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Two new Wronskian conditions for the (3 + 1)-dimensional Jimbo–Miwa equation

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ABSTRACT

In this paper, we obtain two kinds of sufficient conditions consisting of systems of linear partial differential equations, which guarantee that the corresponding Wroskian determinant solves the (3 + 1)-dimensional Jimbo–Miwa equation in the Hirota bilinear form. Our results suggest that more general conditions could be derived by further study.

1. Introduction

Soliton equations are one special kind of nonlinear partial differential equations, which are characterized by the existence of solitary wave solutions. It is challenging to obtain explicit solutions to most nonlinear partial differential equations. One well known soliton equation is the Korteweg-de Vries (KdV) equation, which can go back to the observation by Russell. After the celebrated inverse scattering method was constructed in the 1960s, the area of soliton equations and integrable systems grew very rapidly, and have deeply influenced many branches of mathematics and physics. Compared with the analytical approach, the Hirota bilinear method is more straightforward and easily handled to get the explicit soliton solutions. Furthermore, the beauty of algebra hidden in the soliton equations is found by M. Sato, who got his τ function and Grassmannian solutions from the Hirota bilinear form.

The (3 + 1)-dimensional Jimbo–Miwa equation

$$u_{xxxy} + 3u_{xx}u_y + 3u_xu_{xy} + 2u_{yt} - 3u_{xz} = 0 \tag{1.1}$$

was firstly investigated by Jimbo–Miwa and its soliton solutions were obtained in [11]. It is the second member in the entire KP hierarchy and it was studied in a series of papers [12–16]. Ma [15] proposed a direct approach to solve Eq. (1.1). Wazwaz [16] employed the Hirota's bilinear method to obtain multiple-soliton solutions.

We recall Hirota's bilinear operators [17] defined by

$$D_xD_yf\cdot g = (\partial_{x'} - \partial_{y'})(\partial_{y'} - \partial_{x'})f(x',y')g(x',y')\big|_{x'=x,y'=y} = \partial_{x'}\partial_{y'}f(x+x',y+y')g(x-x',y-y')\big|_{x'=y'=0}. \tag{1.2}$$

For instance, we have

$$D_xf\cdot g = f_xg - fg_x, \quad D_xD_tf\cdot g = f_{xt}g - f_xg_t - f_tg_x + fg_{xt}.$$

Consider the Cole–Hopf transformation

$$u = 2(\ln f)_x, \tag{1.3}$$

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we get the Hirota bilinear form of Eq. (1.1)

$$(D_x^3D_y + 2D_yD_t - 3D_xD_z)f\cdot f = (f_{xxxy} + 2f_{yt} - 3f_{xz})f - f_{xxx}f_y - 3f_{xxy}f_x - 2f_yf_t + 3f_xf_z + 3f_{xx}f_{xy} = 0. \tag{1.4}$$

Notice that the following (3 + 1)-dimensional nonlinear evolution equation

$$3w_{xz} - (2w_t + w_{xxx} - 2ww_x)_y + 2\left(w_x\partial_x^{-1}w_y\right)_x = 0 \tag{1.5}$$

has the same Hirota bilinear equation (1.4) as Eq. (1.1) under a dependent variable transformation

$$u = -3(\ln f)_{xx}. \tag{1.6}$$

This equation is widely studied by many authors in [5,18,19], our results also provide two new solutions to this equation.

The Wronskian technique is a useful tool to construct exact solutions to bilinear differential equations [1–4]. It has been applied to many soliton equations such as MKdV, NLS, derivative NLS, sine-Gordon and other equations [5–10]. In the process of utilizing Wronskian technique, the main difficulty lies in the construction of a system of linear differential conditions, which are not unique. In [5,19], the authors presented different linear differential conditions for the N-th order Wronskian determinant solutions of Eq. (1.4). In [20], the Wronskian determinant solutions of Eq. (1.4) under a set of linear differential conditions were obtained.

In this paper, we get another two new linear differential conditions for the N-th order Wronskian determinant solutions of Eq. (1.4). Our results show that Eq. (1.4) has diverse Wronskian determinant solutions under different linear differential conditions, and there is a promise that we can find a broader class of linear differential conditions for Eq. (1.4).

2. The first Wronskian conditions for Eq. (1.4)

We use the Wronskian technique in the compact notation introduced by Freeman and Nimmo [1, 21]:

$$(\phi_1,\phi_2,\ldots,\phi_N) = \left|\widehat{N-1}\right| = \begin{vmatrix} \phi_1^{(0)} & \phi_1^{(1)} & \cdots & \phi_1^{(N-1)} \\ \phi_2^{(0)} & \phi_2^{(1)} & \cdots & \phi_2^{(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N^{(0)} & \phi_N^{(1)} & \cdots & \phi_N^{(N-1)} \end{vmatrix}, \quad N \geqslant 1, \tag{2.1}$$

where

$$\phi_i^{(j)} = \frac{\partial^j}{\partial x^j}\phi_i, \quad 0 \leqslant j \leqslant N-1,\ 1 \leqslant i \leqslant N. \tag{2.2}$$

Solutions determined by $f = |\widehat{N-1}|$ to Eq. (1.4) are called Wronskian determinant solutions.

Theorem 2.1. *Let a set of functions $\phi_i = \phi_i(x,y,z,t),\ 1 \leqslant i \leqslant N$ satisfy the following linear conditions:*

$$\phi_{i,xx} = \sum_{j=1}^{N}\lambda_{ij}\phi_j, \tag{2.3}$$

$$\begin{cases} \phi_{i,y} = k\phi_{i,x}, \\ \phi_{i,t} = \phi_{i,xx}, \\ \phi_{i,z} = \frac{4k}{3}\phi_{i,xxx} + \frac{2k}{3}\phi_{i,xx}, \end{cases} \tag{2.4}$$

where the coefficient matrix $A = (\lambda_{ij})_{1\leqslant i,j\leqslant N}$ is an arbitrary real constant matrix and k is an arbitrary nonzero constant. Then the Wronskian determinant $f = |\widehat{N-1}|$ defined by (2.1) solves Eq. (1.4).

The proof of Theorem 2.1 need the following two Lemmas, which can be derived by straight computation.

Lemma 2.1 [1]. *Set $a_j, j = 1,2,\ldots,N$ to be an N-dimensional column vector, and $b_j, j = 1,2,\ldots,N$ to be a real constant but not to be zero. Then we have*

$$\sum_{i=1}^{N} b_i|a_1,a_2,\ldots,a_N| = \sum_{j=1}^{N}|a_1,a_2,\ldots,ba_j,\ldots,a_N|,$$

where $ba_j = (b_1a_{1j}, b_2a_{2j},\ldots,b_Na_{Nj})^T$.

Lemma 2.2 [22]. *Under the condition (2.3) and Lemma 2.1, the following equalities hold:*

$$(\widehat{N-1})\sum_{i=1}^{N}\lambda_{ii}\left(\sum_{i=1}^{N}\lambda_{ii}(\widehat{N-1})\right)=\left(\sum_{i=1}^{N}\lambda_{ii}|\widehat{N-1}|\right)^2=\left(-|\widehat{N-3},N-1,N|+|\widehat{N-2},N+1|\right)^2$$
$$=|\widehat{N-1}|\Big(|\widehat{N-5},N-3,N-2,N-1,N|-|\widehat{N-4},N-2,N-1,N+1|$$
$$+2|\widehat{N-3},N,N+1|-|\widehat{N-3},N-1,N+2|+|\widehat{N-2},N+3|\Big).$$

Proof of Theorem 2.1. Under the properties of the Wronskian determinant and the conditions (2.3) and (2.4), we can compute various derivatives of the Wronskian determinant $f=|\widehat{N-1}|$ with respect to the variables x,y,z,t as follows:

$$f_x=|\widehat{N-2},N|,\ f_{xx}=|\widehat{N-3},N-1,N|+|\widehat{N-2},N+1|,$$
$$f_{xxx}=|\widehat{N-4},N-2,N-1,N|+2|\widehat{N-3},N-1,N+1|+|\widehat{N-2},N+2|,$$
$$f_y=k|\widehat{N-2},N|,\ f_{xy}=k(|\widehat{N-3},N-1,N|+|\widehat{N-2},N+1|),$$
$$f_{xxy}=k(|\widehat{N-4},N-2,N-1,N|+2|\widehat{N-3},N-1,N+1|+|\widehat{N-2},N+2|),$$
$$f_{xxxy}=k(|\widehat{N-5},N-3,N-2,N-1,N|+3|\widehat{N-4},N-2,N-1,N+1|+2|\widehat{N-3},N,N+1|+3|\widehat{N-3},N-1,N+2|+|\widehat{N-2},N+3|),$$
$$f_t=|\widehat{N-2},N+1|-|\widehat{N-3},N-1,N|,$$
$$f_{yt}=k(|\widehat{N-2},N+2|-|\widehat{N-4},N-2,N-1,N|),$$
$$f_z=\frac{4k}{3}(|\widehat{N-4},N-2,N-1,N|-|\widehat{N-3},N-1,N+1|+|\widehat{N-2},N+2|)+\frac{2k}{3}(|\widehat{N-2},N+1|-|\widehat{N-3},N-1,N|),$$
$$f_{xz}=\frac{4k}{3}(|\widehat{N-5},N-3,N-2,N-1,N|-|\widehat{N-3},N,N+1|+|\widehat{N-2},N+3|)+\frac{2k}{3}(|\widehat{N-2},N+2|-|\widehat{N-4},N-2,N-1,N|),$$

Therefore,

$$(f_{xxxy}+2f_{yt}-3f_{xz})f=3k(-|\widehat{N-5},N-3,N-2,N-1,N|+|\widehat{N-4},N-2,N-1,N+1|+2|\widehat{N-3},N,N+1|+|\widehat{N-3},N-1,N+2|-|\widehat{N-2},N+3|)|\widehat{N-1}|,$$
$$-f_{xxx}f_y-3f_{xxy}f_x-2f_yf_t+3f_xf_z=-12k|\widehat{N-3},N-1,N+1||\widehat{N-2},N|,$$
$$3f_{xx}f_{xy}=3k(|\widehat{N-3},N-1,N|+|\widehat{N-2},N+1|)^2=3k(-|\widehat{N-3},N-1,N|+|\widehat{N-2},N+1|)^2+12k|\widehat{N-3},N-1,N||\widehat{N-2},N+1|,$$

Thus we obtain by using Lemma 2.2,

$$\left(D_x^3D_y+2D_yD_t-3D_xD_z\right)f\cdot f$$
$$=12k(|\widehat{N-3},N,N+1||\widehat{N-1}|-|\widehat{N-3},N-1,N+1||\widehat{N-2},N|+|\widehat{N-3},N-1,N||\widehat{N-2},N+1|)=0. \tag{2.5}$$

This last equality is nothing but the Plücker relation for determinants:

$$|B,A_1,A_2||B,A_3,A_4|-|B,A_1,A_3||B,A_2,A_4|+|B,A_1,A_4||B,A_2,A_3|=0, \tag{2.6}$$

where B denotes an $N\times(N-2)$ matrix, and A_i, $1\leqslant i\leqslant 4$ are four N-dimensional column vectors. Therefore, we have shown that $f=|\widehat{N-1}|$ solves Eq. (1.4) under the linear differential conditions (2.3) and (2.4).

The corresponding solution of Eq. (1.1) is

$$u=2(\ln f)_x,\quad f=|\widehat{N-1}|,$$

and the corresponding solution of Eq. (1.5) is

$$u=-3(\ln f)_{xx},\quad f=|\widehat{N-1}|.$$

From the linear differential conditions (2.3) and (2.4) as well as the transformation (1.3), we can compute the exact Wronskian solutions including rational solutions, solitons, negatons and positons of Eq. (1.1).

For example, if we let the coefficient matrix $A=(\lambda_{ij})_{1\leqslant i,j\leqslant N}$ in Theorem 2.1 has the following form of matrix (cf. [10,20,22,23] for details)

$$A=\begin{pmatrix} J(\lambda_1) & & & 0\\ 0 & J(\lambda_2) & & \\ & \ddots & \ddots & \\ 0 & & 0 & J(\lambda_m)\end{pmatrix}_{N\times N} \tag{2.7}$$

and

$$J(\lambda_1)=\begin{pmatrix}\lambda_1 & & & 0\\ 0 & \lambda_1 & & \\ & \ddots & \ddots & \\ 0 & & 0 & \lambda_1\end{pmatrix}_{k_1\times k_1}, \tag{2.8}$$

where $\lambda_1\neq 0$. Using the same method as that in [20], we can compute that two 1-solitons of zero-order for Eq. (1.1)

$$u=2\partial_x\ln\left(\cosh\left(\frac{4k}{3}\lambda_1^{\frac{3}{2}}z+\frac{2k}{3}\lambda_1 z+k\sqrt{\lambda_1}y+\sqrt{\lambda_1}x+\lambda_1 t+\gamma_1\right)\right)=2\sqrt{\lambda_1}\tanh\left(\frac{4k}{3}\lambda_1^{\frac{3}{2}}z+\frac{2k}{3}\lambda_1 z+k\sqrt{\lambda_1}y+\sqrt{\lambda_1}x+\lambda_1 t+\gamma_1\right) \tag{2.9}$$

and

$$u=2\partial_x\ln\left(\sinh\left(\frac{4k}{3}\lambda_1^{\frac{3}{2}}z+\frac{2k}{3}\lambda_1 z+k\sqrt{\lambda_1}y+\sqrt{\lambda_1}x+\lambda_1 t+\gamma_1\right)\right)=2\sqrt{\lambda_1}\coth\left(\frac{4k}{3}\lambda_1^{\frac{3}{2}}z+\frac{2k}{3}\lambda_1 z+k\sqrt{\lambda_1}y+\sqrt{\lambda_1}x+\lambda_1 t+\gamma_1\right), \tag{2.10}$$

with γ_1 being a constant and k being an arbitrary nonzero number. It is obvious that solutions (2.9) and (2.10) are different to the two 1-solitons (3.9) and (3.10) of zero-order in [20]. We can construct the other exact solutions of Eq. (1.1) in the same way as in [20]. Using the linear differential conditions (2.3) and (2.4) as well as the transformation (1.6), we can also obtain the exact Wronskian solutions to Eq. (1.5).

3. The second Wronskian conditions for Eq. (1.4)

In this section, we present another linear differential conditions to obtain one new Wronskian determinant solutions of Eq. (1.4).

Theorem 3.1. *Let a group of functions $\phi_i=\phi_i(x,y,z,t)$, $1\leqslant i\leqslant N$, satisfy the following linear differential condition:*

$$\phi_{i,xx}=\sum_{j=1}^{N}\lambda_{ij}\phi_j, \tag{3.1}$$

$$\begin{cases}\phi_{i,y}=k\phi_{i,x},\\ \phi_{i,t}=\phi_{i,xxx},\\ \phi_{i,z}=2k\phi_{i,xxx},\end{cases} \tag{3.2}$$

where the coefficient matrix $A=(\lambda_{ij})_{1\leqslant i,j\leqslant N}$ is an arbitrary real constant matrix and k is also an arbitrary nonzero constant. Then the Wronskian determinant $f=|\widehat{N-1}|$ defined by (2.1) solves Eq. (1.4).

Proof. Under the properties of the Wronskian determinant and the conditions (3.1) and (3.2), we get partial derivatives of the Wronskian determinant $f=|\widehat{N-1}|$ with respect to the variables x,y,z,t as follows

$$\begin{aligned}
&f_x=|\widehat{N-2},N|,\quad f_{xx}=|\widehat{N-3},N-1,N|+|\widehat{N-2},N+1|,\\
&f_{xxx}=|\widehat{N-4},N-2,N-1,N|+2|\widehat{N-3},N-1,N+1|+|\widehat{N-2},N+2|,\\
&f_y=k|\widehat{N-2},N|,\quad f_{xy}=k(|\widehat{N-3},N-1,N|+|\widehat{N-2},N+1|),\\
&f_{xxy}=k(|\widehat{N-4},N-2,N-1,N|+2|\widehat{N-3},N-1,N+1|+|\widehat{N-2},N+2|),\\
&f_{xxxy}=k(|\widehat{N-5},N-3,N-2,N-1,N|+3|\widehat{N-4},N-2,N-1,N+1|+2|\widehat{N-3},N,N+1|+3|\widehat{N-3},N-1,N+2|+|\widehat{N-2},N+3|),\\
&f_t=|\widehat{N-4},N-2,N-1,N|-|\widehat{N-3},N-1,N+1|+|\widehat{N-2},N+2|,\\
&f_{yt}=k(|\widehat{N-5},N-3,N-2,N-1,N|-|\widehat{N-3},N,N+1|+|\widehat{N-2},N+3|),\\
&f_z=2k(|\widehat{N-4},N-2,N-1,N|-|\widehat{N-3},N-1,N+1|+|\widehat{N-2},N+2|),\\
&f_{xz}=2k(|\widehat{N-5},N-3,N-2,N-1,N|-|\widehat{N-3},N,N+1|+|\widehat{N-2},N+3|).
\end{aligned}$$

Therefore,

$$\begin{aligned}
(f_{xxxy}+2f_{yt}-3f_{xz})f&=3k(-|\widehat{N-5},N-3,N-2,N-1,N|+|\widehat{N-4},N-2,N-1,N+1|+2|\widehat{N-3},N,N+1|\\
&\quad+|\widehat{N-3},N-1,N+2|-|\widehat{N-2},N+3|)|\widehat{N-1}|,\\
-f_{xxx}f_y-3f_{xxy}f_x-2f_yf_t+3f_xf_z&=-12k|\widehat{N-3},N-1,N+1||\widehat{N-2},N|,\\
3f_{xx}f_{xy}&=3k(|\widehat{N-3},N-1,N|+|\widehat{N-2},N+1|)^2\\
&=3k(-|\widehat{N-3},N-1,N|+|\widehat{N-2},N+1|)^2+12k|\widehat{N-3},N-1,N||\widehat{N-2},N+1|,
\end{aligned}$$

and further we obtain by using Lemma 2.2,

$$(D_x^3D_y + 2D_yD_t - 3D_xD_z)f \cdot f = 12k(|\widehat{N-3}, N, N+1||\widehat{N-1}| - |\widehat{N-3}, N-1, N+1||\widehat{N-2}, N| + |\widehat{N-3}, N-1, N||\widehat{N-2}, N+1|) = 0.$$

Therefore, $f = |\widehat{N-1}|$ solves Eq. (1.4) under the conditions (3.1) and (3.2).

Similarly, using the linear differential conditions (3.1) and (3.2) as well as the transformation (1.3), we can also compute the exact Wronskian solutions including rational solutions, solitons, negatons and positons of Eq. (1.1). For example, if we let the coefficient matrix A has the following form of matrix

$$A = \begin{pmatrix} J(\lambda_1) & & & 0 \\ 0 & J(\lambda_2) & & \\ & \ddots & \ddots & \\ 0 & & 0 & J(\lambda_m) \end{pmatrix}_{N\times N} \tag{3.3}$$

and

$$J(\lambda_1) = \begin{pmatrix} 0 & & & 0 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{pmatrix}_{k_1\times k_1}. \tag{3.4}$$

Using the same method as that in [20], we can compute a rational solution of zero-order and a rational solution of first-order for Eq. (1.1)

$$u = \frac{2}{x+ky} \tag{3.5}$$

and

$$u = \frac{2(k^2y^2 + 2kxy + x^2)}{k^2xy^2 + kx^2y + \frac{1}{3}x^3 + \frac{1}{3}k^3y^3 - 2kz - t - 1}, \tag{3.6}$$

with k being an arbitrary nonzero number. It is easy to see that solutions (3.5) and (3.6) are different to the rational solution (3.2) of zero-order and the rational solution (3.3) of first-order in [20]. For the other exact solutions of Eq. (1.1), we omit here. Similarly, using the linear differential conditions (3.1) and (3.2) as well as the transformation (1.6), we can also compute the exact Wronskian solutions of Eq. (1.5).

4. Conclusion remarks

In this paper, we establish two new conditions consisting of systems of linear partial differential equations, which guarantee that the corresponding Wroskian determinant solves the (3 + 1)-dimensional Jimbo–Miwa Eq. (1.1). Moreover, these conditions also work for Eq. (1.5). Our results are different from that in [5,19,20]. The question is how can we find a unified way to derive a set of conditions, and what is the relation between them. There should be some more generalized conditions involving combined equations for Wronskian solutions. Searching for such a generalized conditions is under way.

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References

[1] N.C. Freeman, J.J.C. Nimmo, Soliton solutions of the Korteweg-de Vries and Kadomtsev–Petviashvili equations: the Wronskian technique, Phys. Lett. A 95 (1983) 1–3.
[2] V.B. Matveev, Positon–positon and soliton–positon collisions: KdV case, Phys. Lett. A 166 (1992) 205–208. Generalized Wronskian formula forsolutions of the KdV equations: first applications, 166 (1992) 209–212.
[3] W.X. Ma, Wronskians, generalized Wronskians and solutions to the Korteweg-de Vries equation, Chaos Soliton Fract. 19 (2004) 163–170.
[4] W.X. Ma, K. Maruno, Complexiton solutions of the Toda lattice equation, Physica A 343 (2004) 219–237.
[5] X.G. Geng, Y.L. Ma, N-soliton solution and its Wronskian form of a (3 + 1)-dimensional nonlinear evolution equation, Phys. Lett. A 369 (2007) 285–289.
[6] Z.Y. Yan, Multiple solution profiles to the higher-dimensional Kadomtsev–Petviashvilli equations via Wronskian determinant, Chaos Soliton Fract. 33 (2007) 951–957.
[7] J.P. Wu, N-soliton solution, generalized double Wronskian determinant solution and rational solution for a (2 + 1)-dimensional nonlinear evolution equation, Phys. Lett. A 373 (2008) 83–88.

[8] J. Ji, The double Wronskian solutions of a non-isospectral Kadomtsev–Petviashvili equation, Phys. Lett. A 372 (2008) 6074–6081.
[9] Y. Zhang, Y.N. Lv, L.Y. Ye, H.Q. Zhao, The exact solutions to the complex KdV equation, Phys. Lett. A 367 (2007) 447–465.
[10] C.X. Li, W.X. Ma, X.J. Liu, Y.B. Zeng, Wronskian solutions of the Boussinesq equation-solitons, negatons, positons and complexitons, Inverse Probl. 23 (2007) 279–296.
[11] M. Jimbo, T. Miwa, Solitons and infinite dimensional Lie algebra, Publ. Res. I. Math. Sci. 19 (1983) 943–948.
[12] E.G. Fan, An algebraic method for finding a series of exact solutions to integrable and nonintegrable nonlinear evolution equations, J. Phys. A 36 (25) (2003) 7009–7026.
[13] X.Y. Tang, Z.F. Liang, Variable separation solutions for the (3 + 1)-dimensional Jimbo–Miwa equation, Phys. Lett. A 351 (6) (2006) 398–402.
[14] X.B. Hu, D.L. Wang, H.W. Tam, W.M. Xue, Soliton solutions to the Jimbo–Miwa equations and the Fordy–Gibbons–Jimbo–Miwa equation, Phys. Lett. A 262 (1999) 310–320.
[15] W.X. Ma, A transformed rational function method and exact solutions to the 3 + 1 dimensional Jimbo–Miwa equation, Chaos Soliton Fract. 42 (2009) 1356–1363.
[16] A.M. Wazwaz, Multiple-soliton solutions for the Calogero–Bogoyavlenskii–Schiff, Jimbo–Miwa and YTSF equations, Appl. Math. Comput. 203 (2008) 592–597.
[17] R. Hirota, The Direct Method in Soliton Theory, Cambridge University Press, 2004.
[18] X.G. Geng, Algebraic-geometrical solutions of some multidimensional nonlinear evolution equations, J. Phys. A 36 (9) (2003) 2289–2303.
[19] J.P. Wu, A new Wronskian condition for a (3 + 1)-dimensional nonlinear evolution equation, Chin. Phys. Lett. 28 (5) (2011) 050501.
[20] Y.N. Tang, W.X. Ma, W. Xu, L. Gao, Wronskian determinant solutions of the (3 + 1)-dimensional Jimbo–Miwa equation, Appl. Math. Comput. 217 (2011) 8722–8730.
[21] J.J.C. Nimmo, N.C. Freeman, A method of obtaining the N-soliton solution of the Boussinesq equation in terms of a Wronskian, Phys. Lett. A 95 (1983) 4–6.
[22] W.X. Ma, Y. You, Solving the Korteweg-de Vries equation by its bilinear form: Wronskian solutions, Trans. Am. Math. Soc. 357 (2005) 1753–1778.
[23] W.X. Ma, C.X. Li, J.S. He, A second Wronskian formulation of the Boussinesq equation, Nonlinear Anal. 70 (2009) 4245–4258.