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Integrable coupling hierarchy and Hamiltonian structure for a matrix spectral problem with arbitrary-order

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ABSTRACT

We presented an integrable coupling hierarchy of a matrix spectral problem with arbitrary order zero matrix r by using semi-direct sums of matrix Lie algebra. The Hamiltonian structure of the resulting integrable couplings hierarchy is established by means of the component trace identities. As an example, when r is 2×2 zero matrix specially, the integrable coupling hierarchy and its Hamiltonian structure of the matrix spectral problem are computed.

1. Introduction

In recent years, the study of integrable couplings of soliton equations has received considerable attention [1–6]. Integrable couplings are coupled systems of integrable equations which contain given integrable equations as their sub-systems, is one of foremost and pretty interesting topics in the soliton theory. There are much richer mathematical structures behind integrable couplings than scalar integrable equations. Moreover, the study of integrable couplings generalizes the symmetry problem and provides clues towards complete classification of integrable equations.

Mathematically, for a given integrable system of evolution type $u_t = K(u)$, we actually need to construct a new bigger triangular integrable system as follows:

$$\bar{u}_t = \begin{bmatrix} u \\ v \end{bmatrix}_t = \overline{K}(\bar{u}) = \begin{bmatrix} K(u) \\ S(u,v) \end{bmatrix}.$$

The vector-valued function S should satisfy the non-triviality condition $\frac{\partial S}{\partial [u]} \neq 0$, where $[u]$ denotes a vector consisting of all derivatives of u with respect to the space variable. A few ways to construct integrable coupling of solitons equations are presented by perturbation [1,2], enlarging spectral problems [3], constructing new loop Lie algebra [4] and creating semi-direct sums of Lie algebra [5]. Many integrable couplings systems of the well-known integrable hierarchies have been worked out such as AKNS hierarchy, Toda hierarchy, JM hierarchy, KN hierarchy and so on.

Recently, Ma [7–9] studied the following iso-spectral problem with an arbitrary order matrix:

$$\phi_x = U\phi = U(u,\lambda)\phi, \quad U = \begin{bmatrix} \lambda J_2 & q \\ -q^T & r \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{1.1}$$

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where λ is a spectral parameter, r is a skew-symmetric (i.e., $r^T = -r$) matrix of arbitrary order and $u = p(q,r)$ is a vector potential (p is a kind of arrangement of all entries in q and r into a vector).

The author obtained the associated hierarchy of multi-component Lax integrable equations of Eq. (1.1) and presented a Hamiltonian structure of the resulting hierarchy. Moreover, he studied three different special forms of integrable properties:

I: When r is a zero matrix [9], the obtained hierarchy possesses a bi-Hamiltonian formulation.
II: For the case of r not being a zero matrix, when $\delta_n = \alpha c_n$ (see [7]), the obtained hierarchy possesses Hamiltonian formulation.
III: In *II*, when $\delta_n = \alpha c_{n+1}$ (see [8]), the obtained hierarchy has another different Hamiltonian structure.

Now, the problem is, whether the above integrable systems have integrable couplings? Further, whether the integrable couplings possess Hamiltonian structure? In this paper, we will study these problems based on the works [9].

The paper is organized as follows. In Section 2, we first recall the integrable hierarchy of an arbitrary order matrix spectral problem [9] when $r = \mathbf{0}$. In Section 3, an integrable coupling hierarchy is established by semi-direct sums of Lie algebra and its Hamiltonian form is obtained by means of the component trace identities. In Section 4, an example for the case r being 2×2 matrix, the integrable coupling and its Hamiltonian structure of the matrix spectral problem is computed. Finally, a few concluding remarks are given and some further problems are presented.

2. Multi-component integrable hierarchy when $r = \mathbf{0}$

For the spectral problem (1.1), when $r = \mathbf{0}$, we assume that

$$V = \begin{bmatrix} aJ_2 & b \\ -b^T & c \end{bmatrix} = \sum_{j=0}^{\infty} V_j \lambda^{-j}, \quad V_j = \begin{bmatrix} a_jJ_2 & b_j \\ -b_j^T & c_j \end{bmatrix}, \tag{2.1}$$

where a is scalar, c and c_j, $j \geqslant 0$ are skew-symmetric matrices of the same size as r, then we have

$$[U,V] = \begin{bmatrix} q^Tb - qb^T & \lambda J_2 b - aJ_2 q + qc \\ \lambda b^T J_2 - aq^T J_2 + cq^T & -q^T b + qb^T \end{bmatrix}.$$

Therefore, the stationary zero curvature equation

$$V_x = [U,V] \tag{2.2}$$

generates

$$\begin{cases} a_{j,x}J_2 = q^Tb_j - qb_j^T, \\ b_{j,x} = J_2 b_{j+1} - a_jJ_2 q + qc_j, \quad j \geqslant 0, \\ c_{j,x} = -q^T b_j + qb_j^T \end{cases} \tag{2.3}$$

with the initial values satisfying $a_0 = \text{const}$, $b_0 = c_0 = \mathbf{0}$.

Now for any integer $n \geqslant 1$, we introduce

$$V^{(n)} = \sum_{j=0}^{n} V_j \lambda^{n-j}, \tag{2.4}$$

and write

$$U = \lambda U_0 + U_1, \quad U_0 = \begin{bmatrix} J_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad U_1 = \begin{bmatrix} \mathbf{0} & q \\ -q^T & \mathbf{0} \end{bmatrix}.$$

Then we have

$$V_x^{(n)} - \left[U, V^{(n)}\right] = V_x^{(n)} - \left[\lambda U_0, V^{(n)}\right] - \left[U_1, V^{(n)}\right] = V_{n,x} - [U_1, V_n] = \begin{bmatrix} a_{n,x}J_2 + qb_n^T - q^Tb_n & b_{n,x} - qc_n + a_nJ_2 q \\ -b_{n,x}^T - c_n^T q + q^T a_n J_2 & c_{n,x}J_2 + q^T b_n - qb_n^T \end{bmatrix},$$

From (2.3), the above matrix becomes

$$V_x^{(n)} - \left[U, V^{(n)}\right] = \begin{bmatrix} \mathbf{0} & J_2 b_{n+1} \\ -b_{n+1}^T J_2 & \mathbf{0} \end{bmatrix}. \tag{2.5}$$

So, the compatibility conditions of the matrix spectral problems

$$\phi_x = U\phi, \quad \phi_{t_n} = V^{(n)}\phi \tag{2.6}$$

are

$$U_{t_n} = V_x^{(n)} - \left[U, V^{(n)}\right], \tag{2.7}$$

which give rise to a hierarchy of Lax integrable evolution equations

$$q_{t_n} = J_c b_{n+1} = J_2 b_{n+1}, \tag{2.8}$$

where J_c denote the compact form of the matrix operator J:

$$(p(q))_{t_n} = p(J_2 b_{n+1}) = Jp(b_{n+1}), \quad n \geqslant 1. \tag{2.9}$$

On one hand, note that the inner product between two vectors $p(A)$ and $p(B)$ is

$$(p(A), p(B)) = \int \mathrm{tr}(A^T B) dx,$$

where A and B denote matrices of the same size as q. Obviously, J is skew-symmetric. That is, for any two matrices A and B of the same size as q, we have

$$(p(A), p(J_c B)) = -(p(J_c A), p(B)).$$

Since the operator J is independent of the dependent variables in q, J automatically satisfies the Jacobi identity

$$\left(p(A), p\left((J_c)'[J_c B]C\right)\right) + \mathrm{cycle}(A, B, C) = 0,$$

where $(J_c)'$ denotes the Gateaux derivative, and A, B and C are arbitrary matrices of the same size as q. Therefore, J is a Hamiltonian operator.

On the other hand, let us recall the variational trace identity

$$\frac{\delta}{\delta u} \int \left\langle V, \frac{\partial U}{\partial \lambda} \right\rangle dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} \left\langle V, \frac{\partial U}{\partial u} \right\rangle, \tag{2.10}$$

where killing form $\langle P, Q \rangle = \mathrm{tr}(PQ)$, and γ is a constant. In our case, we have

$$\left\langle V, \frac{\partial U}{\partial q} \right\rangle = -2b, \quad \left\langle V, \frac{\partial U}{\partial \lambda} \right\rangle = -2a. \tag{2.11}$$

An application of the trace identity leads to

$$\frac{\delta}{\delta q} \int (-2a) dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} (-2b),$$

which is equivalent to

$$\frac{\delta}{\delta q} \int (-2a_{n+1}) dx = (\gamma - n)(-2b_n),$$

by comparing the coefficients of λ^{-n-1}, $n \geqslant 0$. Taking $n = 0$ yields the constant $\gamma = 0$, and

$$\frac{\delta}{\delta q} H_{n+1} = b_{n+1}, \quad H_{n+1} = \int \frac{-a_{n+2}}{n+1} dx, \quad n \geqslant 0. \tag{2.12}$$

It naturally follows that the integrable hierarchy (2.8) has a Hamiltonian structure.

3. Integrable coupling hierarchy and its Hamiltonian structure

3.1. Integrable coupling hierarchy

As in [5,10], introduce two Lie algebras of matrices:

$$G = \left\{ \begin{bmatrix} U & \mathbf{0} \\ \mathbf{0} & U \end{bmatrix} \middle| A \in \mathbb{R}[\lambda] \otimes sl(2) \right\}, \quad G_c = \left\{ \begin{bmatrix} \mathbf{0} & B \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \middle| B \in \mathbb{R}[\lambda] \otimes sl(2) \right\}, \tag{3.1}$$

where the loop algebra $\mathbb{R}[\lambda] \otimes sl(2)$ is defined by span $\{\lambda^n A | n \geqslant 0, A \in sl(2)\}$, and form a semi-direct sum $\overline{G} = G \uplus G_c$ of these two Lie algebras G and G_c. In this case, G_c is an Abelian ideal of $\overline{G}$. For the spectral problem (1.1), we define the corresponding enlarged spectral matrix as follows:

$$\overline{U} = \overline{U}(\bar{u}, \lambda) = \begin{bmatrix} U & U_a \\ \mathbf{0} & U \end{bmatrix} \in G \uplus G_c, \quad U_a = \begin{bmatrix} \mathbf{0} & d \\ -d^T & \mathbf{0} \end{bmatrix}, \tag{3.2}$$

where $\bar{u} = (p(q), p(d))^T$ is a vector potential, d is a matrix of the same size as q.

To solve the corresponding enlarged stationary zero curvature equation $\overline{V}_x = [\overline{U}, \overline{V}]$, we set

$$\overline{V} = \begin{bmatrix} V & V_a \\ \mathbf{0} & V \end{bmatrix}, \quad V_a = V_a(\bar{u}, \lambda) = \begin{bmatrix} fJ_2 & e \\ -e^T & \mathbf{0} \end{bmatrix}, \tag{3.3}$$

where V is a solution to (2.2), e is a matrix of the same size as b. Then, the enlarged stationary zero curvature equation becomes

$$V_{a,x} = [U, V_a] + [U_a, V]. \tag{3.4}$$

This equation is equivalent to

$$V_{a,x} = \begin{bmatrix} -qe^T + q^T e - db^T + d^T b & \lambda J_2 e - fJ_2 q + dc - aJ_2 d \\ e^T \lambda J_2 - q^T fJ_2 - d^T aJ_2 + cd^T & -q^T e + qe^T - d^T b + db^T \end{bmatrix}.$$

Expand V_a as

$$V_a = \sum_{j=0}^{\infty} V_{a,j} \lambda^{-j}, \quad V_{a,j} = \begin{bmatrix} f_j J_2 & e_j \\ -e_j^T & \mathbf{0} \end{bmatrix}, \tag{3.5}$$

so we obtain

$$\begin{cases} f_{j,x} J_2 = -qe_j^T + q^T e_j - db_j^T + d^T b_j, \\ e_{j,x} = J_2 e_{j+1} - f_j J_2 q + dc_j - a_j J_2 d, \end{cases} \quad j \geqslant 0, \tag{3.6}$$

upon setting $e_0 = \mathbf{0}, f_0 = \text{const}$. Assuming $e_j|_{u=0} = f_j|_{u=0} = 0, j \geqslant 1$, we see that f_i and sets of matrices e_j are uniquely determined. From (3.4),

$$(V_{a,j})_x = [U_0, V_{a,j+1}] + [U_1, V_{a,j}] + [U_a, V_j] \quad j \geqslant 0. \tag{3.7}$$

Now, we define

$$\overline{V}^{(n)} = \begin{bmatrix} V^{(n)} & V_a^{(n)} \\ \mathbf{0} & V^{(n)} \end{bmatrix} \in \overline{G}, \quad V_a^{(n)} = \lambda^n V_a, \quad n \geqslant 0. \tag{3.8}$$

Then, the nth enlarged zero curvature equation

$$\overline{U}_{t_n} - \left(\overline{V}^{(n)}\right)_x + \left[\overline{U}, \overline{V}^{(n)}\right] = 0 \tag{3.9}$$

leads to

$$U_{a,t_n} - \left(V_a^{(n)}\right)_x + \left[U, V_a^{(n)}\right] + \left[U_a, V^{(n)}\right] = 0, \tag{3.10}$$

together with the nth equation (2.7). Based on (3.7), this can be simplified to

$$U_{a,t_n} - [U_0, V_{a,n+1}] = 0, \tag{3.11}$$

which gives rise to

$$d_{t_n} = S_n(q, d) = J_2 e_{n+1}. \tag{3.12}$$

Therefore, we obtain a hierarchy of coupling system

$$\bar{u}_{t_n} = \begin{bmatrix} p(q) \\ p(d) \end{bmatrix}_{t_n} = \overline{K}_n(\bar{u}) = \begin{bmatrix} K_n(q) \\ S_n(q, d) \end{bmatrix} = \begin{bmatrix} p(J_2 b_{n+1}) \\ p(J_2 e_{n+1}) \end{bmatrix}, \quad n \geqslant 0 \tag{3.13}$$

for the original hierarchy equation (2.8).

3.2. Hamiltonian structure

In the theory of the integrable couplings, an important topic is to establish Hamiltonian structures of integrable couplings. As a rule, the Hamiltonian structures of integrable systems may be established by the trace identity [11,12]. However, it usually cannot be used to the case of integrable couplings. Ma and Chen [10] have done a series of original work on considerably improving the Tu trace identity to construct a Hamiltonian structure of integrable coupling systems by using semi-direct sums of Lie algebras. In Ref. [13], Ma and Zhang presented a component-trace identity based on variational identities on semi-direct sums of Lie algebras for generating Hamiltonian structures of multi-component integrable couplings. In this subsection, we are going to establish the Hamiltonian structure of Eq. (3.13) by means of the component-trace identity. In what follows, we firstly list the identity.

Lemma 1 (*Component-trace Identity [13]*). *Let g be a matrix Lie algebra consisting of block matrices. For a given spectral matrix* $\overline{U} = \overline{U}(\bar{u}, \lambda) = (U_0, U_1, \ldots, U_N) \in g$, *we have the variational identity*

$$\frac{\delta}{\delta u}\int \sum_{k=0}^{N} \alpha_k tr\left(\sum_{i+j=k} V_i \frac{\partial U_j}{\partial \lambda}\right)dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\sum_{k=0}^{N}\alpha_k tr\left(\sum_{i+j=k} V_i \frac{\partial U_j}{\partial u}\right),$$

where $\overline{V} = \overline{V}(\bar{u}, \lambda) = (V_0, V_1, \ldots, V_N) \in g$ *satisfies the stationary zero-curvature equation* $V_x = [U, V]$, *all* α_k'*s are arbitrary constants with* $\alpha_N \neq 0$ *and* γ *is a constant.*

Using this lemma, note that $N = 1$ in our spectral matrix, so we have

$$\frac{\delta}{\delta u}\int \mathrm{tr}\left(V\frac{\partial U}{\partial \lambda} + V\frac{\partial U_a}{\partial \lambda} + V_a\frac{\partial U}{\partial \lambda}\right)dx = \frac{\delta}{\delta u}\int \mathrm{tr}\left(V\frac{\partial U}{\partial \bar{u}} + V\frac{\partial U_a}{\partial \bar{u}} + V_a\frac{\partial U}{\partial \bar{u}}\right), \tag{3.14}$$

by setting $\alpha_0 = \alpha_1 = 1$. In this case, we can directly compute that

$$\mathrm{tr}\left(V\frac{\partial U}{\partial q} + V\frac{\partial U_a}{\partial q} + V_a\frac{\partial U}{\partial q}\right) = -2b - 2e, \tag{3.15}$$

$$\mathrm{tr}\left(V\frac{\partial U}{\partial d} + V\frac{\partial U_a}{\partial d} + V_a\frac{\partial U}{\partial d}\right) = -2b, \tag{3.16}$$

$$\mathrm{tr}\left(V\frac{\partial U}{\partial \lambda} + V\frac{\partial U_a}{\partial \lambda} + V_a\frac{\partial U}{\partial \lambda}\right) = -2a - 2f. \tag{3.17}$$

Therefore, the variational identity leads to

$$\frac{\delta}{\delta u}\int(-2a - 2f)dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}(p(-2b - 2e), p(-2b))^T,$$

which is equivalent to

$$\frac{\delta}{\delta u}\int(-2a_{n+1} - 2f_{n+1})dx = (\gamma - n)(p(-2b_n - 2e_n), p(-2b_n))^T, \tag{3.18}$$

by comparing the coefficients of λ^{-n-1}, $n \geqslant 0$. Taking $n = 0$ yields the constant $\gamma = 0$. Therefore, we have

$$(p(b_{n+1} + e_{n+1}), p(b_{n+1}))^T = \frac{\delta}{\delta \bar{u}}\int \frac{-(a_{n+2} + f_{n+2})}{n+1}dx, \quad n \geqslant 0. \tag{3.19}$$

Since we have

$$\begin{pmatrix} p(J_2 b_{n+1}) \\ p(J_2 e_{n+1}) \end{pmatrix} = \begin{bmatrix} \mathbf{0} & J \\ J & -J \end{bmatrix}\begin{pmatrix} p(b_{n+1}) + p(e_{n+1}) \\ p(b_{n+1}) \end{pmatrix} = \bar{J}\begin{pmatrix} p(b_{n+1}) + p(e_{n+1}) \\ p(b_{n+1}) \end{pmatrix}, \tag{3.20}$$

where $\bar{J}$ obviously is a Hamiltonian operator because J is a Hamiltonian operator. It then follows that the enlarged hierarchy possesses the following Hamiltonian structure:

$$\bar{u}_{t_n} = \bar{K}_n = \bar{J}\frac{\delta \overline{H}_{n+1}}{\delta \bar{u}}, \quad \overline{H}_{n+1} = \int \frac{-(a_{n+2} + f_{n+2})}{n+1}dx, \quad n \geqslant 0. \tag{3.21}$$

4. An example when *r* is a 2 × 2 zero matrix

In particular, when r is a 2×2 zero matrix, let us denote

$$q = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = \sum_{j=0}^{\infty} b_j = \sum_{j=0}^{\infty}\begin{bmatrix} b_{1,j} & b_{2,j} \\ b_{3,j} & b_{4,j} \end{bmatrix}, \quad c = \begin{bmatrix} 0 & h \\ -h & 0 \end{bmatrix} = \sum_{j=0}^{\infty}\begin{bmatrix} 0 & h_j \\ -h_j & 0 \end{bmatrix},$$

where u_i, $1 \leqslant i \leqslant 4$, are scalar variables, $b_{i,j}$, $1 \leqslant i \leqslant 4$ and h_j, $j \geqslant 0$ are scalar functions. Then the recursion relations (2.3) become

$$\begin{cases} b_{1,j+1} = -(b_{3,j})_x + u_1 a_j - u_4 h_j, \\ b_{2,j+1} = -(b_{4,j})_x + u_2 a_j + u_3 h_j, \\ b_{3,j+1} = (b_{1,j})_x + u_3 a_j + u_2 h_j, \\ b_{4,j+1} = (b_{2,j})_x + u_4 a_j - u_1 h_j, \\ (a_{j+1})_x = -u_1 b_{3,j+1} - u_2 b_{4,j+1} + u_3 b_{1,j+1} + u_4 b_{2,j+1}, \\ (h_{j+1})_x = -u_1 b_{2,j+1} + u_2 b_{1,j+1} - u_3 b_{4,j+1} + u_4 b_{3,j+1}, \end{cases} \tag{4.1}$$

where $j \geqslant 0$, and the integrable hierarchy (2.9) gives rise to

$$u_{t_n} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}_{t_n} = \begin{bmatrix} b_{3,n+1} \\ b_{4,n+1} \\ -b_{1,n+1} \\ -b_{2,n+1} \end{bmatrix} = \Phi \begin{bmatrix} b_{3,n} \\ b_{4,n} \\ -b_{1,n} \\ -b_{2,n} \end{bmatrix} = \Phi J \begin{bmatrix} b_{1,n} \\ b_{2,n} \\ b_{3,n} \\ b_{4,n} \end{bmatrix} = \Phi J \frac{\delta}{\delta u} H_n, \quad n \geqslant 1, \tag{4.2}$$

where the Hamiltonian operator J, the hereditary recursion operator Φ are given by

$$J = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \tag{4.3}$$

and

$$\Phi = \begin{bmatrix} p_{24} - p_{31} & -p_{23} - p_{32} & -\partial - p_{22} - p_{33} & p_{21} - p_{34} \\ -p_{14} - p_{41} & p_{13} - p_{42} & p_{12} - p_{43} & -\partial - p_{11} - p_{44} \\ \partial + p_{11} + p_{44} & p_{12} - p_{43} & p_{13} - p_{42} & p_{14} + p_{41} \\ p_{21} - p_{34} & \partial + p_{22} + p_{33} & p_{23} + p_{32} & p_{24} - p_{31} \end{bmatrix} \tag{4.4}$$

respectively, where $p_{ij} = u_i \partial^{-1} u_j$, $1 \leqslant i, j \leqslant 4$.

Upon choosing $a_0 = 1$, $h_0 = 0$ and $b_0 = \mathbf{0}$, as well as setting every integration constant to zero, the first nonlinear system of integrable equations in the hierarchy (4.2) reads

$$\begin{cases} u_{1,t_2} = -u_{3,xx} + u_3 a_2 + u_2 h_2, \\ u_{2,t_2} = -u_{4,xx} + u_4 a_2 - u_1 h_2, \\ u_{3,t_2} = u_{1,xx} - u_1 a_2 + u_4 h_2, \\ u_{4,t_2} = u_{2,xx} - u_2 a_2 - u_3 h_2, \end{cases} \tag{4.5}$$

where $a_2 = -\frac{1}{2}(u_1^2 + u_2^2 + u_3^2 + u_4^2),\ h_2 = u_1 u_4 - u_2 u_3$. It has a Hamiltonian structure:

$$u_{t_2} = \Phi J \frac{\delta}{\delta u} H_2$$

with the Hamiltonian functional being given by

$$H_2 = \int \frac{1}{2}(u_{1,x} u_3 - u_1 u_{3,x} + u_{2,x} u_4 - u_2 u_{4,x}) dx. \tag{4.6}$$

For the integrable coupling, we write

$$U_a = \begin{bmatrix} 0 & 0 & d_1 & d_2 \\ 0 & 0 & d_3 & d_4 \\ -d_1 & -d_3 & 0 & 0 \\ -d_2 & -d_4 & 0 & 0 \end{bmatrix}, \quad V_a = \begin{bmatrix} 0 & f & e_1 & e_2 \\ -f & 0 & e_3 & e_4 \\ -e_1 & -e_3 & 0 & 0 \\ -e_2 & -e_4 & 0 & 0 \end{bmatrix}, \tag{4.7}$$

where d_i, $1 \leqslant i \leqslant 4$, are scalar variables, e_i, $1 \leqslant i \leqslant 4$, and f are scalar functions.

From (3.6), we can get

$$\begin{cases} f_{j,x} = u_3 e_{1,j} + u_4 e_{2,j} - u_1 e_{3,j} - u_2 e_{4,j} + d_3 b_{1,j} + d_4 b_{2,j} - d_1 b_{3,j} - d_2 b_{4,j}, \\ e_{1,j+1} = -(e_{3,j})_x - d_4 h_j + u_1 f_j + d_1 a_j, \\ e_{2,j+1} = -(e_{4,j})_x + d_3 h_j + u_2 f_j + d_2 a_j, \\ e_{3,j+1} = (e_{1,j})_x + d_2 h_j + u_3 f_j + d_3 a_j, \\ e_{4,j+1} = (e_{2,j})_x - d_1 h_j + u_4 f_j + d_4 a_j, \end{cases} \tag{4.8}$$

where $j \geqslant 0$. From (3.12), we also can get

$$d_{t_n} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix}_{t_n} = \begin{bmatrix} e_{3,n+1} \\ e_{4,n+1} \\ -e_{1,n+1} \\ -e_{2,n+1} \end{bmatrix}. \tag{4.9}$$

Therefore, we obtain a hierarchy of coupling system

$$\bar{u}_{t_n} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix}_{t_n} = \begin{bmatrix} b_{3,n+1} \\ b_{4,n+1} \\ -b_{1,n+1} \\ -b_{2,n+1} \\ e_{3,n+1} \\ e_{4,n+1} \\ -e_{1,n+1} \\ -e_{2,n+1} \end{bmatrix} \tag{4.10}$$

together with (4.2). From (3.15) and (3.16),

$$p(-2b-2e) = \begin{bmatrix} -2b_1 - 2e_1 \\ -2b_2 - 2e_2 \\ -2b_3 - 2e_3 \\ -2b_4 - 2e_4 \end{bmatrix}, \quad p(-2b) = \begin{bmatrix} -2b_1 \\ -2b_2 \\ -2b_3 \\ -2b_4 \end{bmatrix}. \tag{4.11}$$

So, (3.19) and (3.20) lead to

$$\begin{aligned}(b_{1,n+1}+e_{1,n+1}, b_{2,n+1}+e_{2,n+1}, b_{3,n+1}+e_{3,n+1}, b_{4,n+1}+e_{4,n+1}, b_{1,n+1}, b_{2,n+1}, b_{3,n+1}, b_{4,n+1})^T &= \frac{\delta}{\delta\bar{u}}\int \frac{-(a_{n+2}+f_{n+2})}{n+1}dx \\ &= \frac{\delta}{\delta\bar{u}}\overline{H}_{n+1}, \quad n \geqslant 0,\end{aligned} \tag{4.12}$$

$$\bar{u}_{t_n} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix}_{t_n} = \begin{bmatrix} b_{3,n+1} \\ b_{4,n+1} \\ -b_{1,n+1} \\ -b_{2,n+1} \\ e_{3,n+1} \\ e_{4,n+1} \\ -e_{1,n+1} \\ -e_{2,n+1} \end{bmatrix} = \overline{\Phi} \begin{bmatrix} b_{3,n} \\ b_{4,n} \\ -b_{1,n} \\ -b_{2,n} \\ e_{3,n} \\ e_{4,n} \\ -e_{1,n} \\ -e_{2,n} \end{bmatrix} = \overline{\Phi J} \begin{bmatrix} b_{1,n}+e_{1,n} \\ b_{2,n}+e_{2,n} \\ b_{3,n}+e_{3,n} \\ b_{4,n}+e_{4,n} \\ b_{1,n} \\ b_{2,n} \\ b_{3,n} \\ b_{4,n} \end{bmatrix} = \overline{\Phi J}\frac{\delta}{\delta\bar{u}}\bar{H}_n, \tag{4.13}$$

where $n \geqslant 1$, the Hamiltonian operator $\bar{J}$, the hereditary recursion operator $\overline{\Phi}$ are given by

$$\bar{J} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} = \begin{pmatrix} \mathbf{0} & J \\ J & -J \end{pmatrix}, \tag{4.14}$$

and

$$\overline{\Phi} = \begin{bmatrix} \Phi & \mathbf{0} \\ \Phi_1 & \Phi_2 \end{bmatrix}, \tag{4.15}$$

where

$$\Phi_1 = \begin{bmatrix} -k_{31}-l_{31}+l_{24} & -k_{32}-l_{32}-l_{23} & -k_{33}-l_{33}-l_{22} & -k_{34}-l_{34}+l_{21} \\ -k_{41}-l_{41}-l_{14} & -k_{42}-l_{42}+l_{13} & -k_{43}-l_{43}+l_{12} & -k_{44}-l_{44}-l_{11} \\ k_{11}+l_{11}+l_{44} & k_{12}+l_{12}-l_{43} & k_{13}+l_{13}-l_{42} & k_{14}+l_{14}+l_{41} \\ k_{21}+l_{21}-l_{34} & k_{22}+l_{22}+l_{33} & k_{23}+l_{23}+l_{32} & k_{24}+l_{24}-l_{31} \end{bmatrix}$$

with $k_{ij} = u_i\partial^{-1}d_j$, $l_{ij} = d_i\partial^{-1}u_j$, $1 \leqslant i, j \leqslant 4$, and

$$\Phi_2 = \begin{bmatrix} -p_{31} & -p_{32} & -\partial - p_{33} & -p_{34} \\ -p_{41} & -p_{42} & -p_{43} & -\partial - p_{44} \\ \partial + p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & \partial + p_{22} & p_{23} & p_{24} \end{bmatrix}.$$

Upon choosing $a_0 = 1$, $h_0 = 0$, $f_0 = 0$, $b_0 = \mathbf{0}$, $e_0 = \mathbf{0}$, as well as setting every integration constant to zero, the first nonlinear system of integrable equations in the hierarchy (4.13) reads

$$\begin{cases} u_{1,t_2} = -u_{3,xx} + u_3 a_2 + u_2 h_2, \\ u_{2,t_2} = -u_{4,xx} + u_4 a_2 - u_1 h_2, \\ u_{3,t_2} = u_{1,xx} - u_1 a_2 + u_4 h_2, \\ u_{4,t_2} = u_{2,xx} - u_2 a_2 - u_3 h_2, \\ d_{1,t_2} = -d_{3,xx} + d_3 a_2 + d_2 h_2 + u_3 f_2, \\ d_{2,t_2} = -d_{4,xx} + d_4 a_2 - d_1 h_2 + u_4 f_2, \\ d_{3,t_2} = d_{1,xx} - d_1 a_2 + d_4 h_2 - u_1 f_2, \\ d_{4,t_2} = d_{2,xx} - d_2 a_2 - d_3 h_2 - u_2 f_2, \end{cases} \tag{4.16}$$

where $a_2 = -\frac{1}{2}(u_1^2 + u_2^2 + u_3^2 + u_4^2)$, $h_2 = u_1 u_4 - u_2 u_3$, and $f_2 = -(u_1 d_1 + u_2 d_2 + u_3 d_3 + u_4 d_4)$.

It has a Hamiltonian structure:

$$\bar{u}_{t_2} = \overline{\Phi J}\frac{\delta}{\delta \bar{u}}\overline{H}_2$$

with the Hamiltonian functional being given by

$$\overline{H}_2 = -\int \frac{1}{2}(a_3 + f_3)dx, \tag{4.17}$$

where $a_3 = u_1 u_{3,x} - u_{1,x} u_3 + u_2 u_{4,x} - u_{2,x} u_4$, $f_3 = u_1 d_{3,x} - u_{1,x} d_3 + u_2 d_{4,x} - u_{2,x} d_4 - u_3 d_{1,x} + u_{3,x} d_1 - u_4 d_{2,x} + u_{4,x} d_2$.

5. Conclusion

In summary, we used semi-direct sums of matrix Lie algebra to construct an integrable coupling hierarchy of matrix spectral problem with arbitrary order zero matrix r. Then, we also presented the Hamiltonian structure of the integrable coupling hierarchy by means of the component-trace identities. Specially, when r is 2×2 zero matrix, the integrable coupling hierarchy and its Hamiltonian structure of the matrix spectral problem are computed as an example.

We must point out, we only considered the special case when r is an arbitrary order zero matrix in this paper. For the more general case, that is, when r is non-zero matrix, whether the matrix spectral problem still possess integrable coupling hierarchy and its Hamiltonian structure is more challenging problem and is worthy of being studied further.

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