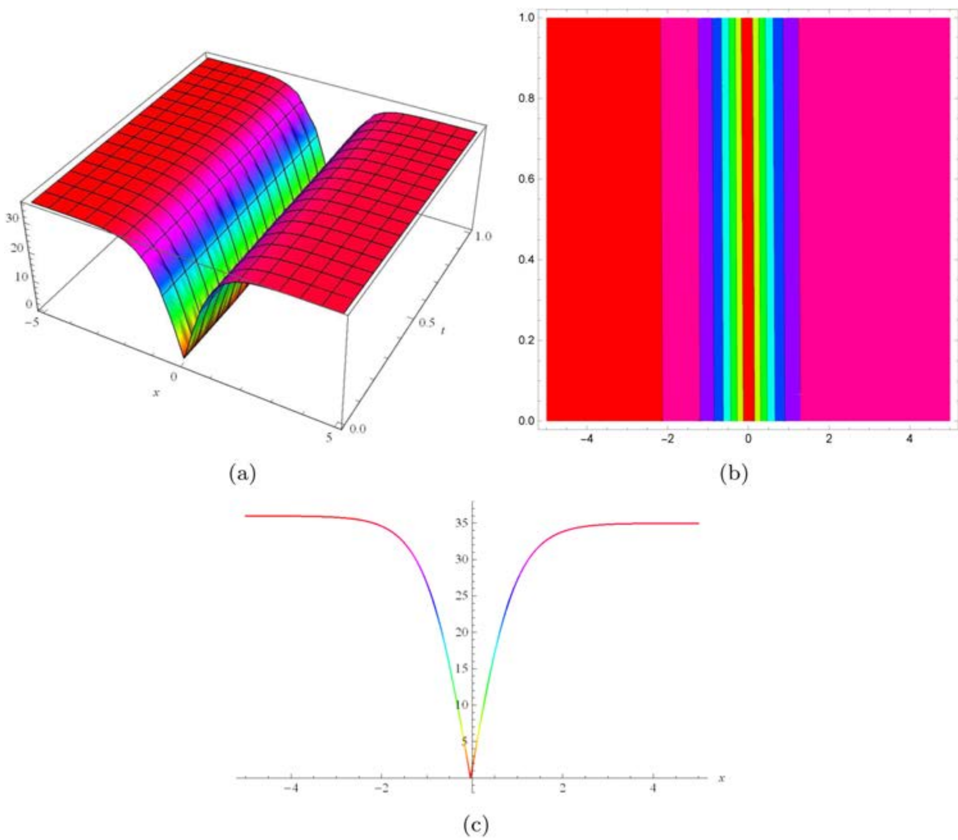


Fig. 7. (Color online) 3D, Contour and 2D plots for $|u_1^{\mathrm{I}}(x,t)|$ when $\alpha = 5, \beta = 3, \omega = -0.5, \lambda = -2, \delta = 3, \kappa = 1$.

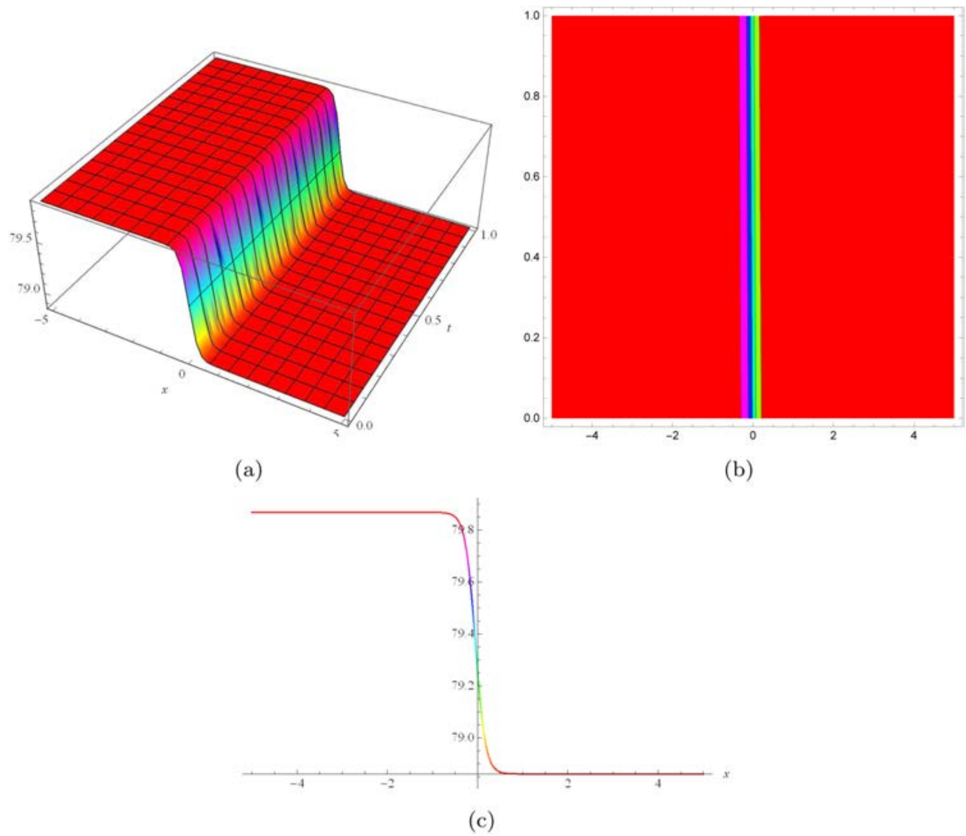


Fig. 8. (Color online) 3D, Contour and 2D plots for $|u_5^{\mathrm{I}}(x,t)|$ when $\alpha = 5, \beta = 3, \omega = -0.5, \lambda = -2, \delta = 3, \kappa = 1$.

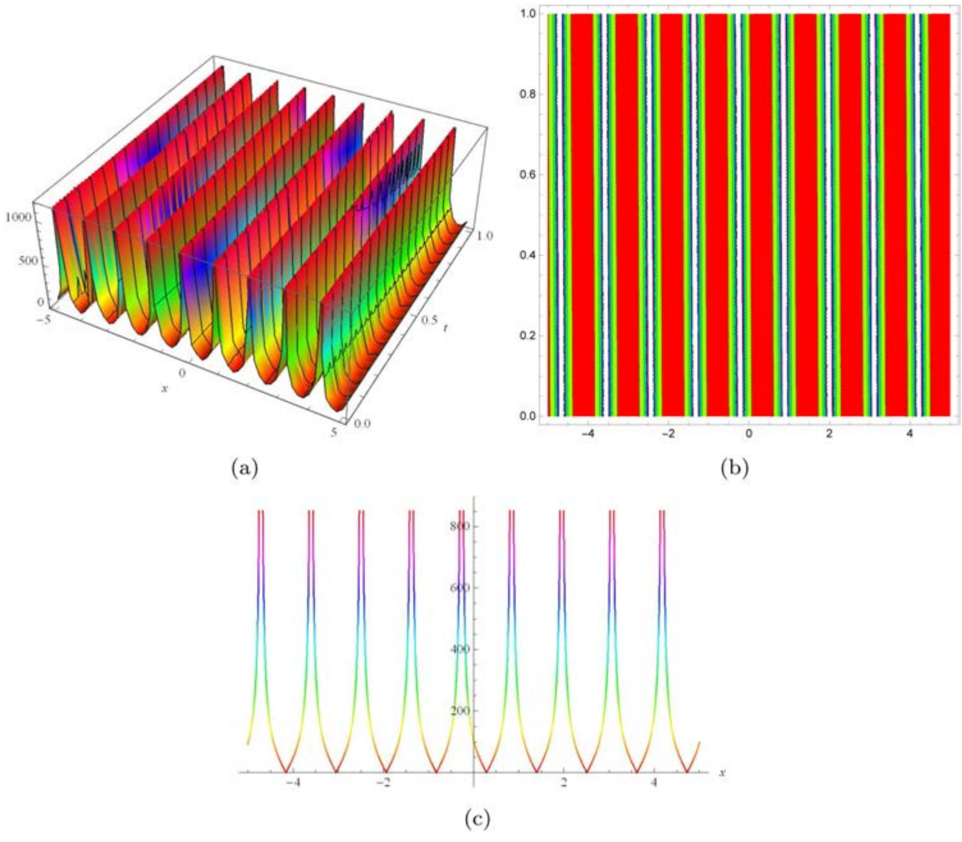


Fig. 9. (Color online) 3D, Contour and 2D plots for $|u_{13}^{\mathrm{I}}(x,t)|$ when $\alpha = 5, \beta = 3, \omega = -0.5, \lambda = -2, \delta = 3, \kappa = 1$.

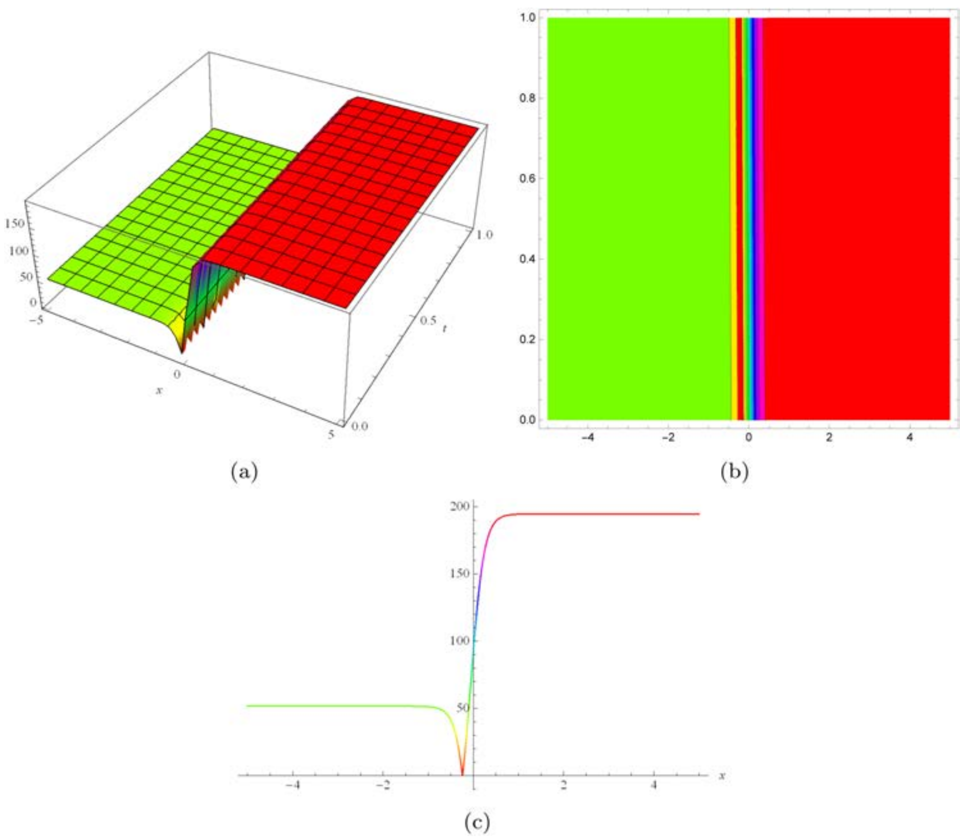
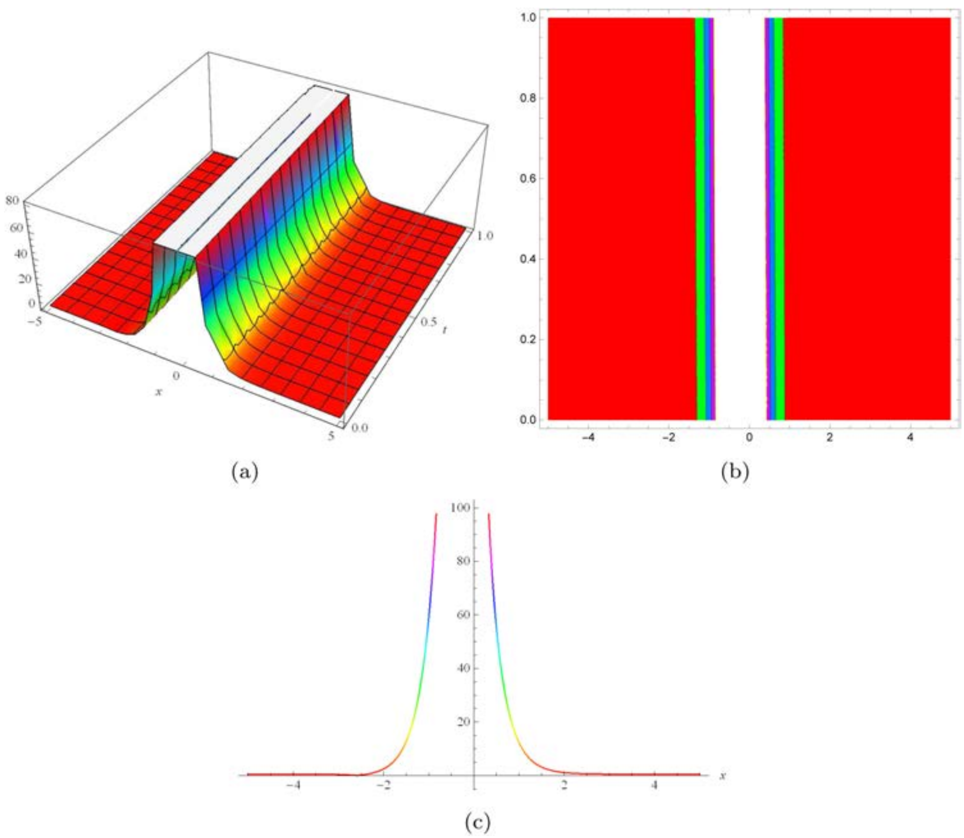


Fig. 10. (Color online) 3D, Contour and 2D plots for $|u_{24}^{\mathrm{I}}(x,t)|$ when $\alpha = 5, \beta = 3, \omega = -0.5, \lambda = -2, \delta = 3, \kappa = 1$.



(a) (b) (c)

Fig. 11. (Color online) 3D, Contour and 2D plots for $|u_1^{\mathrm{II}}(x,t)|$ when $\alpha = 5, \beta = 3, \omega = -0.5, \lambda = -2, \delta = 3, \kappa = 1$.

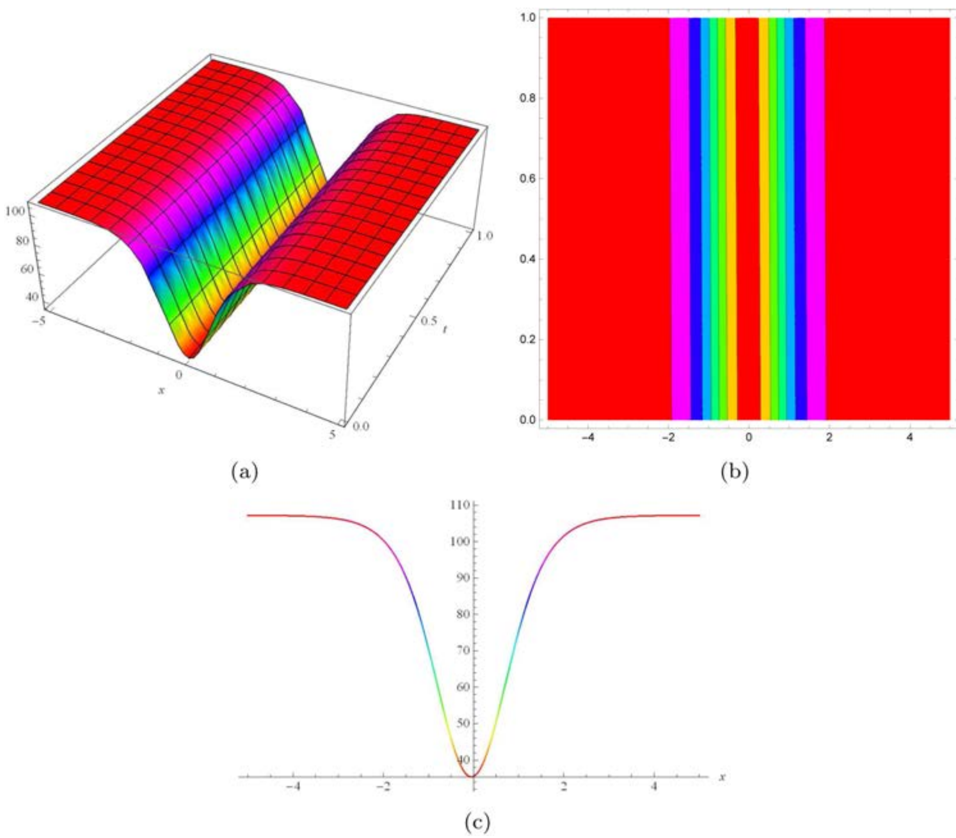


Fig. 12. (Color online) 3D, Contour and 2D plots for $|u_1^{\mathrm{III}}(x,t)|$ when $\alpha = 5, \beta = 3, \omega = -0.5, \lambda = -2, \delta = 3, \kappa = 1$.

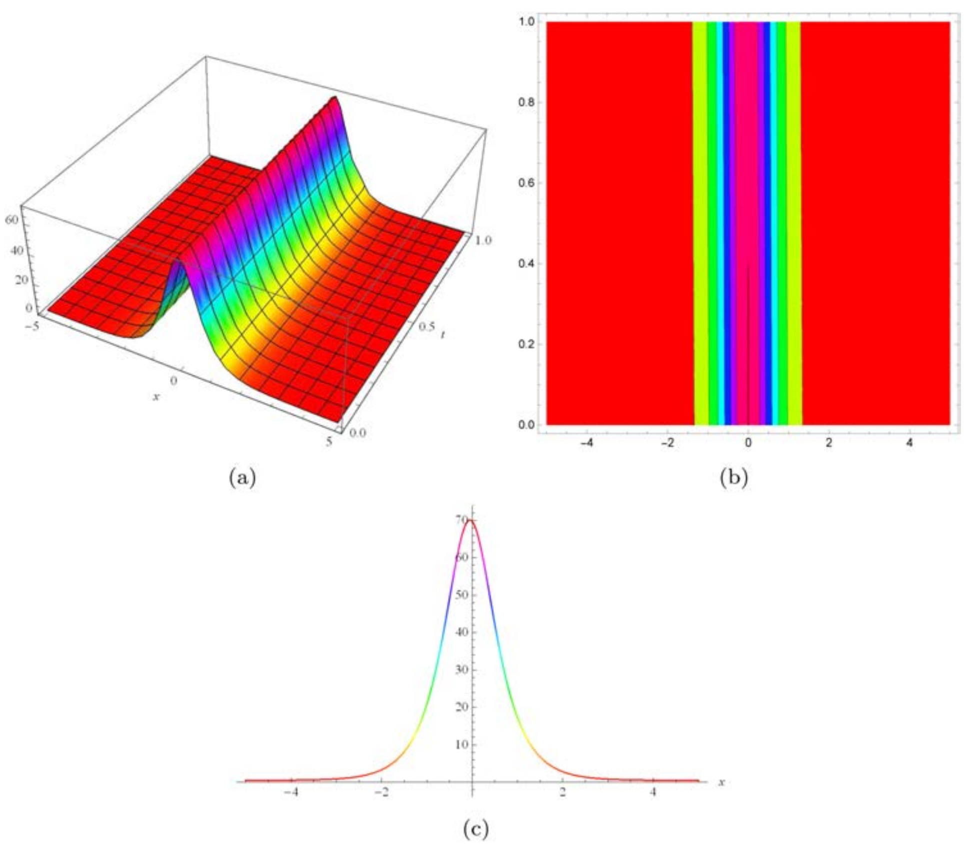
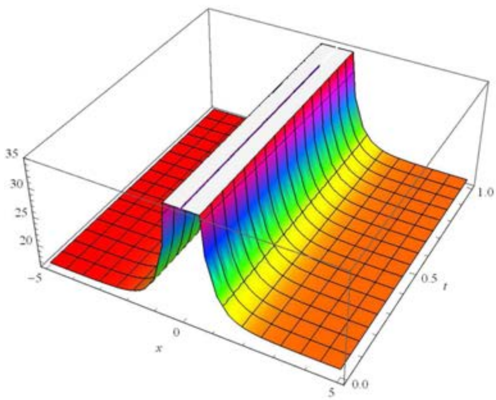
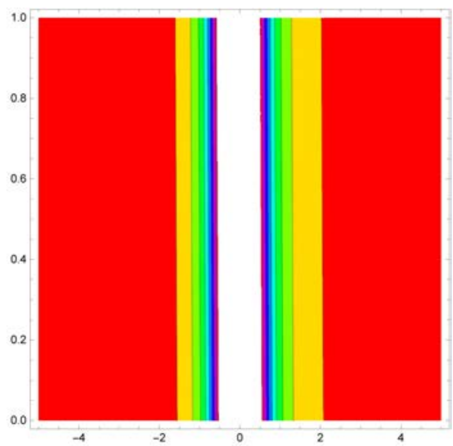


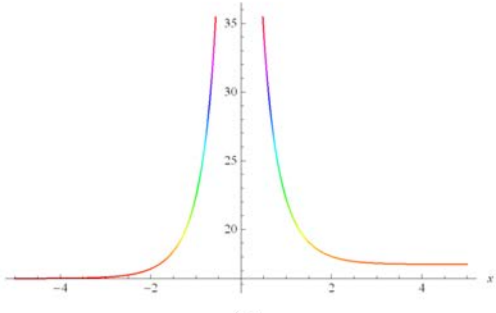
Fig. 13. (Color online) 3D, Contour and 2D plots for $|u_3^{\mathrm{IV}}(x,t)|$ when $\alpha = 5, \beta = 3, \omega = -0.5, \lambda = -2, \delta = 3, \kappa = 1$.



(a)



(b)



(c)

Fig. 14. (Color online) 3D, Contour and 2D plots for $|u_{13}^{\mathrm{IV}}(x,t)|$ when $\alpha = 5, \beta = 3, \omega = -0.5, \lambda = -2, \delta = 3, \kappa = 1$.

Similarly, Fig. 5 visualizes a W-shaped solitary wave for $|u_7(x,t)|$ when $\alpha = 5, \beta = 3, \omega = -0.5, \lambda = -2, \delta = 3, \kappa = 1$ while Figs. 6, 11 and 14 display singular wave for $|u_8(x,t)|, |u_1^{\mathrm{II}}(x,t)|$ and $|u_{13}^{\mathrm{IV}}(x,t)|$ when $\alpha = 5, \beta = 3, \omega = -0.5, \lambda = -2, \delta = 3, \kappa = 1$ whereas Figs. 7 and 12 illustrate dark soliton structures for $|u_1^{\mathrm{I}}(x,t)|$ and $|u_1^{\mathrm{III}}(x,t)|$ when $\alpha = 5, \beta = 3, \omega = -0.5, \lambda = -2, \delta = 3, \kappa = 1$ whilst Figs. 8 and 10 display the solitary wave structure for $|u_5^{\mathrm{I}}(x,t)|$ and $|u_{24}^{\mathrm{I}}(x,t)|$ when $\alpha = 5, \beta = 3, \omega = -0.5, \lambda = -2, \delta = 3, \kappa = 1$.

4. Conclusion

In a cold-ion plasma, the Schamel equation regulates the propagation of ion-acoustic waves where certain electrons do not behave isothermally throughout the wave's passage but are imprisoned. On the other hand, the KdV equation is a mathematical description of shallow water waves. To predict the influence of surface for deep water in the presence of solitary waves, the Schamel-KdV model is considered. This is a general model of weakly nonlinear long waves that contains leading order nonlinearity and dispersion. The extended modified AEM method and the extended FAN sub-equation method are successfully employed to demonstrate the behavior of the nonlinear wave structure by visualizing the 3D, contour and 2D graphical solutions respectively. Mathematica 11.0 is utilized to authenticate the computations and to visualize various wave structures. For a different set of parameters we attained different graphical interpretation of the obtained solutions such as bright, dark, singular, explicit, periodic, combined and peakon type solitary waves. The results established in Ref. 64 are compared and it is observed that this study displayed various new wave structures while some of the solutions have displayed the identical behaviors such as the dark solitons $|u_1^{\mathrm{I}}(x,t)|, |u_1^{\mathrm{III}}(x,t)|$ represented in Fig. 7, 12.

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