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Wronskian determinant solutions of the (3 + 1)-dimensional Jimbo–Miwa equation

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ABSTRACT

A set of sufficient conditions consisting of systems of linear partial differential equations is obtained which guarantees that the Wronskian determinant solves the (3 + 1)-dimensional Jimbo–Miwa equation in the bilinear form. Upon solving the linear conditions, the resulting Wronskian formulations bring solution formulas, which can yield rational solutions, solitons, negatons, positons and interaction solutions.

1. Introduction

Wronskian formulations are a common feature for soliton equations, and it is a powerful tool to construct exact solutions to the corresponding Hirota bilinear equations of the soliton equations [1–4]. The resulting technique has been applied to many soliton equations such as the MKdV, NLS, derivative NLS, sine-Gordon and other equations [5–10]. Within Wronskian formulations, soliton solutions and rational solutions are usually expressed as some kind of logarithmic derivatives of Wronskian type determinants [11–14].

The (3 + 1)-dimensional Jimbo–Miwa equation

$$u_{xxxy} + 3u_{xx}u_y + 3u_xu_{xy} + 2u_{yt} - 3u_{xz} = 0 \tag{1.1}$$

was firstly investigated by Jimbo–Miwa and its soliton solutions were obtained in [15]. It is the second member in the entire Kadomtsev–Petviashvili hierarchy. Ma [16] proposed a direct approach to exact solutions of nonlinear partial differential equation by using rational function transformations to solve Eq. (1.1). Wazwaz [17] employed the Hirota's bilinear method to this equation and confirmed that it is completely integrable and it admits multiple-soliton solutions of any order. In [18], the traveling wave solutions of Eq. (1.1) expressed by hyperbolic, trigonometric and rational functions were constructed by the G'/G-expansion method, where $G = G(\xi)$ satisfies a second order linear ordinary differential equation.

A Hirota bilinear form of Eq. (1.1) is

$$\begin{aligned}\left(D_x^3D_y + 2D_yD_t - 3D_xD_z\right)f\cdot f &= f_{xxxy}f - f_{xxx}f_y - 3f_{xxy}f_x + 3f_{xx}f_{xy} + 2f_{yt}f - 2f_yf_t - 3f_{xz}f + 3f_xf_z \\ &= f(f_{xxxy} + 2f_{yt} - 3f_{xz}) + (-f_{xxx}f_y - 3f_{xxy}f_x - 2f_yf_t + 3f_xf_z) + 3f_{xx}f_{xy} = 0\end{aligned} \tag{1.2}$$

after the Cole–Hopf transformation

$$u = 2(\ln f)_x = 2f_x/f,$$

where D_x, D_y, D_z and D_t are the Hirota operators [19].

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In this paper, we aim to present the Wronskian determinant solutions of the above Eq. (1.1), which will particularly lead to an approach for constructing rational solutions and solitons to the Eq. (1.1). Our results will also show the richness and diversity of solution structures of the Eq. (1.1).

The paper is organized as follows. In Section 2, the Wronskian determinant solution is presented for the bilinear Eq. (1.2). In Section 3, an approach for constructing exact solutions including rational solutions is furnished, and many examples of solutions such as rational solutions, positons and negatons are provided. Finally, some conclusions are given in Section 4.

2. Wronskian formulation

The Wronskian technique is introduced by Freeman and Nimmo [1,20], they set

$$W(\phi_1,\phi_2,\ldots,\phi_n)=(\widehat{N-1};\Phi)=(\widehat{N-1})=\begin{vmatrix}\phi_1^{(0)} & \phi_1^{(1)} & \cdots & \phi_1^{(N-1)}\\ \phi_2^{(0)} & \phi_2^{(1)} & \cdots & \phi_2^{(N-1)}\\ \vdots & \vdots & \ddots & \vdots\\ \phi_N^{(0)} & \phi_N^{(1)} & \cdots & \phi_N^{(N-1)}\end{vmatrix},\quad N\geqslant 1, \tag{2.1}$$

where

$$\Phi=(\phi_1,\phi_2,\ldots,\phi_n)^T,\phi_i^{(0)}=\phi_i,\quad \phi_i^{(j)}=\frac{\partial^j}{\partial x^j}\phi_i,\quad j\geqslant 1,\quad 1\leqslant i\leqslant N. \tag{2.2}$$

Solutions determined by $u=2(\ln f)_x$ with $f=(\widehat{N-1})$to the Eq. (1.1) are called Wronskian solutions.

Proposition 1. *Assuming that $\phi_i=\phi_i(x,y,z,t)$ (where $t\geqslant 0,\infty<x,y,z<\infty$, $i=1,2,\ldots,N$) has continuous derivative up to any order, and satisfies the following linear differential conditions*

$$\phi_{i,xx}=\sum_{j=1}^{N}\lambda_{ij}(t)\phi_j, \tag{2.3}$$

$$\phi_{i,y}=3\phi_{i,x}, \tag{2.4}$$

$$\phi_{i,z}=4\phi_{i,xxx}+2\phi_{i,x}, \tag{2.5}$$

$$\phi_{i,t}=\phi_{i,x}, \tag{2.6}$$

then $f=(\widehat{N-1})$ defined by (2.1) solves the bilinear Eq. (1.2).

Before proving the above results, we state the following three known useful Lemmas.

Lemma 1

$$|D,a,b||D,c,d|-||D,a,c|||D,b,d|+|D,a,d||D,b,c|=0, \tag{2.7}$$

where D is $N\times(N-2)$ matrix, and a, b, c, d are n-dimensional column vectors.

Lemma 2. *Set $a_j(j=1,\ldots,n)$ to be an n-dimensional column vector, and b_j $(j=1,\ldots,n)$ to be a real constant but not to be zero. Then we have*

$$\sum_{i=1}^{N}b_i|a_1,a_2,\ldots,a_N|=\sum_{j=1}^{N}|a_1,a_2,\ldots,ba_j,\ldots,a_N|, \tag{2.8}$$

where $ba_j=(b_1a_{1j},b_2a_{2j},\ldots,b_Na_{Nj})^T$.

Lemma 3 [11]. *Under the condition* (2.3) and Lemma 2, *the following equalities hold:*

$$\begin{aligned}(\widehat{N-1})\sum_{i=1}^{N}\lambda_{ii}(t)\left(\sum_{i=1}^{N}\lambda_{ii}(t)(\widehat{N-1})\right)&=\left(\sum_{i=1}^{N}\lambda_{ii}(t)(\widehat{N-1})\right)^2=[(\widehat{N-3},N-1,N)-(\widehat{N-2},N+1)]^2\\&=(\widehat{N-5},N-3,N-2,N-1,N)-(\widehat{N-4},N-2,N-1,N+1)\\&\quad-(\widehat{N-3},N-1,N+2)+2(\widehat{N-3},N,N+1)+(\widehat{N-2},N+3).\end{aligned} \tag{2.9}$$

Proof of Proposition 1. Obviously, we always have

$$f_x=(\widehat{N-2},N),$$

$$f_{xx}=(\widehat{N-3},N-1,N)+(\widehat{N-2},N+1),$$

$$f_{xxx}=(\widehat{N-4},N-2,N-1,N)+2(\widehat{N-3},N-1,N+1)+(\widehat{N-2},N+2).$$

Using the conditions (2.4)–(2.6), we get that

$$
\begin{aligned}
&f_y=3(\widehat{N-2},N), f_{yx}=3(\widehat{N-3},N-1,N)+3(\widehat{N-2},N+1),\\
&f_{xxy}=3(\widehat{N-4},N-2,N-1,N)+6(\widehat{N-3},N-1,N+1)+3(\widehat{N-2},N+2),\\
&f_{xxxy}=3(\widehat{N-5},N-3,N-2,N-1,N)+9(\widehat{N-4},N-2,N-1,N+1)\\
&\qquad\quad+6(\widehat{N-3},N,N+1)+9(\widehat{N-3},N-1,N+2)+3(\widehat{N-2},N+3),\\
&f_z=4(\widehat{N-4},N-2,N-1,N)-4(\widehat{N-3},N-1,N+1)+4(\widehat{N-2},N+2)+2(\widehat{N-2},N),\\
&f_{xz}=4(\widehat{N-5},N-3,N-2,N-1,N)-4(\widehat{N-3},N,N+1)+4(\widehat{N-2},N+3)+2(\widehat{N-3},N-1,N)+2(\widehat{N-2},N+1),\\
&f_t=(\widehat{N-2},N),\\
&f_{yt}=3(\widehat{N-3},N-1,N)+3(\widehat{N-2},N+1).
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
f(f_{xxxy}+2f_{yt}-3f_{xz})=9(\widehat{N-1})[&-(\widehat{N-5},N-3,N-2,N-1,N)+(\widehat{N-4},N-2,N-1,N+1)\\
&+2(\widehat{N-3},N,N+1)+(\widehat{N-3},N-1,N+2)-(\widehat{N-2},N+3)],
\end{aligned}
$$

$$
(-f_{xxx}f_y-3f_{xxy}f_x-2f_yf_t+3f_xf_z)=f_x(-3f_{xxx}-3f_{xxy}-6f_t+3f_z)=-36(\widehat{N-2},N)(\widehat{N-3},N-1,N+1),
$$

$$
\begin{aligned}
3f_{xx}f_{xy}&=9[-(\widehat{N-3},N-1,N)+(\widehat{N-2},N+1)+2(\widehat{N-3},N-1,N)]^2\\
&=9[-(\widehat{N-3},N-1,N)+(\widehat{N-2},N+1)]^2+36(\widehat{N-3},N-1,N)(\widehat{N-2},N+1).
\end{aligned}
$$

Using Lemma 3, we obtain

$$
\begin{aligned}
\left(D_x^3D_y+2D_yD_t-3D_xD_z\right)f\cdot f=9(\widehat{N-1})[&-(\widehat{N-5},N-3,N-2,N-1,N)+(\widehat{N-4},N-2,N-1,N+1)\\
&+2(\widehat{N-3},N,N+1)+(\widehat{N-3},N-1,N+2)-(\widehat{N-2},N+3)]\\
&-36(\widehat{N-2},N)(\widehat{N-3},N-1,N+1)+9[-(\widehat{N-3},N-1,N)\\
&+(\widehat{N-2},N+1)]^2+36(\widehat{N-3},N-1,N)(\widehat{N-2},N+1)=36(\widehat{N-3},N,N+1)(\widehat{N-1})\\
&-36(\widehat{N-2},N)(\widehat{N-3},N-1,N+1)+36(\widehat{N-3},N-1,N)(\widehat{N-2},N+1)=0.
\end{aligned}
$$

This shows that $f=(\widehat{N-1})$ solve the bilinear Eq. (1.2). The corresponding solution of Eq. (1.1) is

$$
u=2\frac{f_x}{f}=2\frac{(\widehat{N-2},N)}{(\widehat{N-1})}.
$$

Observation 1. From the compatibility conditions $\phi_{i,xxt}=\phi_{i,txx}$ $(i=1,\ldots,N)$ of the conditions (2.3)–(2.6), we have

$$
\sum_{j=1}^{N}\lambda_{ij,t}(t)\phi_j=0,\quad (i=1,\ldots,N)
$$

and thus it is easy to see that the Wronskian determinant $W(\phi_1,\phi_2,\ldots,\phi_N)$ becomes zero if there is at least one entry λ_{ij} satisfying $\lambda_{ij,t}(t)\neq 0$.

Observation 2. If the coefficient matrix $A=(\lambda_{ij})$ is similar to another matrix M under an invertible constant matrix P, i.e., we have $A=P^{-1}MP$, then $\widetilde{\Phi}=P\Phi$ solves

$$
\widetilde{\Phi}_{xx}=M\widetilde{\Phi},\quad \widetilde{\Phi}_y=3\widetilde{\Phi}_x,\quad \widetilde{\Phi}_z=4\widetilde{\Phi}_{xxx}+2\widetilde{\Phi}_x,\quad \widetilde{\Phi}_t=\widetilde{\Phi}_x
$$

and the resulting Wronskian solutions to the Eq. (1.1) are the same:

$$
u(A)=2\partial_x\ln|\Phi^{(0)},\Phi^{(1)},\ldots,\Phi^{(N-1)}|=2\partial_x\ln|P\Phi^{(0)},P\Phi^{(1)},\ldots,P\Phi^{(N-1)}|=u(M).
$$

Based on Observation I, we only need to consider the reduced case of (2.3)–(2.6) under $dA/dt=0$, i.e., the following conditions:

$$
\phi_{i,xx}=\sum_{j=1}^{N}\lambda_{ij}\phi_j,\quad \phi_{i,y}=3\phi_{j,x},\quad \phi_{i,z}=4\phi_{i,xxx}+2\phi_{i,x},\quad \phi_{i,t}=\phi_{i,x},\tag{2.10}
$$

where $A = (\lambda_{ij})$ is an arbitrary real constant matrix. Moreover, Observation II tells us that an invertible constant linear transformation on Φ in the Wronskian determinant does not change the corresponding Wronskian solution, and thus, we only have to solve (2.10) under the Jordan form of A.

3. Wronskian solutions

In principle, we can construct general solutions of the Eq. (1.2) by solving the linear conditions (2.10). But it is not easy. In this section we will present a few special Wronskian solutions to the Eq. (1.2).

It is well known that the corresponding Jordan form of a real matrix

$$\boldsymbol{A} = \begin{bmatrix} J(\lambda_1) & & & 0 \\ 1 & J(\lambda_2) & & \\ & \ddots & \ddots & \\ 0 & & 1 & J(\lambda_m) \end{bmatrix}_{n\times n}$$

have the following two types of blocks:

I.

$$J(\lambda_i) = \begin{bmatrix} \lambda_i & & & 0 \\ 1 & \lambda_i & & \\ & \ddots & \ddots & \\ 0 & & 1 & \lambda_i \end{bmatrix}_{k_i\times k_i},$$

II.

$$J(\lambda_i) = \begin{bmatrix} A_i & & & 0 \\ I_2 & A_i & & \\ & \ddots & \ddots & \\ 0 & & I_2 & A_i \end{bmatrix}_{l_i\times l_i}, \quad A_i = \begin{bmatrix} \alpha_i & -\beta_i \\ \beta_i & \alpha_i \end{bmatrix}, \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

where λ_i, α_i and $\beta_i > 0$ are all real constants. The first type of blocks have the real eigenvalue λ_i with algebraic multiplicity $k_i(\sum_{i=1}^{m} k_i = N)$, and the second type of blocks have the complex eigenvalue $\lambda_i^{\pm} = \alpha_i \pm \beta_i\sqrt{-1}$ with algebraic multiplicity l_i.

3.1. Rational solutions

Suppose A have the first type of Jordan blocks. Without loss of generality, let

$$J(\lambda_1) = \begin{bmatrix} \lambda_1 & & & 0 \\ 1 & \lambda_1 & & \\ & \ddots & \ddots & \\ 0 & & 1 & \lambda_1 \end{bmatrix}_{k_1\times k_1}.$$

In this case, if the eigenvalue $\lambda_1 = 0$, $J(\lambda_1)$ becomes to the following form:

$$\begin{bmatrix} 0 & & & 0 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{bmatrix}_{k_1\times k_1}$$

from the condition (2.10), we get

$$\phi_{1,xx} = 0, \quad \phi_{i+1,xx} = \phi_i, \quad \phi_{i,y} = 3\phi_{i,x}, \quad \phi_{i,z} = 4\phi_{i,xxx} + 2\phi_{i,x}, \quad \phi_{i,t} = \phi_{i,x}.i \geqslant 1. \tag{3.1}$$

Such functions $\phi_i(i \geqslant 1)$ are all polynomials in x, y, z and t, and a general Wronskian solution to the (3 + 1)-dimensional Jimbo–Miwa Eq. (1.1)

$$u = 2\partial_x \ln W(\phi_1, \phi_2, \ldots, \phi_{k_1})$$

is rational and is called a rational Wronskian solution of order k_1.

From (3.1), we solve $\phi_{1,xx} = 0, \phi_{1,y} = 3\phi_{1,x}$, $\phi_{1,z} = 4\phi_{1,xxx} + 2\phi_{1,x}$, $\phi_{1,t} = \phi_{1,x}$, and have

$$\phi_1 = c_1(x + t + 3y + 2z) + c_2.$$

Similarly, by solving $\phi_{i+1,xx} = \phi_i$, $\phi_{i+1,y} = 3\phi_{i+1,x}$, $\phi_{i+1,z} = 4\phi_{i+1,xxx} + 2\phi_{i+1,x}$, $\phi_{i+1,t} = \phi_{i+1,x}$, $i \geqslant 2$, then two special rational solutions of lower-order are obtained after setting some integral constants to be zero.

1) **Zero-order:** when $c_1 = 1$, $c_2 = 0$, $\phi_1 = x + 3y + 2z + t$, $f = W(\phi_1) = x + 3y + 2z + t$,

$$u = 2\partial_x \ln W(\phi_1) = \frac{2}{x + 3y + 2z + t}. \tag{3.2}$$

2) **First-order:** $\phi_1 = x + 3y + 2z + t$,

$$\phi_2 = \frac{1}{6}t^3 + \frac{t^2}{2}(x+3y) + \frac{t}{2}(x+3y)^2 + \frac{1}{6}(x+3y)^3 + \frac{4z^3}{3} + 2z^2(x+3y+t) + z(x+3y+t)^2 + 4z,$$
$$f = W(\phi_1, \phi_2) = p,$$
$$u = 2\partial_x \ln W(\phi_1, \phi_2) = \frac{2(x+3y+2z+t)^2}{p}, \tag{3.3}$$

where

$$p = \frac{1}{3}t^3 + t^2(x+3y) + t(x+3y)^2 + \frac{1}{3}(x+3y)^3 + \frac{8z^3}{3} + 4z^2(x+3y+t) + 2z(x+3y+t)^2 - 4z.$$

2) **Second-order:** $\phi_1 = x + 3y + 2z + t$,

$$\phi_2 = \frac{1}{6}t^3 + \frac{t^2}{2}(x+3y) + \frac{t}{2}(x+3y)^2 + \frac{1}{6}(x+3y)^3 + \frac{4z^3}{3} + 2z^2(x+3y+t) + z(x+3y+t)^2 + 4z,$$

$$\begin{aligned}\phi_3 = {} & \frac{t^5}{120} + \frac{5t^4}{120}(x+3y) + \frac{t^3}{12}(x+3y)^2 + \frac{t^2}{12}(x+3y)^3 + \frac{5t}{120}(x+3y)^4 + \frac{1}{120}(x+3y)^5 \\ & + \frac{4z^5}{15} + \frac{2z^4}{3}(x+t+3y) + \frac{2z^3}{3}(x+t+3y)^2 + \frac{z^2}{3}(x+t+3y)^3 + \frac{z}{12}(x+t+3y)^4 + 8z^3 \\ & + 8z^2(x+t+3y) + 2z(x+t+3y)^2\end{aligned}$$

$$f = W(\phi_1, \phi_2, \phi_3) = p_1, \quad u = 2\partial_x \ln W(\phi_0, \phi_1, \phi_2) = \frac{2q_1}{p_1}, \tag{3.4}$$

where

$$\begin{aligned}p_1 = {} & \frac{64z^6}{45} + \frac{64z^5}{15}(x+3y+t) + \frac{16z^4}{3}(x+3y+t)^2 - \frac{32}{3}z^4 + \frac{32z^3}{9}(x+3y+t)^3 \\ & - 16z^3(x+3y+t) + \frac{4z^2}{3}(x+3y+t)^4 - 8z^2(x+3y+t)^2 - 16z^2 + \frac{4z}{15}(x+3y+t)^5 \\ & - \frac{4}{3}z(x+3y+t)^3 + \frac{2t^5}{15}(x+3y) + \frac{t^4}{3}(x+3y)^2 + \frac{4t^3}{9}(x+3y)^3 + \frac{t^2}{3}(x+3y)^4 \\ & + \frac{2t}{15}(x+3y)^5 + \frac{1}{45}(x+3y)^6 - \frac{x^6}{45} - \frac{81y^6}{5}, \\ q_1 = {} & \frac{64z^5}{15} + \frac{32z^4}{3}(x+3y+t) + \frac{32z^3}{3}(x+3y+t)^2 - 16z^3 - 16z^2(x+3y+t) + \frac{4z}{3}(x+3y+t)^4 \\ & - 4z(x+3y+t)^2 + \frac{2t^5}{15} + \frac{2t^4}{3}(x+3y) + \frac{4t^3}{3}(x+3y)^2 + \frac{4t^2}{3}(x+3y)^3 + \frac{2t}{3}(x+3y)^4 + \frac{2}{15}(x+3y)^5 - \frac{2x^5}{15}.\end{aligned}$$

3.2. Solitons, negatons and positons

If the eigenvalue $\lambda_1 \neq 0$, $J(\lambda_1)$ becomes to the following form

$$\begin{bmatrix} \lambda_1 & & & 0 \\ 1 & \lambda_1 & & \\ & \ddots & \ddots & \\ 0 & & 1 & \lambda_1 \end{bmatrix}_{k_1 \times k_1}.$$

We start from the eigenfuction $\phi_1(\lambda_1)$, which is determined by

$$(\phi_1(\lambda_1))_{xx} = \lambda_1\phi_1(\lambda_1), \quad (\phi_1(\lambda_1))_y = 3(\phi_1(\lambda_1))_x,$$
$$(\phi_1(\lambda_1))_z = 4(\phi_1(\lambda_1))_{xxx} + 2(\phi_1(\lambda_1))_x, \quad (\phi_1(\lambda_1))_t = (\phi_1(\lambda_1))_x. \tag{3.5}$$

General solutions to this system in two cases of $\lambda_1 > 0$ and $\lambda_1 < 0$ are

$$\phi_1(\lambda_1) = C_1\sinh(\sqrt{\lambda_1}(x+3y+2(2\lambda_1+1)z+t)) + C_2\cosh(\sqrt{\lambda_1}(x+3y+2(2\lambda_1+1)z+t)), \quad \lambda_1 > 0, \tag{3.6}$$

$$\phi_1(\lambda_1) = C_3\cos(\sqrt{-\lambda_1}(3y+x+2(2\lambda_1+1)z+t)) - C_4\sin(\sqrt{-\lambda_1}(3y+x+2(2\lambda_1+1)z+t)), \quad \text{for } -\frac{1}{2} < \lambda_1 < 0, \tag{3.7}$$

$$\phi_1(\lambda_1) = -C_3\cos(\sqrt{-\lambda_1}(3y+x+2(2\lambda_1+1)z+t)) + C_4\sin(\sqrt{-\lambda_1}(3y+x+2(2\lambda_1+1)z+t)), \quad \text{for } \lambda_1 < -\frac{1}{2}, \tag{3.8}$$

respectively, where C_1, C_2, C_3 and C_4 are arbitrary real constants. But obviously the solutions (3.7) and (3.8) have the opposite sign, so we will only consider the solution (3.7) later. By an inspection, we find that

$$\begin{bmatrix} \phi_1(\lambda_1) \\ \frac{1}{1!}\partial_{\lambda_1}\phi_1(\lambda_1) \\ \vdots \\ \frac{1}{(k_1-1)!}\partial_{\lambda_1}^{k_1-1}\phi_1(\lambda_1) \end{bmatrix}_{xx} = \begin{bmatrix} \lambda_1 & & & 0 \\ 1 & \lambda_1 & & \\ & \ddots & \ddots & \\ 0 & & 1 & \lambda_1 \end{bmatrix}_{k_1\times k_1} \begin{bmatrix} \phi_1(\lambda_1) \\ \frac{1}{1!}\partial_{\lambda_1}\phi_1(\lambda_1) \\ \vdots \\ \frac{1}{(k_1-1)!}\partial_{\lambda_1}^{k_1-1}\phi_1(\lambda_1) \end{bmatrix}$$

and

$$\left(\frac{1}{j!}\partial_{\lambda_1}^{j}\phi_1(\lambda_1)\right)_y = 3\left(\frac{1}{j!}\partial_{\lambda_1}^{j}\phi_1(\lambda_1)\right)_x,$$
$$\left(\frac{1}{j!}\partial_{\lambda_1}^{j}\phi_1(\lambda_1)\right)_z = 4\left(\frac{1}{j!}\partial_{\lambda_1}^{j}\phi_1(\lambda_1)\right)_{xxx} + 2\left(\frac{1}{j!}\partial_{\lambda_1}^{j}\phi_1(\lambda_1)\right)_x,$$
$$\left(\frac{1}{j!}\partial_{\lambda_1}^{j}\phi_1(\lambda_1)\right)_t = \left(\frac{1}{j!}\partial_{\lambda_1}^{j}\phi_1(\lambda_1)\right)_x, \quad 0 \leqslant j \leqslant k_1-1.$$

Therefore, through this set of eigenfunctions, we obtain a Wronskian solution to the Eq. (1.1):

$$u = 2\partial_x \ln W(\phi_1(\lambda_1), \quad \frac{1}{1!}\partial_{\lambda_1}\phi_1(\lambda_1), \ldots, \frac{1}{(k_1-1)!}\partial_{\lambda_1}^{k_1-1}\phi_1(\lambda_1)),$$

which corresponds to the first type of Jordan blocks with a nonzero real eigenvalue.

When $\lambda_1 > 0$, we get negaton solutions, and when $\lambda_1 < 0$, we get positon solutions.

If we suppose A have m different nonzero real eigenvalues, in which there are l positive real eigenvalues and $m-l$ negative real eigenvalues, then more general negaton can be obtained by combining l sets of eigenfunctions associated with different $\lambda_i > 0$:

$$u = 2\partial_x \ln W\left(\phi_1(\lambda_1), \frac{1}{1!}\partial_{\lambda_1}\phi_1(\lambda_1), \ldots, \frac{1}{(k_1-1)!}\partial_{\lambda_1}^{k_1-1}\phi_1(\lambda_1); \cdots; \phi_l(\lambda_l), \frac{1}{1!}\partial_{\lambda_l}\phi_l(\lambda_l), \ldots, \frac{1}{(k_l-1)!}\partial_{\lambda_l}^{k_l-1}\phi_l(\lambda_l)\right).$$

Similarly, more general positon can be obtained by combining $m-l$ sets of eigenfunctions associated with different $\lambda_i < 0$:

$$u = 2\partial_x \ln W\left(\phi_1(\lambda_1), \frac{1}{1!}\partial_{\lambda_1}\phi_1(\lambda_1), \ldots, \frac{1}{(k_1-1)!}\partial_{\lambda_1}^{k_1-1}\phi_1(\lambda_1); \cdots; \phi_{m-l}(\lambda_{m-l}),\right.$$
$$\left.\frac{1}{1!}\partial_{\lambda_{m-l}}\phi_{m-l}(\lambda_{m-l}), \ldots, \frac{1}{(k_{m-l}-1)!}\partial_{\lambda_{m-l}}^{k_{m-l}-1}\phi_{m-l}(\lambda_{m-l})\right).$$

This solution is called an n-negaton of order $(k_1-1, k_2-1, \ldots, k_l-1)$ or n-positon of order $(k_1-1, k_2-1, \ldots, k_{m-l}-1)$. If $l = n$ or $l = 0$, we simply say that it is an n-negaton of order n or an n-positon of order n.

An n-soliton solution is a special n-negaton:

$$u = 2\partial_x \ln W(\phi_1, \phi_2, \ldots, \phi_n)$$

with ϕ_i given by

$$\phi_i = \cosh\left(\sqrt{\lambda_i}(x+3y+2(2\lambda_i+1)z+t)+\gamma_i\right), \quad i \text{ odd},$$
$$\phi_i = \sinh\left(\sqrt{\lambda_i}(x+3y+2(2\lambda_i+1)z+t)+\gamma_i\right), \quad i \text{ even},$$

where $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n$ and γ_i $(1 \leqslant i \leqslant n)$ are arbitrary real constants.

Two kinds of special negatons of order k are

$$u = 2\partial_x \ln W(\phi, \partial_\lambda \phi, \ldots, \partial_\lambda^{k-1}\phi), \quad \phi = \cosh(\sqrt{\lambda}(x + 3y + 2(2\lambda + 1)z + t) + \gamma),$$
$$u = 2\partial_x \ln W(\phi, \partial_\lambda \phi, \ldots, \partial_\lambda^{k-1}\phi), \quad \phi = \sinh(\sqrt{\lambda}(x + 3y + 2(2\lambda + 1)z + t) + \gamma),$$

where $\lambda > 0$ and γ is an arbitrary constant.

Two kinds of special positons of order k are

$$u = 2\partial_x \ln W(\phi, \partial_\lambda \phi, \ldots, \partial_\lambda^{k-1}\phi),$$
$$\phi(\lambda) = \cos(\sqrt{-\lambda}(3y + x + 2(2\lambda + 1)z + t) + \gamma), \quad \lambda < 0,$$
$$u = 2\partial_x \ln W(\phi, \partial_\lambda \phi, \ldots, \partial_\lambda^{k-1}\phi),$$
$$\phi(\lambda) = \sin(\sqrt{-\lambda}(3y + x + 2(2\lambda + 1)z + t) + \gamma), \quad \lambda < 0.$$

To understand the above results better, we shall give several exact solitons, positons and negatons of lower-order as follows:

(1) *Two* solitons of zero-order:

$$u = 2\partial_x \ln W(\phi_1) = 2\partial_x \ln(\cosh(\sqrt{\lambda_1}(x + 3y + 2(2\lambda_1 + 1)z + t) + \gamma_1)) = 2\sqrt{\lambda_1}\tanh(\theta_1), \tag{3.9}$$

$$u = 2\partial_x \ln W(\phi_1) = 2\partial_x \ln(\sinh(\sqrt{\lambda_1}(x + 3y + 2(2\lambda_1 + 1)z + t) + \gamma_1)) = 2\sqrt{\lambda_1}\coth(\theta_1), \tag{3.10}$$

where $\theta_1 = \sqrt{\lambda_1}(x + 3y + 2(2\lambda_1 + 1)z + t) + \gamma_1, \lambda_1 > 0$.

One soliton of first-order:

$$u = 2\partial_x \ln W(\cosh(\theta_1), \sinh(\theta_2)) = \frac{2(\lambda_1 - \lambda_2)(\sinh(\theta_1 + \theta_1) - \sinh(\theta_1 - \theta_2))}{(\sqrt{\lambda_1} - \sqrt{\lambda_2})\cosh(\theta_1 + \theta_2) - (\sqrt{\lambda_1} + \sqrt{\lambda_2})\cosh(\theta_1 - \theta_2)}, \tag{3.11}$$

where $\theta_i = \sqrt{\lambda_i}(x + 3y + 2(2\lambda_i + 1)z + t) + \gamma_i$, $\lambda_i > 0, i = 1,2$.

(2) *One* negaton of first-order:

$$u = 2\partial_x \ln W(\cosh(\theta), \partial_{\lambda_1}\cosh(\theta)) = \frac{4\sqrt{\lambda_1}(1 + \cosh(2\theta))}{2\sqrt{\lambda_1}(x + 3y + 12\lambda_1 z + 2z + t) + \sinh(2\theta)}, \tag{3.12}$$

where $\theta = (\sqrt{\lambda_1}(x + 3y + 2(2\lambda_1 + 1)z + t) + \gamma_1)$.

(3) *Two* positon of zero-order:

$$u = 2\partial_x \ln W(\phi_1) = 2\partial_x \ln(\cos(\sqrt{-\lambda_1}(x + 3y + 2(2\lambda_1 + 1)z + t) + \gamma_1)) = -2\sqrt{-\lambda_1}\tan(\theta_3). \tag{3.13}$$

$$u = 2\partial_x \ln W(\phi_1) = 2\partial_x \ln(-\sin(\sqrt{-\lambda_1}(x + 3y + 2(2\lambda_1 + 1)z + t) + \gamma_1)) = 2\sqrt{-\lambda_1}\cot(\theta_3)). \tag{3.14}$$

where $\theta_3 = \sqrt{-\lambda_1}(x + 3y + 2(2\lambda_1 + 1)z + t) + \gamma_1$.

One positon of first-order:

$$u = 2\partial_x \ln W(\cos(\theta), \partial_{\lambda_1}\cos(\theta)) = \frac{4\sqrt{-\lambda_1}(1 + \cos(2\theta))}{2\sqrt{-\lambda_1}(x + 3y + 12\lambda_1 z + 2z + t) + \sin(2\theta)}, \tag{3.15}$$

where $\theta = \sqrt{-\lambda_1}(x + 3y + 2(2\lambda_1 + 1)z + t) + \gamma_1$.

3.3. Interaction solutions

Let us assume that there are two sets of eigenfunctions

$$(\phi_1(\lambda), \phi_2(\lambda), \ldots, \phi_k(\lambda); \psi_1(\mu), \ldots, \psi_l(\mu)) \tag{3.16}$$

associated with two different eigenvalues λ and μ, respectively. A Wronskian solution $u = 2\partial_x \ln W(\phi_1(\lambda), \phi_2(\lambda), \ldots, \phi_k(\lambda); \psi_1(\mu), \ldots, \psi_l(\mu))$ is said to be a Wronskian interaction solution between two solutions determined by the two sets of eigenfunctions in (3.16). Of course, we can have more general Wronskian interaction solutions among three kinds of solutions such as rational solutions, negatons, positons.

In what follows, we shall show a few special Wronskian interaction solutions. Let us first choose different sets of special eigenfunctions:

$$\phi_{\text{rational}} = x + 3y + 2z + t,$$
$$\phi_{\text{soliton}} = \cosh(\sqrt{\lambda_1}(x + 3y + 2(2\lambda_1 + 1)z + t)),$$
$$\phi_{\text{positon}} = \cos(\sqrt{-\lambda_2}(x + 3y + 2(2\lambda_2 + 1)z + t)),$$

where $\lambda_1 > 0$, $\lambda_2 < 0$ are constants.

Three Wronskian interaction determinants between any two of a rational solution, a single soliton and a single positon are

$$W(\phi_{\text{rational}}, \phi_{\text{soliton}}) = \sqrt{\lambda_1}(x + 3y + 2z + t)\sinh(\theta_1) - \cosh(\theta_1), \tag{3.17}$$
$$W(\phi_{\text{rational}}, \phi_{\text{positon}}) = -\sqrt{-\lambda_2}(x + 3y + 2z + t)\sin(\theta_2) - \cos(\theta_2), \tag{3.18}$$
$$W(\phi_{\text{soliton}}, \phi_{\text{positon}}) = -\sqrt{-\lambda_2}\cosh(\theta_1)\sin(\theta_2) - \sqrt{\lambda_1}\cos(\theta_2)\sinh(\theta_1), \tag{3.19}$$

where $\theta_1 = \sqrt{\lambda_1}(x + 3y + 2(2\lambda_1 + 1)z + t), \theta_2 = \sqrt{-\lambda_2}(x + 3y + 2(2\lambda_2 + 1)z + t)$.

Further, the corresponding Wronskian interaction solutions are

$$u_{rs} = 2\partial_x \ln W(\phi_{\text{rational}}, \phi_{\text{soliton}}) = \frac{2\lambda_1(x + 3y + 2z + t)\cosh(\theta_1)}{\sqrt{\lambda_1}(x + 3y + 2z + t)\sinh(\theta_1) - \cosh(\theta_1)}, \tag{3.20}$$
$$u_{rp} = 2\partial_x \ln W(\phi_{\text{rational}}, \phi_{\text{positon}}) = \frac{-2\lambda_2(x + 3y + 2z + t)\cos(\theta_2)}{\sqrt{-\lambda_2}(x + 3y + 2z + t)\sin(\theta_2) + \cos(\theta_2)}, \tag{3.21}$$
$$u_{sp} = 2\partial_x \ln W(\phi_{\text{soliton}}, \phi_{\text{ positon}}) = \frac{2(\lambda_1 - \lambda_2)\cosh(\theta_1)\cos(\theta_2)}{\sqrt{-\lambda_2}\cosh(\theta_1)\sin(\theta_2) + \sqrt{\lambda_1}\sinh(\theta_1)\cos(\theta_2)}, \tag{3.22}$$

where $\theta_1 = \sqrt{\lambda_1}(x + 3y + 2(2\lambda_1 + 1)z + t), \theta_2 = \sqrt{-\lambda_2}(x + 3y + 2(2\lambda_2 + 1)z + t)$.

The following is one Wronskian interaction determinant and solution involving the three eigenfunctions.

$$\begin{aligned} W(\phi_{\text{rational}}, \phi_{\text{soliton}}, \phi_{\text{positon}}) = {} & (x + 3y + 2z + t)\Big(\lambda_2\sqrt{\lambda_1}\sinh(\theta_1)\cos(\theta_2) + \lambda_1\sqrt{-\lambda_2}\cosh(\theta_1)\sin(\theta_2)\Big) \\ & + (\lambda_1 - \lambda_2)\cosh(\theta_1)\cos(\theta_2) = p_2, \end{aligned} \tag{3.23}$$

$$u_{rsp} = 2\partial_x \ln W(\phi_{\text{rational}}, \phi_{\text{soliton}}, \phi_{\text{positon}}) = \frac{2q_2}{p_2}, \tag{3.24}$$

where

$$q_2 = (x + 3y + 2z + t)\sqrt{-\lambda_1\lambda_2}(\lambda_1 - \lambda_2)\sinh(\theta_1)\sin(\theta_2) + \Big(\lambda_1\sqrt{\lambda_1}\sinh(\theta_1)\cos(\theta_2) + \lambda_2\sqrt{-\lambda_2}\cosh(\theta_1)\sin(\theta_2)\Big),$$

$$\theta_1 = \sqrt{\lambda_1}(x + 3y + 2(2\lambda_1 + 1)z + t), \quad \text{and} \quad \theta_2 = \sqrt{-\lambda_2}(x + 3y + 2(2\lambda_2 + 1)z + t).$$

When the corresponding Jordan form of a real matrix is the second type of block, the solutions of the Eq. (2.10) will become very complicated, so we omit that here.

4. Conclusion

In sum, we gave the Wronskian determinant solutions of the (3 + 1)-dimensional Jimbo–Miwa equation through the Wronskian technique. Moreover, we obtained some rational solutions, soliton solutions, positons and negatons of this equation by solving the resultant systems of linear partial differential equations which guarantee that the Wronskian determinant solves the equation in the bilinear form. All these show the richness of the solution space of the (3 + 1)-dimensional Jimbo–Miwa equation and the resulting solutions are expected to help understand wave dynamics in weakly nonlinear and dispersive media.

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