GENERAL

Grammian and Pfaffian solutions as well as Pfaffianization for a (3+1)-dimensional generalized shallow water equation

Chin. Phys. B Vol. 21, No. 7 (2012) 070212

Grammian and Pfaffian solutions as well as Pfaffianization for a (3+1)-dimensional generalized shallow water equation*

Tang Ya-Ning(唐亚宁)[a)†], Ma Wen-Xiu(马文秀)[b)], and Xu Wei(徐 伟)[a)]

a) Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an 710072, China

b) Department of Mathematics and Statistics, University of South Florida, Tampa FL 33620-5700, USA

(Received 21 November 2011; revised manuscript received 5 January 2012)

Based on the Grammian and Pfaffian derivative formulae, Grammian and Pfaffian solutions are obtained for a (3+1)-dimensional generalized shallow water equation in the Hirota bilinear form. Moreover, a Pfaffian extension is made for the equation by means of the Pfaffianization procedure, the Wronski-type and Gramm-type Pfaffian solutions of the resulting coupled system are presented.

Keywords: Hirota bilinear form, Grammian and Pfaffian solutions, Wronski-type and Gramm-type Pfaffian solutions

PACS: 02.90.+p, 02.30.Ik, 05.45.Yv **DOI:** 10.1088/1674-1056/21/7/070212

1. Introduction

The investigation of the exact traveling wave solutions to nonlinear partial differential equations plays an important role in the study of nonlinear physical phenomena. During the past several decades, many powerful and efficient methods have been proposed to obtain the exact traveling wave and solitary wave solutions of nonlinear evolution equations.[1−4]

It is known that nonlinear equations such as the Korteweg–de Vries (KdV) equation, the Boussinesq equation, and the Kadomtsev–Petviashvili (KP) equation possess multi-soliton solutions generated from a combination of exponential waves. Solitons and positons can be expressed in terms of Wronskian determinants.[5−15] For higher-dimensional soliton equations, there exist Grammian solutions and Pfaffian solutions.[16−18] The Grammian solutions to the KP equation were constructed by Nakamura[19] and the Pfaffian solutions to the B-type Kadomtsev-Petviashvili (BKP) equation were presented by Hirota.[20] The Hirota bilinear form plays a crucial role in the construction of these solutions.[16,21]

The following (3+1)-dimensional generalized shallow water equation:

$$u_{xxxy} - 3u_{xx}u_y - 3u_xu_{xy} + u_{yt} - u_{xz} = 0 \qquad (1)$$

was investigated in different ways.[22,23] In Ref. [22], the soliton-type solutions for Eq. (1) were constructed by using a generalized tanh algorithm with symbolic computation. In Ref. [23], the traveling wave solutions of Eq. (1) expressed by hyperbolic, trigonometric, and rational functions were established with the G'/G-expansion method, where $G = G(\xi)$ satisfies a second-order linear ordinary differential equation.

Equation (1) can be written as

$$u_{xxxy} + 3u_{xx}u_y + 3u_xu_{xy} - u_{yt} - u_{xz} = 0, \qquad (2)$$

under a scale transformation $x \to -x$, and thus we can discuss the solutions of Eq. (2) equivalently.

Utilizing the Cole–Hopf transformation

$$u = 2(\ln f)_x, \qquad (3)$$

we obtain the Hirota bilinear form of Eq. (2)

$$\begin{aligned}&(D_x^3D_y - D_yD_t - D_xD_z)f\cdot f\\ =\ &(f_{xxxy} - f_{yt} - f_{xz})f - f_{xxx}f_y\\ &-3f_{xxy}f_x + f_yf_t + f_xf_z + 3f_{xx}f_{xy}\end{aligned}$$

*Project supported by the National Natural Science Foundation of China (Grant Nos. 10932009 and 11172233), the Northwestern Polytechnical University Foundation for Fundamental Research, China (Grant No. GBKY1034), the State Administration of Foreign Experts Affairs of China, and the Chunhui Plan of the Ministry of Education of China.

†Corresponding author. E-mail: tyaning@nwpu.edu.cn

$$= 0, \tag{4}$$

where D_x, D_y, D_z, and D_t are the Hirota operators.[16]

In this paper, we will show that Eq. (2) has a class of Grammian solutions and a class of Pfaffian solutions. In addition, we will carry out the Pfaffianization procedure[24] to extend Eq. (2) to a new coupled system, whose Wronski-type and Gramm-type Pfaffian solutions exist.

2. Grammian solutions

Let us now introduce the following Grammian determinant:

$$f = \det(a_{ij})_{1\le i,j\le N},\quad a_{ij} = c_{ij} + \int^x \phi_i\psi_j\,\mathrm{d}x, \tag{5}$$

where c_{ij} is constant, and functions $\phi_i = \phi_i(x,y,z,t)$ and $\psi_j = \psi_j(x,y,z,t)$ satisfy

$$\begin{aligned}&\phi_{i,y} = k\phi_{i,xx},\quad \phi_{i,t} = -2\phi_{i,xxx},\\ &\phi_{i,z} = 3k\phi_{i,xxxx},\quad \psi_{j,y} = -k\psi_{j,xx},\\ &\psi_{j,t} = -2\psi_{j,xxx},\quad \psi_{j,z} = -3k\psi_{j,xxxx},\end{aligned} \tag{6}$$

where $i \ge 1$, $j \le N$, and k is an arbitrary nonzero number. In the above definition of a_{ij}, the lower limit of integration is chosen to make sure that the functions ϕ_i and ψ_j as well as their derivatives are zero in the lower limit.

Theorem 1 If ϕ_i and ψ_j satisfy Eq. (6), then the Grammian determinant f defined by Eq. (5) is the solution of Eq. (4).

Proof Let us express the determinant f by means of an N-th order Pfaffian as

$$f = (1, 2, \ldots, N, N^*, \ldots, 2^*, 1^*), \tag{7}$$

where $(i, j^*) = a_{ij}$, and $(i, j) = (i^*, j^*) = 0$.

To compute the derivatives of the entries a_{ij} and the Grammian determinant f, we introduce new Pfaffian entries as usual

$$\begin{aligned}&(d_n^*, i) = \frac{\partial^n}{\partial x^n}\phi_i,\quad (d_n, j^*) = \frac{\partial^n}{\partial x^n}\psi_j,\\ &(d_m, d_n^*) = (d_n, i) = (d_m^*, j^*) = 0,\quad m, n \ge 0.\end{aligned} \tag{8}$$

In terms of these new entries and Eq. (6), the derivatives of the entries $a_{ij} = (i, j^*)$ are given by

$$\begin{aligned}
\frac{\partial}{\partial x}a_{i,j} &= \phi_i\psi_j = (d_0, d_0^*, i, j^*),\\
\frac{\partial}{\partial y}a_{i,j} &= \int^x (\phi_{i,y}\psi_j + \phi_i\psi_{j,y})\mathrm{d}x\\
&= k\int^x (\phi_{i,xx}\psi_j - \phi_i\psi_{j,xx})\mathrm{d}x\\
&= k(\phi_{i,x}\psi_j - \phi_i\psi_{j,x})\\
&= k[-(d_1, d_0^*, i, j^*) + (d_0, d_1^*, i, j^*)],\\
\frac{\partial}{\partial t}a_{i,j} &= \int^x (\phi_{i,t}\psi_j + \phi_i\psi_{j,t})\mathrm{d}x\\
&= -2\int^x (\phi_{i,xxx}\psi_j + \phi_i\psi_{j,xxx})\mathrm{d}x\\
&= -2(\phi_{i,xx}\psi_j - \phi_{i,x}\psi_{j,x} + \phi_i\psi_{j,xx})\\
&= -2[(d_2, d_0^*, i, j^*) - (d_1, d_1^*, i, j^*)\\
&\quad + (d_0, d_2^*, i, j^*)],\\
\frac{\partial}{\partial z}a_{i,j} &= \int^x (\phi_{i,z}\psi_j + \phi_i\psi_{j,z})\mathrm{d}x\\
&= 3k\int^x (\phi_{i,xxxx}\psi_j - \phi_i\psi_{j,xxxx})\mathrm{d}x\\
&= 3k(\phi_{i,xxx}\psi_j - \phi_{i,xx}\psi_{j,x}\\
&\quad - \phi_i\psi_{j,xxx} + \phi_{i,x}\psi_{j,xx})\\
&= 3k[-(d_3, d_0^*, i, j^*) + (d_0, d_3^*, i, j^*)\\
&\quad + (d_2, d_1^*, i, j^*) - (d_1, d_2^*, i, j^*)].
\end{aligned}$$

Then we can develop differential rules for Pfaffians as in Ref. [16], and compute various derivatives of the Grammian determinant f with respect to variables x, y, z, and t as follows:

$$\begin{aligned}
f_x &= (d_0, d_0^*, \bullet),\\
f_{xx} &= (d_1, d_0^*, \bullet) + (d_0, d_1^*, \bullet),\\
f_{xxx} &= (d_2, d_0^*, \bullet) + 2(d_1, d_1^*, \bullet) + (d_0, d_2^*, \bullet),\\
f_y &= k[-(d_1, d_0^*, \bullet) + (d_0, d_1^*, \bullet)],\\
f_{xy} &= k[(d_0, d_2^*, \bullet) - (d_2, d_0^*, \bullet)],\\
f_{xxy} &= k[-(d_3, d_0^*, \bullet) + (d_0, d_3^*, \bullet)\\
&\quad - (d_2, d_1^*, \bullet) + (d_1, d_2^*, \bullet)],\\
f_{xxxy} &= k[(d_0, d_4^*, \bullet) - (d_4, d_0^*, \bullet)\\
&\quad - 2(d_3, d_1^*, \bullet) + 2(d_1, d_3^*, \bullet)\\
&\quad - (d_0, d_2^*, d_1, d_0^*, \bullet) + (d_2, d_0^*, d_0, d_1^*, \bullet)],\\
f_z &= 3k[-(d_3, d_0^*, \bullet) + (d_0, d_3^*, \bullet)\\
&\quad + (d_2, d_1^*, \bullet) - (d_1, d_2^*, \bullet)],\\
f_{xz} &= 3k[(d_0, d_4^*, \bullet) - (d_4, d_0^*, \bullet)\\
&\quad + (d_2, d_1^*, d_0, d_0^*, \bullet) - (d_1, d_2^*, d_0, d_0^*, \bullet)],\\
f_t &= -2[(d_2, d_0^*, \bullet) - (d_1, d_1^*, \bullet) + (d_0, d_2^*, \bullet)],\\
f_{yt} &= -2k[(d_0, d_4^*, \bullet) - (d_4, d_0^*, \bullet)\\
&\quad +(d_3, d_1^*, \bullet) - (d_1, d_3^*, \bullet)\\
&\quad - (d_0, d_2^*, d_1, d_0^*, \bullet) + (d_2, d_0^*, d_0, d_1^*, \bullet)],
\end{aligned}$$

where the abbreviated notation $\bullet$ denotes the list of indices $1, 2, \ldots, N, N^*, (N-1)^*, \ldots, 1^*$ that are common to each Pfaffian. Substituting the above derivatives of f into Eq. (3), we obtain

$$
\begin{aligned}
&(f_{xxxy} - f_{yt} - f_{xz})f \\
&= 6k[-(d_0, d_2^*, d_1, d_0^*, \bullet) + (d_2, d_0^*, d_0, d_1^*, \bullet)](\bullet), \\
&- 3f_{xxy}f_x + f_x f_z = 6k[(d_2, d_1^*, \bullet)(d_0, d_0^*, \bullet) \\
&- (d_1, d_2^*, \bullet)(d_0, d_0^*, \bullet)], \\
&- f_{xxx}f_y + f_y f_t + 3f_{xx}f_{xy} \\
&= 6k[-(d_2, d_0^*, \bullet)(d_0, d_1^*, \bullet) \\
&+ (d_0, d_2^*, \bullet)(d_1, d_0^*, \bullet)],
\end{aligned}
$$

and further obtain that

$$
(D_x^3 D_y - D_y D_t - D_x D_z) f \cdot f = 6k(A_1 + A_2), \quad (9)
$$

where

$$
\begin{aligned}
A_1 &= -(d_0, d_2^*, d_1, d_0^*, \bullet)(\bullet) - (d_1, d_2^*, \bullet)(d_0, d_0^*, \bullet) \\
&\quad + (d_0, d_2^*, \bullet)(d_1, d_0^*, \bullet) \\
&= -(d_0, d_2^*, d_1, d_0^*, \bullet)(\bullet) + (d_0, d_2^*, \bullet)(d_1, d_0^*, \bullet) \\
&\quad + (d_0, d_0^*, \bullet)(d_2^*, d_1, \bullet) \\
&= 0,
\end{aligned}
$$

and

$$
\begin{aligned}
A_2 &= (d_2, d_0^*, d_0, d_1^*, \bullet)(\bullet) + (d_2, d_1^*, \bullet)(d_0, d_0^*, \bullet) \\
&\quad - (d_2, d_0^*, \bullet)(d_0, d_1^*, \bullet) \\
&= (d_2, d_0^*, d_0, d_1^*, \bullet)(\bullet) - (d_2, d_0^*, \bullet)(d_0, d_1^*, \bullet) \\
&\quad - (d_2, d_1^*, \bullet)(d_0^*, d_0, \bullet) \\
&= 0.
\end{aligned}
$$

It is easy to see that Eq. (9) is nothing but the Jacobi identity for determinants. Therefore, we show that f defined by Eq. (5) is the solution of Eq. (4) under the linear differential conditions (6).

The corresponding solution of Eq. (2) is

$$
\begin{aligned}
&u = 2(\ln f)_x, \\
&f = \det\left(c_{ij} + \int^x \phi_i \psi_j \,\mathrm{d}x\right)_{1 \le i,j \le N}, \quad (10)
\end{aligned}
$$

where c_{ij} is a constant.

3. Pfaffian solutions

Let us now introduce the following Pfaffian:

$$
\begin{aligned}
&f_N = \mathrm{Pf.}(a_{ij})_{1 \le i,j \le 2N}, \\
&a_{ij} = c_{ij} + \int^x D_x \phi_i \cdot \phi_j \,\mathrm{d}x, \quad (11)
\end{aligned}
$$

where c_{ij} is constant, and function $\phi_i = \phi_i(x, y, z, t)$ satisfies

$$
\begin{aligned}
&\phi_{i,y} = l \int^x \phi_i \,\mathrm{d}x, \quad \phi_{i,z} = 3l\phi_{i,x}, \\
&\phi_{i,t} = \phi_{i,xxx}, \quad (12)
\end{aligned}
$$

where $1 \le i \le 2N$ and l is an arbitrary nonzero number.

Theorem 2 If ϕ_i $(1 \le i \le 2N)$ satisfies Eq. (12), then the Pfaffian f_N defined by Eq. (11) is the solution of Eq. (4).

Proof Let us express the determinant f_N by means of an N-th order Pfaffian as

$$
f_N = (1, 2, \ldots, 2N), \quad (13)
$$

where the Pfaffian entries $(i, j) = a_{ij}$.

Based on Eq. (12), the derivatives of the entries $a_{ij} = (i, j)$ are given by

$$
\begin{aligned}
\frac{\partial}{\partial x} a_{ij} &= \frac{\partial}{\partial x}\left(\int^x D_x \phi_i \cdot \phi_j \,\mathrm{d}x\right) \\
&= \phi_{i,x}\phi_j - \phi_i \phi_{j,x} = (d_0, d_1, i, j), \\
\frac{\partial}{\partial y} a_{ij} &= l \int^x (\phi_{i,xy}\phi_j + \phi_{i,x}\phi_{j,y} \\
&\quad - \phi_{i,y}\phi_{j,x} - \phi_i \phi_{j,xy}) \,\mathrm{d}x \\
&= l\,(\phi_i \phi_{j,y} - \phi_{i,y}\phi_j) = l(d_{-1}, d_0, i, j), \\
\frac{\partial}{\partial z} a_{ij} &= 3l \frac{\partial}{\partial x}\left(\int^x D_x \phi_i \cdot \phi_j \,\mathrm{d}x\right) \\
&= 3l\,(\phi_{i,x}\phi_j - \phi_i \phi_{j,x}) = 3l(d_0, d_1, i, j), \\
\frac{\partial}{\partial t} a_{ij} &= \int^x (\phi_{i,xt}\phi_j + \phi_{i,x}\phi_{j,t} - \phi_{i,t}\phi_{j,x} - \phi_i \phi_{j,xt}) \,\mathrm{d}x \\
&= \phi_{i,xxx}\phi_j - \phi_i \phi_{j,xxx} - 2\,(\phi_{i,xx}\phi_{j,x} - \phi_{i,x}\phi_{j,xx}) \\
&= (d_0, d_3, i, j) - 2(d_1, d_2, i, j),
\end{aligned}
$$

with the definition that

$$
\begin{aligned}
&(d_{-1}, i) = \int^x \phi_i \,\mathrm{d}x, \quad (d_n, i) = \frac{\partial^n}{\partial x^n}\phi_i, \\
&(d_m, d_n) = 0, \quad m, n = 0, 1, 2, 3. \quad (14)
\end{aligned}
$$

Then we can develop differential rules for Pfaffians as in Ref. [16], and compute various derivatives of the Pfaffian $f_N = (1, 2, \ldots, 2N) = (\bullet)$ with respect to the variables x, y, z, and t as follows:

$$
\begin{aligned}
f_{N,x} &= (d_0, d_1, \bullet), \quad f_{N,xx} = (d_0, d_2, \bullet), \\
f_{N,xxx} &= (d_1, d_2, \bullet) + (d_0, d_3, \bullet), \\
f_{N,y} &= l(d_{-1}, d_0, \bullet), \quad f_{N,yx} = l(d_{-1}, d_1, \bullet), \\
f_{N,yxx} &= l[(d_{-1}, d_2, \bullet) + (d_0, d_1, \bullet)], \\
f_{N,yxxx} &= l[(d_{-1}, d_3, \bullet) + 2(d_0, d_2, \bullet) \\
&\quad + (d_{-1}, d_0, d_1, d_2, \bullet)], \\
f_{N,t} &= (d_0, d_3, \bullet) - 2(d_1, d_2, \bullet), \\
f_{N,yt} &= l[(d_{-1}, d_3, \bullet) - (d_0, d_2, \bullet)
\end{aligned}
$$

$$-2(d_{-1},d_0,d_1,d_2,\bullet)],$$
$$f_{N,z} = 3l(d_0,d_1,\bullet), \quad f_{N,xz} = 3l(d_0,d_2,\bullet).$$

Substituting the above derivatives of f_N into Eq. (3), we can now solve the following equations:

$$\begin{aligned}
&(f_{N,xxxy} - f_{N,yt} - f_{N,xz})f_N \\
&= 3l(d_{-1},d_0,d_1,d_2,\bullet)(\bullet), \\
&- f_{N,xxx}f_{N,y} - 3f_{N,xxy}f_{N,x} + f_{N,y}f_{N,t} + f_{N,x}f_{N,z} \\
&= -3l(d_{-1},d_0,\bullet)(d_1,d_2,\bullet) - 3l(d_{-1},d_2,\bullet)(d_0,d_1,\bullet), \\
&3f_{N,xx}f_{N,xy} = 3l(d_{-1},d_1,\bullet)(d_0,d_2,\bullet),
\end{aligned}$$

and further obtain

$$\begin{aligned}
&(D_x^3D_y - D_yD_t - D_xD_z)f_N\cdot f_N \\
&= 3l[(d_{-1},d_0,d_1,d_2,\bullet)(\bullet) - (d_{-1},d_0,\bullet)(d_1,d_2,\bullet) \\
&\quad + (d_{-1},d_1,\bullet)(d_0,d_2,\bullet) - (d_{-1},d_2,\bullet)(d_0,d_1,\bullet)] \\
&= 0.
\end{aligned}$$

This last equality is nothing but the Pfaffian identity. Therefore, we show that f_N defined by Eq.(11) is the solution of the bilinear equation (4).

The corresponding solution of Eq. (2) is

$$\begin{aligned}
&u = 2(\ln f_N)_x, \\
&f_N = \mathrm{Pf.}\left(c_{ij} + \int^x D_x\phi_i\cdot\phi_j\,\mathrm{d}x\right)_{1\le i,j\le 2N}, \quad (15)
\end{aligned}$$

where c_{ij} is a constant.

4. Pfaffianization

Recently, a procedure called Pfaffianization was developed by Hirota and Ohta to produce new coupled systems of soliton equations from known soliton equations.[24] These Pfaffianized equations appear as generalized systems of original equations and have solutions expressed in terms of Pfaffians.

4.1. Wronski-type Pfaffian solutions

In what follows, we shall apply the Pfaffianization procedure to the bilinear equation (3) to generate a new coupled system. Let us now consider the Wronski-type Pfaffian $f_N = (1,2,\ldots,2N)$ with its elements satisfying

$$\begin{aligned}
&\frac{\partial}{\partial x}(i,j) = (i+1,j) + (i,j+1), \\
&\frac{\partial}{\partial y}(i,j) = k[(i+2,j) + (i,j+2)], \\
&\frac{\partial}{\partial t}(i,j) = -2[(i+3,j) + (i,j+3)], \\
&\frac{\partial}{\partial z}(i,j) = 3k[(i+4,j) + (i,j+4)], \qquad (16)
\end{aligned}$$

where k is an arbitrary nonzero number. Taking the above assumption into account, we can compute various derivatives of the Pfaffian $f_N = (1,2,\ldots,2N)$ ($f_N = (\widehat{2N})$ for short) with respect to the variables x, y, z, and t, i.e.,

$$\begin{aligned}
f_{N,x} &= (\widehat{2N-1}, 2N+1), \\
f_{N,xx} &= (\widehat{2N-2}, 2N, 2N+1) + (\widehat{2N-1}, 2N+2), \\
f_{N,xxx} &= (\widehat{2N-3}, 2N-1, 2N, 2N+1) \\
&\quad + 2(\widehat{2N-2}, 2N, 2N+2) \\
&\quad + (\widehat{2N-1}, 2N+3), \\
f_{N,xxxx} &= (\widehat{2N-4}, 2N-2, 2N-1, 2N, 2N+1) \\
&\quad + 3(\widehat{2N-3}, 2N-1, 2N, 2N+2) \\
&\quad + 2(\widehat{2N-2}, 2N+1, 2N+2) \\
&\quad + 3(\widehat{2N-2}, 2N, 2N+3) \\
&\quad + (\widehat{2N-1}, 2N+4), \\
f_{N,y} &= k[-(\widehat{2N-2}, 2N, 2N+1) \\
&\quad + (\widehat{2N-1}, 2N+2)], \\
f_{N,xy} &= k[-(\widehat{2N-3}, 2N-1, 2N, 2N+1) \\
&\quad + (\widehat{2N-1}, 2N+3)], \\
f_{N,xxy} &= k[-(\widehat{2N-4}, 2N-2, 2N-1, 2N, 2N+1) \\
&\quad - (\widehat{2N-3}, 2N-1, 2N, 2N+2) \\
&\quad + (\widehat{2N-2}, 2N, 2N+3) + (\widehat{2N-1}, 2N+4)], \\
f_{N,xxxy} &= k[-(\widehat{2N-5}, 2N-3, \ldots, 2N+1) \\
&\quad + (\widehat{2N-1}, 2N+5) \\
&\quad - (\widehat{2N-3}, 2N-1, 2N+1, 2N+2) \\
&\quad + (\widehat{2N-2}, 2N+1, 2N+3) \\
&\quad - 2(\widehat{2N-4}, 2N-2, 2N-1, 2N, 2N+2)], \\
f_{N,yy} &= k^2[(\widehat{2N-4}, 2N-2, 2N-1, 2N, 2N+1) \\
&\quad - (\widehat{2N-3}, 2N-1, 2N, 2N+2) \\
&\quad - (\widehat{2N-2}, 2N, 2N+3) \\
&\quad + 2(\widehat{2N-2}, 2N+1, 2N+2) \\
&\quad + (\widehat{2N-1}, 2N+4)], \\
f_{N,t} &= -2[(\widehat{2N-3}, 2N-1, 2N, 2N+1) \\
&\quad - (\widehat{2N-2}, 2N, 2N+2) + (\widehat{2N-1}, 2N+3)], \\
f_{N,xt} &= -2[(\widehat{2N-4}, 2N-2, 2N-1, 2N, 2N+1) \\
&\quad - (\widehat{2N-2}, 2N+1, 2N+2) \\
&\quad + (\widehat{2N-1}, 2N+4)], \\
f_{N,yt} &= -2k[-(\widehat{2N-5}, 2N-3, \ldots, 2N+1) \\
&\quad - (\widehat{2N-3}, 2N-1, 2N+1, 2N+2) \\
&\quad + (\widehat{2N-4}, 2N-2, 2N-1, 2N, 2N+2) \\
&\quad - (\widehat{2N-2}, 2N, 2N+4)
\end{aligned}$$

$$\begin{aligned} &+ (\widehat{2N-2}, 2N+1, 2N+3) \\ &+ (\widehat{2N-1}, 2N+5)], \\ f_{N,z} =&\ 3k[-(\widehat{2N-4}, 2N-2, 2N-1, 2N, 2N+1) \\ &+ (\widehat{2N-3}, 2N-1, 2N, 2N+2) \\ &- (\widehat{2N-2}, 2N, 2N+3) \\ &+ (\widehat{2N-1}, 2N+4)], \\ f_{N,xz} =&\ 3k[-(\widehat{2N-5}, 2N-3, \ldots, 2N+1) \\ &+ (\widehat{2N-3}, 2N-1, 2N+1, 2N+2) \\ &- (\widehat{2N-2}, 2N+1, 2N+3) \\ &+ (\widehat{2N-1}, 2N+5)]. \end{aligned}$$

Therefore, we can now solve the following equations:

$$\begin{aligned} &(f_{N,xxxy} - f_{N,yt} - f_{N,xz})f_N \\ =&\ 6k[-(\widehat{2N-3}, 2N-1, 2N+1, 2N+2) \\ &+ (\widehat{2N-2}, 2N+1, 2N+3)](\widehat{2N}), \\ &- 3f_{N,xxy}f_{N,x} + f_{N,x}f_{N,z} \\ =&\ 6k[(\widehat{2N-3}, 2N-1, 2N, 2N+2) \\ &- (\widehat{2N-2}, 2N, 2N+3)](\widehat{2N-1}, 2N+1), \\ &- f_{N,xxx}f_{N,y} + 3f_{N,xx}f_{N,xy} + f_{N,y}f_{N,t} \\ =&\ 6k[-(\widehat{2N-3}, 2N-1, 2N, 2N+1) \\ &\times (\widehat{2N-1}, 2N+2) \\ &+ (\widehat{2N-1}, 2N+3)(\widehat{2N-2}, 2N, 2N+1)], \end{aligned}$$

and further obtain

$$(D_x^3 D_y - D_y D_t - D_x D_z) f_N \cdot f_N = 6k(B_1 + B_2), \quad (17)$$

where

$$\begin{aligned} B_1 =&\ (\widehat{2N-3}, 2N-1, 2N-2, 2N) \\ &\times (\widehat{2N-3}, 2N-1, 2N+1, 2N+2) \\ &- (\widehat{2N-3}, 2N-1, 2N-2, 2N+1) \\ &\times (\widehat{2N-3}, 2N-1, 2N, 2N+2) \\ &+ (\widehat{2N-3}, 2N-1, 2N-2, 2N+2) \\ &\times (\widehat{2N-3}, 2N-1, 2N, 2N+1), \end{aligned} \quad (18)$$

and

$$\begin{aligned} B_2 =&\ (\widehat{2N-2}, 2N-1, 2N) \\ &\times (\widehat{2N-2}, 2N+1, 2N+3) \\ &- (\widehat{2N-2}, 2N-1, 2N+1) \\ &\times (\widehat{2N-2}, 2N, 2N+3) \\ &+ (\widehat{2N-2}, 2N-1, 2N+3) \\ &\times (\widehat{2N-2}, 2N, 2N+1). \end{aligned} \quad (19)$$

In view of Eqs. (18) and (19), it is obvious that Eq. (17) is not equal to zero. Following the Hirota–Ohta Pfaffianization procedure,[24] introducing

$$\begin{aligned} g_N =&\ (1, 2, \ldots, 2N-2) = (\widehat{2N-2}), \\ h_N =&\ (1, 2, \ldots, 2N+1, 2N+3) \\ =&\ (\widehat{2N+1}, 2N+3), \\ \hat{g}_N =&\ (1, 2, \ldots, 2N-3, 2N-1) \\ =&\ (\widehat{2N-3}, 2N-1), \\ \hat{h}_N =&\ (1, 2, \ldots, 2N+2) = (\widehat{2N+2}), \end{aligned} \quad (20)$$

then employing the Pfaffian identities, we obtain three bilinear equations from Eqs. (17) and (20) as

$$\begin{aligned} &(D_x^3 D_y - D_y D_t - D_x D_z) f_N \cdot f_N \\ &+ 6k(\hat{g}_N \hat{h}_N - g_N h_N) = 0, \\ &(k^2 D_x^4 + 2k^2 D_x D_t - 3D_y^2 - 2kD_z) g_N \cdot f_N = 0, \\ &(k^2 D_x^4 + 2k^2 D_x D_t - 3D_y^2 + 2kD_z) \hat{h}_N \cdot f_N = 0. \end{aligned} \quad (21)$$

The procedures for deducing the last two expressions of Eq. (21), which are similar to that for the first one of Eq. (21), are neglected here. We call Eq. (21) the coupled (3+1)-dimensional generalized shallow water Hirota bilinear equation.

Let us now summarize the above result.

Theorem 3 Assume that the Pfaffian entries (i, j) satisfy the linear differential equations in Eq. (16). Then $f_N = (1, 2, \ldots, 2N)$ as well as g_N, h_N, $\hat{g}_N$, and $\hat{h}_N$ defined by Eq. (20) solve the Pfaffianized coupled Hirota bilinear equation (21).

By further introducing the dependent variable transformation

$$\begin{aligned} &u = 2(\ln f_N)_x, \quad v = \hat{g}_N / f_N, \\ &\hat{v} = \hat{h}_N / f_N, \quad w = g_N / f_N, \quad \hat{w} = h_N / f_N, \end{aligned} \quad (22)$$

then the coupled (3+1)-dimensional generalized bilinear equation (21) is mapped into

$$\begin{cases} u_{xxxy} - 3u_{xx}u_y - 3u_x u_{xy} + u_{yt} - u_{xz} + 6k(v\hat{v} - w\hat{w})_x = 0, \\ k^2 w_{xxxx} + 6k^2 w_{xx}u_x + k^2 w u_{xxx} + 3k^2 w u_x^2 + 2k^2 w_{xt} + 2k^2 w u_t - 3w_{yy} - 3\partial_x^{-1} u_{yy} w - 2k w_z = 0, \\ k^2 \hat{v}_{xxxx} + 6k^2 \hat{v}_{xx}u_x + k^2 \hat{v} u_{xxx} + 3k^2 \hat{v} u_x^2 + 2k^2 \hat{v}_{xt} - 2k^2 \hat{v} u_t - 3\hat{v}_{yy} - 3\partial_x^{-1} u_{yy} \hat{v} + 2k\hat{v}_z = 0. \end{cases} \quad (23)$$

It is easy to see that the coupled system (23) can be reduced to Eq. (2) when $v = \hat{v} = w = \hat{w} = 0$.

4.2. Gramm-type Pfaffian solutions

In this subsection, we would like to present a class of Gramm-type Pfaffian solutions to the Pfaffianized coupled system (23). Consider the following Gramm-type pfaffians

$$\begin{cases} f_N = (1,2,\ldots,2N), \\ g_N = (c_1,c_0,1,2,\ldots,2N), \\ h_N = (d_0,d_2,1,2,\ldots,2N), \\ \hat{g}_N = (c_2,c_0,1,2,\ldots,2N), \\ \hat{h}_N = (d_0,d_1,1,2,\ldots,2N), \end{cases} \tag{24}$$

where different types of Pfaffian entries (i,j) are defined as

$$\begin{cases} (i,j) = c_{ij} + \int^x (\phi_i\psi_j - \phi_j\psi_i)\,\mathrm{d}x, \quad c_{ij} = -c_{ji}, \\ (d_n,i) = \dfrac{\partial^n}{\partial x^n}\phi_i, \quad (c_n,i) = \dfrac{\partial^n}{\partial x^n}\psi_i, \\ (d_m,d_n) = (c_m,c_n) = (c_m,d_n) = 0, \end{cases} \tag{25}$$

where c_{ij} is constant, $\phi_i = \phi_i(x,y,z,t)$, and $\psi_j = \psi_j(x,y,z,t)$ $(i,j = 1,2,\ldots,2N)$. In the above definition of (i,j), the lower limit of integration is chosen to make sure that the functions ϕ_i and ψ_j as well as their derivatives are zero in the lower limit.

Theorem 4 Assume ϕ_i and ψ_j satisfy the following linear differential equations:

$$\begin{cases} \phi_{i,y} = k\phi_{i,xx}, \quad \phi_{i,t} = -2\phi_{i,xxx}, \\ \phi_{i,z} = 3k\phi_{i,xxxx} \qquad 1 \le i \le 2N, \\ \psi_{j,y} = -k\psi_{j,xx}, \quad \psi_{j,t} = -2\psi_{j,xxx}, \\ \psi_{j,z} = -3k\psi_{j,xxxx}, \quad 1 \le j \le 2N. \end{cases} \tag{26}$$

Then f_N, g_N, h_N, $\hat{g}_N$, and $\hat{h}_N$ defined by Eq. (24) are the solutions of the Pfaffianized coupled Hirota bilinear equation (21).

Proof Based on the conditions (26), the derivatives of the Pfaffian entries (i,j) are given by

$$\begin{aligned}
\frac{\partial}{\partial x}(i,j) &= \phi_i\psi_j - \phi_j\psi_i = (c_0,d_0,i,j), \\
\frac{\partial}{\partial y}(i,j) &= \int^x (\phi_{i,y}\psi_j + \phi_i\psi_{j,y})\,\mathrm{d}x \\
&= k\int^x (\phi_{i,xx}\psi_j - \phi_i\psi_{j,xx})\,\mathrm{d}x \\
&= k(\phi_{i,x}\psi_j - \phi_i\psi_{j,x}) \\
&= k[(c_0,d_1,i,j) - (c_1,d_0,i,j)], \\
\frac{\partial}{\partial z}(i,j) &= \int^x (\phi_{i,z}\psi_j + \phi_i\psi_{j,z})\,\mathrm{d}x \\
&= 3k\int^x (\phi_{i,xxxx}\psi_j - \phi_i\psi_{j,xxxx})\,\mathrm{d}x \\
&= 3k(\phi_{i,xxx}\psi_j - \phi_{i,xx}\psi_{j,x} \\
&\quad - \phi_i\psi_{j,xxx} + \phi_{i,x}\psi_{j,xx}) \\
&= 3k[(c_0,d_3,i,j) - (c_3,d_0,i,j) \\
&\quad + (c_2,d_1,i,j) - (c_1,d_2,i,j)], \\
\frac{\partial}{\partial t}(i,j) &= \int^x (\phi_{i,t}\psi_j + \phi_i\psi_{j,t})\,\mathrm{d}x \\
&= -2\int^x (\phi_{i,xxx}\psi_j + \phi_i\psi_{j,xxx})\,\mathrm{d}x \\
&= -2(\phi_{i,xx}\psi_j - \phi_{i,x}\psi_{j,x} + \phi_i\psi_{j,xx}) \\
&= -2[(c_0,d_2,i,j) - (c_1,d_1,i,j) \\
&\quad + (c_2,d_0,i,j)].
\end{aligned}$$

Then we can develop differential rules for Pfaffians, and compute various derivatives of the Gramm-type Pfaffians $f_N = (1,2,\ldots,2N) = (\bullet)$, $g_N = (c_1,c_0,\bullet)$, $h_N = (d_0,d_2,\bullet)$, $\hat{g}_N = (c_2,c_0,\bullet)$, and $\hat{h}_N = (d_0,d_1,\bullet)$ with respect to the variables x, y, z, and t. Here we only list various derivatives of f_N as follows:

$$\begin{aligned}
f_{N,x} &= (c_0,d_0,\bullet), \\
f_{N,xx} &= (c_1,d_0,\bullet) + (c_0,d_1,\bullet), \\
f_{N,xxx} &= (c_2,d_0,\bullet) + 2(c_1,d_1,\bullet) + (c_0,d_2,\bullet), \\
f_{N,xxxx} &= (c_3,d_0,\bullet) + 3(c_2,d_1,\bullet) + 3(c_1,d_2,\bullet) \\
&\quad + (c_0,d_3,\bullet) + 2(c_1,d_1,c_0,d_0,\bullet), \\
f_{N,y} &= k[-(c_1,d_0,\bullet) + (c_0,d_1,\bullet)], \\
f_{N,xy} &= k[(c_0,d_2,\bullet) - (c_2,d_0,\bullet)], \\
f_{N,xxy} &= k[-(c_3,d_0,\bullet) + (c_0,d_3,\bullet) \\
&\quad - (c_2,d_1,\bullet) + (c_1,d_2,\bullet)], \\
f_{N,xxxy} &= k[(c_0,d_4,\bullet) - (c_4,d_0,\bullet) \\
&\quad - 2(c_3,d_1,\bullet) + 2(c_1,d_3,\bullet) \\
&\quad - (c_0,d_2,c_1,d_0,\bullet) + (c_2,d_0,c_0,d_1,\bullet)], \\
f_{N,yy} &= k^2[(c_3,d_0,\bullet) - (c_2,d_1,\bullet) - (c_1,d_2,\bullet) \\
&\quad + (c_0,d_3,\bullet) + 2(c_1,d_1,c_0,d_0,\bullet)], \\
f_{N,z} &= 3k[-(c_3,d_0,\bullet) + (c_0,d_3,\bullet) \\
&\quad + (c_2,d_1,\bullet) - (c_1,d_2,\bullet)], \\
f_{N,xz} &= 3k[(c_0,d_4,\bullet) - (c_4,d_0,\bullet) \\
&\quad + (c_2,d_1,c_0,d_0,\bullet) - (c_1,d_2,c_0,d_0,\bullet)], \\
f_{N,t} &= -2[(c_2,d_0,\bullet) - (c_1,d_1,\bullet) + (c_0,d_2,\bullet)], \\
f_{N,xt} &= -2[(c_3,d_0,\bullet) + (c_0,d_3,\bullet) \\
&\quad - (c_1,d_1,c_0,d_0,\bullet)], \\
f_{N,yt} &= -2k[(c_0,d_4,\bullet) - (c_4,d_0,\bullet) \\
&\quad + (c_3,d_1,\bullet) - (c_1,d_3,\bullet) \\
&\quad - (c_0,d_2,c_1,d_0,\bullet) + (c_2,d_0,c_0,d_1,\bullet)].
\end{aligned}$$

The other derivatives were neglected since their calculation procedures are similar. Substituting the above derivatives into the first equation of Eq. (21) by employing Eq. (24), we obtain

$$\begin{aligned}&(D_x^3D_y - D_yD_t - D_xD_z)f_N\\&\cdot f_N + 6k(\hat{g}_N\hat{h}_N - g_Nh_N)\\=&-6k[(c_0,d_0,c_2,d_1,\bullet)(\bullet) - (c_0,d_0,\bullet)(c_2,d_1,\bullet)\\&+(c_0,c_2,\bullet)(d_0,d_1,\bullet)\\&-(c_0,d_1,\bullet)(d_0,c_2,\bullet)] + 6k[(c_0,d_0,c_1,d_2,\bullet)(\bullet)\\&-(c_0,d_0,\bullet)(c_1,d_2,\bullet)\\&+(c_0,c_1,\bullet)(d_0,d_2,\bullet) - (c_0,d_2,\bullet)(d_0,c_1,\bullet)]\\=&\ 0.\end{aligned}$$

This last equality is equal to zero since it is nothing but the Pfaffian identity. Similarly, the last two equations in Eq. (21) can also be reduced to Pfaffian identities. Hence, we have shown that f_N, g_N, h_N, $\hat{g}_N$, and $\hat{h}_N$ defined by Eq. (24) is the solution of the Pfaffianized coupled Hirota bilinear system (21).

5. Conclusion and remark

In summary, we have established Grammian and Pfaffian solutions for the (3+1)-dimensional generalized shallow water equation (2). In addition, we have applied the Pfaffianization procedure to derive a new coupled system for Eq. (2), and have constructed a Wronski-type and a Gramm-type Pfaffian solution for this new coupled system. Our results show that Eq. (2) not only has Grammian determinant solutions, but also has Pfaffian determinant solutions. This property is completely different from that of the KP equation, which only has Grammian solutions, and from that of the BKP equation, which only has Pfaffian solutions. Resonant soliton solutions[25] will also be an interesting topic of our future investigation and research.

References

[1] Taogetusang 2011 *Acta Phys. Sin.* **60** 070203 (in Chinese)
[2] Yang Z, Ma S H and Fang J P 2011 *Acta Phys. Sin.* **60** 040508 (in Chinese)
[3] Su J, Xu W, Duan D H and Xu G J 2011 *Acta Phys. Sin.* **60** 110203 (in Chinese)
[4] Qian C, Wang L L and Zhang J F 2011 *Acta Phys. Sin.* **60** 064214 (in Chinese)
[5] Satsuma J 1979 *J. Phys. Soc. Jpn.* **46** 359
[6] Matveev V B 1992 *Phys. Lett.* A **166** 200
[7] Freeman N C and Nimmo J J C 1983 *Phys. Lett.* A **95** 1
[8] Nimmo J J C and Freeman N C 1983 *Phys. Lett.* A **95** 4
[9] Ma W X 2002 *Phys. Lett.* A **301** 35
[10] Ma W X and You Y C 2005 *Trans. Amer. Math. Soc.* **357** 1753
[11] Ma W X, Li C X and He J S 2009 *Nonlinear Anal.* **70** 4245
[12] Li C X, Ma W X, Liu X J and Zeng Y B 2007 *Inverse Probl.* **23** 279
[13] Geng X G and Ma Y L 2007 *Phys. Lett.* A **369** 285
[14] Wu J P 2011 *Chin. Phys. Lett.* **28** 050501
[15] Tang Y N, Ma W X, Xu W and Gao L 2011 *Appl. Math. Comput.* **217** 8722
[16] Hirota R 2004 *The Direct Method in Soliton Theory* (Cambridge: Cambridge University Press)
[17] Wu J P and Geng X G 2009 *Commun. Theor. Phys.* **52** 791
[18] Ma W X, Abdeljabbar A and Asaad M G 2011 *Appl. Math. Comput.* **217** 10016
[19] Nakamura A 1989 *J. Phys. Soc. Jpn.* **58** 412
[20] Hirota R 1989 *J. Phys. Soc. Jpn.* **58** 2285
[21] Hietarinta J 2005 *Phys. AUC* **15** 31
[22] Tian B and Gao Y T 1996 *Comput. Phys. Commu.* **95** 139
[23] Zayed E M E 2010 *J. Appl. Math. Informatics* **28** 383
[24] Hirota R and Ohta Y 1991 *J. Phys. Soc. Jpn.* **60** 798
[25] Ma W X and Fan E G 2011 *Comput. Math. Appl.* **61** 950