Modern Physics Letters B
Vol. 33, No. 11 (2019) 1950133 (14 pages)

DOI: 10.1142/S0217984919501331

Lump and interaction solutions of nonlinear partial differential equations

Yong-Li Sun[*,†], Wen-Xiu Ma[†,‡], Jian-Ping Yu[§,†,‖], Bo Ren[¶] and Chaudry Masood Khaliqu[‡]

[*] *Department of Mathematics, Beijing University of Chemical Technology, Beijing 100029, China*
[†] *Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620, USA*
[‡] *Department of Mathematical Sciences, International Institute for Symmetry Analysis and Mathematical Modelling, North-West University, Mafikeng Campus, Private Bag X2046, Mmabatho 2735, South Africa*
[§] *Department of Applied Mathematics, University of Science and Technology Beijing, Beijing 100083, China*
[¶] *Institute of Nonlinear Science, Shaoxing University, Shaoxing, Zhejiang 312000, China*
[‖] *jpyu@ustb.edu.cn*

Received 17 December 2018
Revised 31 January 2019
Accepted 1 February 2019
Published 28 March 2019

In this paper, lump solutions of nonlinear partial differential equations, the generalized $(2+1)$-dimensional KP equation and the Jimbo–Miwa equation, are studied by using the Hirota bilinear method and carrying out symbolic computations in Maple. Moreover, the interaction solutions, i.e. collisions between lump waves and kink waves, are investigated. A group of graphs are plotted to illustrate the dynamics of the obtained results.

Keywords: Lump solution; interaction solution; Hirota bilinear method.

1. Introduction

It is well known that exact solutions of nonlinear partial differential equations and their generalized forms have been playing an essential role in the study of many complex physical phenomena and other nonlinear engineering problems[1]; for example, lump solutions, kink solutions and so on. As some sort of rational solutions, lump solutions are localized in every direction of the space. Lump solutions to a few nonlinear partial differential equations have been found and their dynamics have also been analyzed, for example, the KP and BKP equations,[2–15] the p-gKP and

[‖] Corresponding author.

p-gBKP equations[16] and so on. Recently, a lot of concerns have been paid to the study of lump–kink solutions, which means the interactions of lump solutions and kink solutions. Tang *et al.* studied the interaction of a lump with a stripe of $(2+1)$-dimensional Ito equation and showed that the lump is drowned or swallowed by a stripe soliton.[17–23] This study will aim at the lump and interaction solutions of the following two nonlinear differential equations: the generalized $(2+1)$-dimensional KP equation is

$$\left(u_t - uu_x - \frac{1}{3}u_{xxx} + u_x\right)_x + \frac{1}{3}u_{yy} = 0 \tag{1}$$

and the Jimbo–Miwa equation is

$$u_{xxxy} + 3u_y u_{xx} + 3u_x u_{xy} + 2u_{yt} - 3u_{xy} = 0\,, \tag{2}$$

where $u = u(x, y, t)$ is a function of spatial variables x, y and time variable t. It is noted that Eqs. (1) and (2) come from the second member of a KP hierarchy and describe some interesting $(2+1)$-dimensional waves.[24]

Actually, the KP equations and Jimbo–Miwa equations and their generalized forms have already been studied by using many effective methods, such as the Hirota bilinear method, the generalized bilinear method, the Exp-function method and so on.[25–32] The KP and the Jimbo–Miwa equations have been applied to modeling water waves of long wavelength with weakly nonlinear restoring forces and frequency dispersion, and they can be employed to modeling waves in ferromagnetic media and two-dimensional matter-wave pulses in Bose–Einstein condensates as well.

In this paper, we apply the Hirota bilinear operators to study the lump solutions and interaction solutions of Eqs. (1) and (2). We construct not only the lump solutions of Eqs. (1) and (2) but also the interaction solutions between lump solutions and stripe soliton solutions. Moreover, we illustrate the dynamical properties of these obtained solutions with the corresponding graphs. There are some differences between our method and the unified method and its generalized form,[33–42] since we make use of the quadratic functions.

This paper is arranged as follows. In Sec. 2, we study the lumps and the interaction solutions of Eq. (1), moreover we analyze their dynamics. The lump solutions and the interaction solutions and the dynamics of solutions of Eq. (2) are investigated in Sec. 3. Some discussions of the applications of lump solutions and interaction solutions to physics are presented in Sec. 4, and some conclusions are given in Sec. 5.

2. Lump and Interaction Solutions of Eq. (1)

In this section, we will construct lump solutions and mixed solutions of Eq. (1). Moreover, their dynamics will also be studied.

2.1. *Lump solutions of Eq.* (1)

Applying the transformation $u = 4(\ln f)_{xx}$, we obtain the corresponding bilinear form to Eq. (1) as follows:

$$\left(D_x D_t - \frac{1}{3}D_x^4 + D_x^2 + \frac{1}{3}D_y^2\right) f \cdot f = 0\,, \tag{3}$$

where $f(x, y, t)$ is an unknown real function and D_x, D_y, D_t are the Hirota derivatives defined by

$$D_x^m D_y^n a(x,y) \cdot b(x,y) = \left(\frac{\partial^m}{\partial s^m}\frac{\partial^n}{\partial t^n}\right) a(x+s, y+t) \cdot b(x-s,\, y-t)\big|_{s=0,t=0}\,,$$
$$m, n = 0, 1, \ldots\,.$$

All the details about the Hirota direct method can be found in Ref. 25 and references therein.

To find the lump solutions of Eq. (1), we express f in the following form

$$f = g^2 + h^2 + a_9, \quad g = a_1 x + a_2 y + a_3 t + a_4, \quad h = a_5 x + a_6 y + a_7 t + a_8\,, \tag{4}$$

where a_i's are all real parameters to be determined. Plugging f into Eq. (3) and equating all the coefficients of all the polynomials of x, y, t to zero leads to a system of algebraic equations in a_i's. Solving this system, we obtain all the values of a_i's:

$$\begin{aligned} a_3 &= -\frac{3a_1^3 + a_1 a_2^2 + 3a_1 a_5^2 - a_1 a_6^2 + 2a_2 a_5 a_6}{3(a_1^2 + a_5^2)}\,,\\ a_7 &= -\frac{3a_1^2 a_5 + 2a_1 a_2 a_6 - a_2^2 a_5 + 3a_5^3 + a_5 a_6^2}{3(a_1^2 + a_5^2)}\,,\\ a_9 &= \frac{3(a_1^6 + 3a_1^4 a_5^2 + 3a_1^2 a_5^4 + a_5^6)}{(a_1 a_6 - a_2 a_5)^2}\,. \end{aligned} \tag{5}$$

There are six independent parameters, two of which are completely free. Moreover, the following conditions can guarantee the localized analyticity, positiveness and rationality in all directions of the function f:

$$a_1^2 + a_5^2 \neq 0, \quad a_1 a_6 - a_2 a_5 \neq 0\,. \tag{6}$$

Plugging Eq. (5) into Eq. (4), we obtain a class of quadratic function solution to the bilinear equation (3), then the solutions of Eq. (1) are also obtained via the transformation $u = 4(\ln f)_{xx}$:

$$u = 4\frac{2a_1^2 + a_5^2}{f} - 8\frac{(a_1 g + a_5 h)^2}{f^2}\,, \tag{7}$$

It is observed that these resulting lump solutions contain six independent parameters, two of which are totally free and for different times. The central points can be obtained via seeking the extremum points, which are of importance to study the lump solitons about the velocity, the changes of waveform and so forth. The amplitude of u is also attained, $\frac{4(2a_1^2+a_5^2)(a_1a_6-a_2a_5)^2}{(a_1^2+a_5^2)^3}$, which tell us that the amplitude is determined by a_1, a_2, a_5, a_6. We also noted that the lump wave (7) is analytic in the XY-plane if and only if $a_9 > 0$. The analyticity of the solution (7) is guaranteed if (6) holds. Moreover, it is easy to find that the aforementioned lump solution $u \to 0$ if and only if the sum of squares $g^2 + h^2 \to \infty$, or, equivalently, $x^2 + y^2 \to \infty$ at any given time. Hence, the condition (6) can guarantee both analyticity and localization of the lump wave (7). Based on the aforementioned discussions, the condition

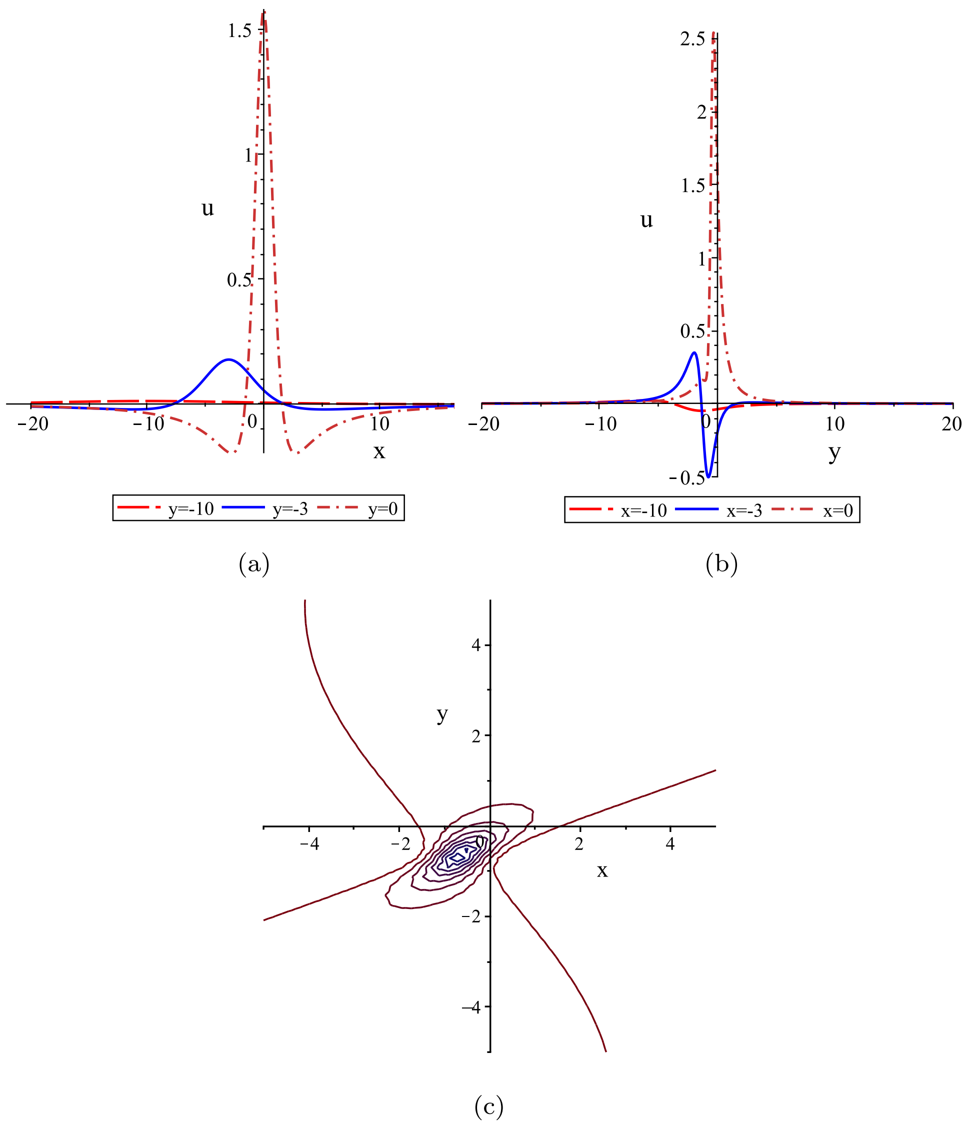


Fig. 1. (Color online) Graphs of lump wave (8) with the specific parameters: $t = 1, a_1 = 0, a_2 = 2, a_4 = 0, a_5 = 1, a_6 = -1$ and $a_8 = 0$: (a) 3D plot, (b) density plot and (c) contour plot.

(6), the analyticity and the localization of the solution (7) are mutually equivalent. Moreover, (6) is also the necessary condition to get lump solitons. The parameters a_1, a_2, a_5 and a_6 determine the expansion and the deflection angles of lump, the smaller the absolute values of which result in a greater expansion of lump. In the meantime, the two totally independent parameters are much more important to the extremum points. Now we choose some specific values of the parameters: $a_1 = 1, a_2 = 2, a_4 = 0, a_5 = 1, a_6 = -1$ and $a_8 = 0$, the graphs of the lump wave (7) are shown vividly by Fig. 1. If we fix time variable $t = 1$, the central points can be obtained via seeking the extremum points as follows: $\left(\frac{1}{6}, \frac{1}{3}\right)$, $\left(\frac{1}{6} \pm \frac{2\sqrt{21}}{3}, \frac{1}{3}\right)$, so that the maximal amplitude of u is attained as $\frac{9}{2}$.

2.2. *Interaction solutions of Eq.* (1)

In this subsection, we are going to investigate the collisions among the lump solution and the kink solution. Now we choose

$$f = g^2 + h^2 + l + a_{12}\,, \tag{8}$$

where $g = a_1x + a_2y + a_3t + a_4, h = a_5x + a_6y + a_7t + a_8$ and $l = e^{a_9x+a_{10}y+a_{11}t}$. Plugging f into Eq. (3) and using the transformation $u = 4(\ln f)_{xx}$, the lump–kink solutions of Eq. (1) are obtained as follows:

Case I. We choose the following parameters:

$$\begin{aligned} a_1 &= 0, \quad a_2 = \pm\sqrt{3}a_9, \quad a_5 = \pm\sqrt{3}a_9\,, \\ a_7 &= \frac{3a_5^2a_9^2 - 3a_5^2 - a_6^2}{3a_5}, \quad a_{10} = \frac{a_6a_9}{a_5}\,, \\ a_{11} &= \frac{a_9(3a_5^2a_9^2 - 3a_5^2 - a_6^2)}{3a_5^2}, \quad a_{12} = \frac{a_5^2}{a_9^2}\,. \end{aligned} \tag{9}$$

In order to guarantee the positiveness, analyticity and the localization of u in all directions in the (x, y)-plane, it is required that $a_5 \neq 0, a_9 \neq 0$. So we have the following lump–kink wave solution:

$$u_1 = \frac{4(2a_5^2 + a_9^2 l)}{g^2 + h^2 + l + \frac{a_5^2}{a_9^2}} - \frac{4(2a_5h + a_9l)^2}{\left(g^2 + h^2 + l + \frac{a_5^2}{a_9^2}\right)^2}\,, \tag{10}$$

which has six independent parameters, four of which are completely independent.

Case II. We select the following parameters:

$$
\begin{aligned}
a_1 &= \frac{\alpha(a_5a_{10}-a_6a_9)}{a_9^2}\,,\\
a_3 &= -\frac{3a_5a_9^4a_{10}+3a_6a_9^5+3a_5a_9^2a_{10}+a_5a_{10}^3-3a_6a_9^3-a_6a_9a_{10}^2}{a_9^4\alpha}\,,\\
a_2 &= \frac{3a_5a_9^4+a_5a_{10}^2-a_6a_9a_{10}}{3\alpha a_9^3}\,,\\
a_7 &= \frac{3a_5a_9^4-3a_5a_9^2+a_5a_{10}^2-2a_6a_9a_{10}}{3a_9^2}\,,\\
a_{11} &= \frac{a_9^4-3a_9^2-a_{10}^2}{3a_9}\,,\\
a_{12} &= \frac{3a_5^2a_9^4+a_5^2a_{10}^2-2a_5a_6a_9a_{10}+a_6^2a_9^2}{3a_9^6}\,,
\end{aligned}
\tag{11}
$$

where $\alpha=\pm\sqrt{\frac{1}{3}}$. In order to guarantee the analyticity, positiveness and rationality of u, the following conditions have to be satisfied:

$$a_9\neq 0,\quad 3a_5^2a_9^4+a_5^2a_{10}^2-2a_5a_6a_9a_{10}+a_6^2a_9^2>0\,.$$

Via the transformation $u=4(\ln f)_{xx}$, we get the following solution to Eq. (1):

$$u_2=\frac{4p_1}{q_1}-\frac{4p_2^2}{q_1^2}\,, \tag{12}$$

where

$$
\begin{aligned}
p_1 &= \frac{2\alpha^2(a_5a_{10}-a_6a_9)^2}{a_9^4}+2a_5^2+a_9^2l\,,\\
p_2 &= \frac{2\alpha(a_5a_{10}-a_6a_9)g}{a_9^2}+2a_5h+a_9l\,,\\
q_1 &= g^2+h^2+l+\frac{3a_5^2a_9^4+a_5^2a_{10}^2-2a_5a_6a_9a_{10}+a_6^2a_9^2}{3a_9^6}\,.
\end{aligned}
$$

It is noted that there are seven independent parameters and three are totally independent. As seen from the above results, the first part g of u_1 in Case I is irrelevant to time variable x for $a_1=0$. In order to gain the collision phenomenon, $a_3^2+a_7^2+a_{11}^2\neq 0$ is required. Therefore, the asymptotic behavior of u is obtained as $u\to 0$ while $t\to\infty$. The asymptotic behavior shows that the lump wave is finally swallowed up by the kink wave with the change of time, which implies they become one kink wave after their collisions, since the lump wave moves faster across the kink wave with higher energy, which is transported by the waves and is directly proportional to the square of the amplitude of the wave. Moreover, we know that the lump–kink wave solution algebraically and exponentially decays at last. Hence,

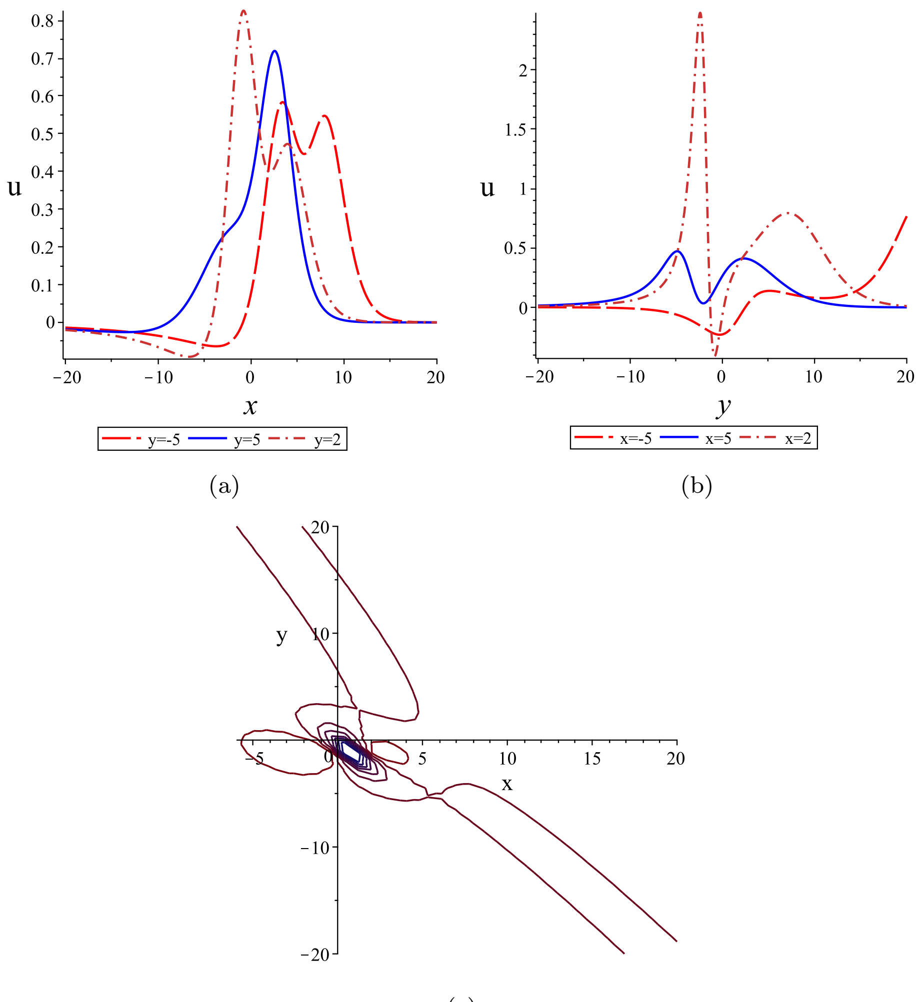


Fig. 2. (Color online) Evolution profiles of lump wave (10) with the specific parameters: $t = 2$, $a_1 = 0, a_3 = 1, a_4 = 0, a_6 = 1, a_8 = 0$, and $a_9 = 1$. (a) x-curves, (b) y-curves and (c) contour plot.

it is a lump–kink solitary wave solution and describes a completely non-elastic interaction between two different solitons. Figure 2 for the solution (10) illustrates the typical phenomena in the interaction between a lump and a kink with the special parameters. This is a completely non-elastic interaction between two different solitons and decays both algebraically and exponentially.

3. Lump and Interaction Solutions of Eq. (2)

In this section, we will find the lump solutions and the lump–kink solutions to the generalized $(2+1)$-dimensional Jimbo–Miwa equation, i.e. Eq. (2), by using the Hirota bilinear forms.

3.1. *Lump solutions of Eq.* (2)

Applying the transformation $u = 2(\ln f)_x$, we obtain the Hirota bilinear form

$$(D_x^3 D_y + 2D_y D_t - 3D_x D_y) f \cdot f = 0 . \tag{13}$$

Employing a similar procedure as in Sec. 2, we obtain a_i's as follows.

$$a_1 = -\frac{a_5 a_6}{a_2}, \quad a_3 = -\frac{3a_5 a_6}{2a_2}, \quad a_7 = \frac{3a_5}{2} . \tag{14}$$

There are six independent parameters, three of which are completely independent. Moreover, the following conditions can guarantee the localized analyticity, positiveness and rationality in all directions of the function f:

$$a_2 \neq 0, \quad a_9 > 0 . \tag{15}$$

Plugging Eq. (14) into the corresponding equations just as we did in Sec. 2, we can get the resulting class of function f, which implies the lump solution of Eq. (2) is

$$u = 4\frac{-a_5 a_6 g + a_2 a_5 h}{a_2 f} . \tag{16}$$

It is observed that these resulting lump solutions contain six independent parameters, two of which are totally free and for different times. We observed that the solution (16) is analytic in the XY-plane if and only if $a_9 > 0$. The analyticity of the solution (16) is guaranteed if (15) holds, which yields $a_2 \neq 0$ and $a_9 > 0$. Moreover, it is easy to find that the aforementioned lump solution $u \to 0$ if and only if the sum of squares $g^2 + h^2 \to \infty$, or, equivalently, $x^2 + y^2 \to \infty$, at any given time. Hence, the condition (15) can guarantee both analyticity and localization of the solution (16). According to the above discussions, the condition (15), the analyticity and the localization of the solution (16) are actually equivalent to each other. The parameters a_2, a_5 and a_6 determine the expansion and the deflection angles of lump, the smaller the absolute values of which result in a greater expansion of lump. In the meantime, the two totally independent parameters are much more important to the extremum points. Now we choose some specific values of the parameters: $a_2 = 1, a_4 = 0, a_5 = 1, a_6 = 1, a_8 = 0$ and $a_9 = 1$, the graphs of the lump wave (16) are given by Fig. 3. If we fix time variable $t = 0$, the central points can be obtained via seeking the extremum points as follows: $\left(\pm\frac{\sqrt{6}}{6}, 0\right)$, so that the maximal amplitude of u is attained as $\frac{3\sqrt{6}}{4}$.

3.2. *Interaction solutions of Eq.* (2)

In this subsection, we investigate the collisions among lump solutions and kink solutions. Therefore, we assume

$$f = g^2 + h^2 + l + a_{13} , \tag{17}$$

where $g = a_1 x + a_2 y + a_3 t + a4, h = a_5 x + a_6 y + a_7 t + a_8$ and $l = e^{a_9 x + a_{10} y + a_{11} t + a_{12}}$.

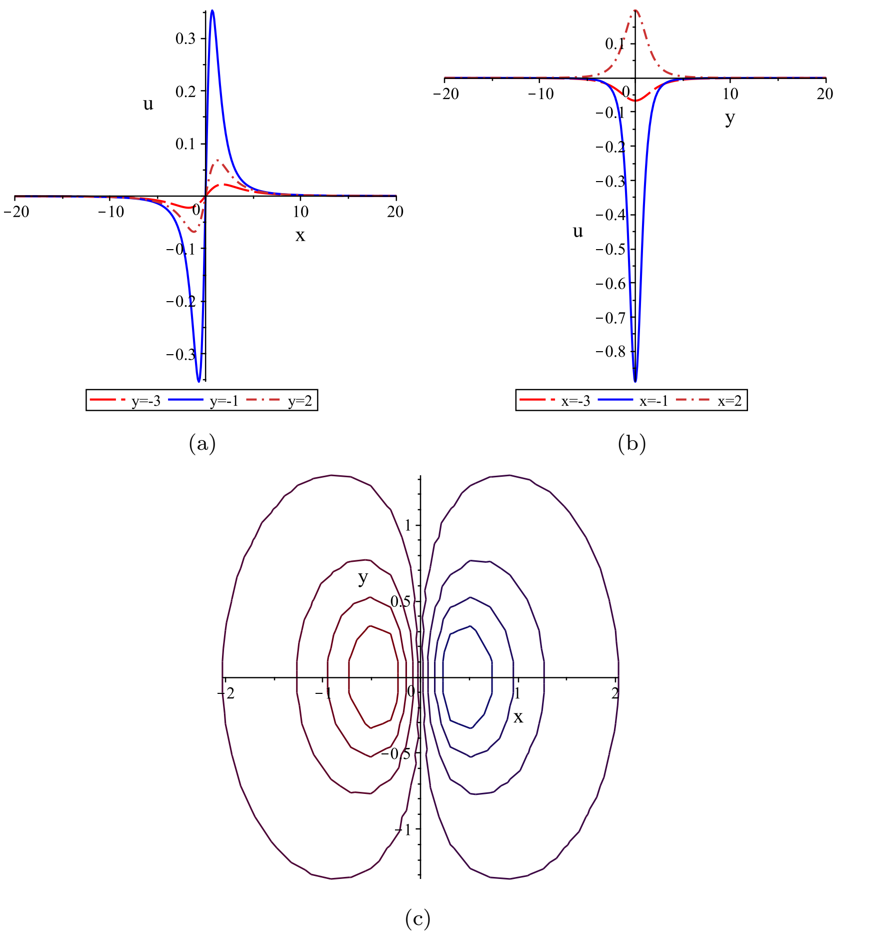


Fig. 3. (Color online) Graphs of lump solution (16) with the specific parameters: $t = 0, a_1 = 1.5, a_2 = 1, a_4 = 0, a_5 = -1, a_6 = 0.4$ and $a_8 = 0$. (a) 3D plot, (b) density plot and (c) contour plot.

Plugging f into Eq. (13), all the aforementioned parameters can be obtained. Therefore, according to the transformation $u = 2(\ln f)_x$, we can obtain five classes of solutions of Eq. (2) as follows.

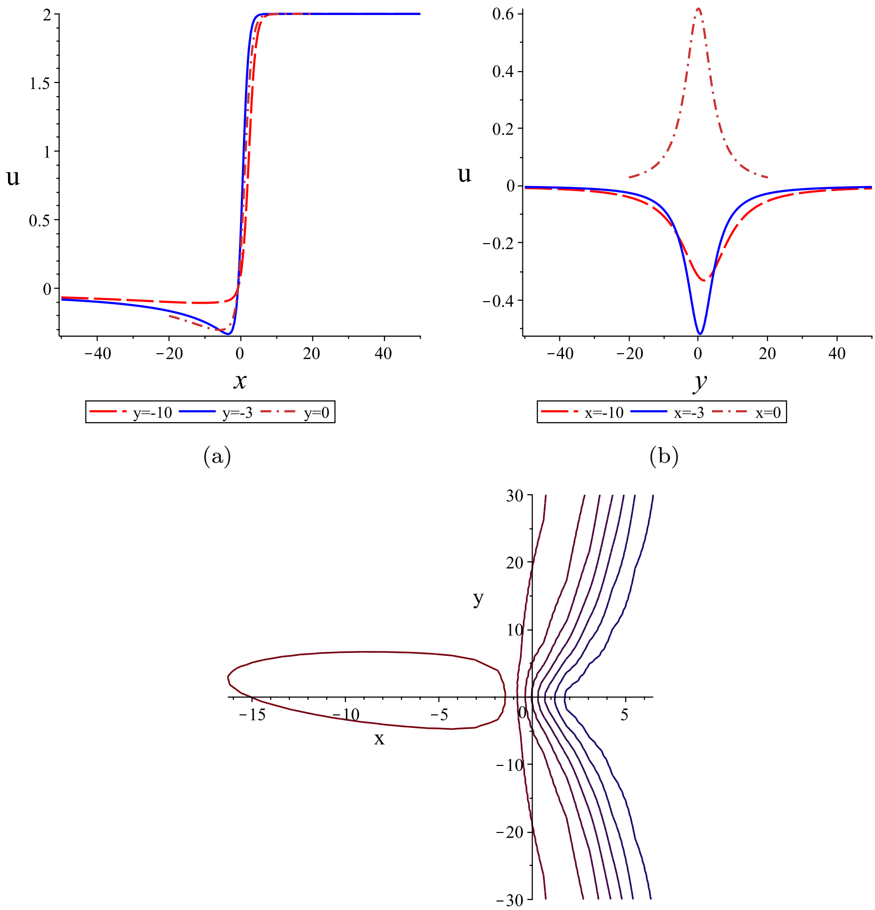
Case I. When the parameters are

$$a_2 = a_5 = a_7 = a_{10} = 0, \quad a_3 = \frac{3a_1}{2}, \quad a_{11} = -\frac{a_9^3}{2} + \frac{3a_9}{2}\,, \tag{18}$$

the lump–kink wave solution is

$$u_1 = \frac{2(2a_1 g + a_9 l)}{\left(\frac{3a_1}{2}t + xa_1 + a_4\right)^2 + (ya_6 + a_8)^2 + e^{t\left(-\frac{a_9^3}{2} + \frac{3a_9}{2}\right) + xa_9 + a_{12}} + a_{13}}\,, \tag{19}$$

with seven independent parameters, five of which are completely independent.



(a) (b) (c)

Fig. 4. (Color online) Evolution profiles of lump wave (20) with the specific parameters: $t = 1$, $a_2 = 1, a_4 = 0, a_6 = 1, a_7 = 1, a_8 = 0, a_9 = 1, a_{12} = 0$ and $a_{13} = 1$. (a) x-curves, (b) y-curves and (c) contour plot.

Case II. If we select the parameters as

$$a_1 = -\frac{2a_6a_7}{3a_2}, \quad a_3 = -\frac{a_6a_7}{a_2}, \quad a_5 = \frac{2a_7}{3}, \quad a_{10} = 0, \quad a_{11} = a_9^3, \tag{20}$$

we obtain the following lump–kink wave solution:

$$u_2 = \frac{2(-4a_6a_7g + 4a_2a_7h + 3a_2a_9l)}{3a_2\left((-t\frac{a_6a_7}{a_2} - x\frac{2a_6a_7}{3a_2} + ya_2 + a_4)^2 + \left(ta_7 + x\frac{2a_7}{3} + ya_6 + a_8\right)^2 + e^{t\left(-\frac{a_9^3}{2} + \frac{3a_9}{2}\right) + xa_9 + a_{12}} + a_{13}\right)}, \tag{21}$$

which has eight independent free parameters, four of which are totally independent.

Case III. We choose the following parameters:

$$a_1 = a_3 = a_5 = a_7 = 0, \quad a_{11} = -\frac{a_9^3}{2} + \frac{3a_9}{2}\,, \tag{22}$$

then the lump–kink wave solution is obtained as

$$u_3 = \frac{2a_9 l}{(ya_2 + a_4)^2 + (ya_6 + a_8)^2 + e^{t\left(-\frac{a_9^3}{2} + \frac{3a_9}{2}\right) + xa_9 + ya_{10} + a_{12}} + a_{13}}\,. \tag{23}$$

There are eight independent parameters, seven of which are completely independent.

Case IV. Selecting the parameters as

$$a_1 = a_3 = a_6 = a_9 = a_{11} = 0, \quad a_7 = \frac{3a_5}{2} \tag{24}$$

leads to the following solution:

$$u_4 = \frac{4a_5 h}{(y + a_4)^2 + \left(\frac{3a_5}{2}t + a_5 x + a_8\right)^2 + e^{a_{10}y + a_{12}} + a_{13}}\,. \tag{25}$$

It is observed that there are seven independent parameters, six of which are totally independent.

Case V. Choosing the following parameters

$$a_2 = -\frac{a_5 a_6}{a_1}, \quad a_3 = \frac{3a_1}{2}, \quad a_7 = \frac{3a_5}{2}, \quad a_9 = a_{11} = 0 \tag{26}$$

results in a lump–kink wave solution

$$u_5 = \frac{4(a_1 g + a_5 h)}{(\frac{3a_1}{2}t + a_1 x - \frac{a_5 a_6}{a_1}y + a_4)^2 + (\frac{3a_5}{2}t + a_5 x + ya_6 + a_8)^2 + e^{ya_{10} + a_{12}} + a_{13}}\,. \tag{27}$$

It is noted that there are eight independent parameters, five of which are totally independent. As seen from the above results, the exponential functions of u_1 and u_2 in Case I and Case II are irrelevant to time variable y for $a_{10} = 0$, respectively. In Case III, the first two parts g and h of the lump–kink solution in u_3 are irrelevant to space variable x and time variable t. The first part g of u_4 in Case IV is irrelevant to space variable x and time variable t as well, and the second part h of u_4 is irrelevant to the space variable y, the third part namely the exponential function is irrelevant to space variable x and time variable t. For Case V, the third part namely the exponential function is irrelevant to x and t. Therefore, in order to gain the collision phenomenon, $a_3^2 + a_7^2 + a_{11}^2 \neq 0$ is required. Therefore, the asymptotic behavior of u is obtained as $u \to 0$ while $t \to \infty$. The asymptotic behavior proves that the lump wave is finally swallowed up by the kink wave, then they become one kink wave after the collisions, since the lump wave moves faster across the kink wave with higher energy, which is transported by the waves and is directly proportional to the square of the amplitude of the wave. Moreover, we know that the lump–kink solution u algebraically and exponentially decays at last. Hence, it is a lump–kink solitary wave solution and describes a completely non-elastic collision between two different soliton waves.

4. Discussions

It has been found that the lump solution can propagate symmetrically in any direction on the XY-plane although it is not axisymmetric, and its intrinsic anisotropy is caused by the anisotropy of the medium.[43] Moreover, for the lump–kink solutions, it is found that the lump wave moves faster across the kink wave with higher energy, then they become one kink wave after their collision. So some studies and applications of lump solitons and lump–kink solutions of the KP equation and the Jimbo–Miwa equation in physics came out. For example, the evolutions of lump solitons and their generations which are from few-cycle input pulses have been numerically simulated by Minzoni and Smyth,[44] and Leblond *et al.*,[45] respectively, see also the references therein. Particularly, rogue wave solutions, which have drawn a lot of concerns from mathematicians and physicists all over the world, are a very interesting class of lump or lump–kink solutions, and they can help analyze the wave propagation of earthquake response of structure. Now, such kind of solutions is often used to describe many significant nonlinear wave phenomena, for instance, oceanography[46] and nonlinear optics.[47]

5. Conclusions

In this research, we have studied the lump solutions and the lump–kink solutions and some dynamic characters of a generalized $(2+1)$-dimensional KP equation and Jimbo–Miwa equation. Based on the bilinear forms and the approach of positive quadratic functions, some lump solutions are derived, in which some important parameters are involved, and in order to guarantee the positiveness, the analyticity and the rational localization, these parameters have to satisfy some conditions. In the meantime, the lump–kink solutions are also investigated via adding one exponential function to the positive quadratic function.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (Nos. 11101029, 11271362 and 11375030), the Fundamental Research Funds for the Central Universities (No. 610806), Beijing City Board of Education Science and Technology Key Project (No. KZ201511232034), Beijing Nova program (No. Z131109000413029) and Beijing Finance Funds of Natural Science Program for Excellent Talents (No. 2014000026833ZK19).

Conflict of Interest

The authors declare that they have no conflict of interest.

References

1. M. J. Ablowitz and P. A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge University Press, New York, 1991).
2. W. X. Ma, *Phys. Lett. A* **379** (2015) 1975.
3. D. S. Wang and J. Liu, *Appl. Math. Lett.* **79** (2018) 211.
4. B. Ren, *Phys. Scripta* **90** (2015) 065206.
5. D. S. Wang and X. L. Wang, *Nonlinear Anal.-Real* **41** (2018) 334.
6. B. Ren, W. X. Ma and J. Yu, *Comput. Math. Appl.* **77**(8) (2019) 2086, doi:10.1016/j.camwa.2018.12.010.
7. D. S. Wang, Y. R. Shi, W. X. Feng and L. Wen, *Physica D* **351–352** (2017) 30.
8. Y. Sun, B. Tian, X. Y. Xie, J. Chai and H. M. Yin, *Waves Random Complex Media* **28** (2018) 544.
9. A. Askari, D. Saadatmand and K. Javidan, *Waves Random Complex Media* **29** (2018) 368, doi:10.1080/17455030.2018.1439203.
10. C. R. Gilson and J. J. C. Nimmo, *Phys. Lett. A* **147** (1990) 472.
11. J. G. Liu and G. P. Ai, *Mod. Phys. Lett. B* **32** (2018) 1850343.
12. L. Zou, Z. B. Yu, S. F. Tian, L. L. Feng and J. Li, *Mod. Phys. Lett. B* **32** (2018) 1850104.
13. J. Y. Yang and W. X. Ma, *Mod. Phys. Lett. B* **30** (2016) 1640028.
14. H. Wang, Y. H. Wang, W. X. Ma and C. L. Temuer, *Mod. Phys. Lett. B* **32** (2018) 1850376.
15. W. X. Ma and T. C. Xia, *Phys. Scr.* **87** (2013) 458.
16. W. X. Ma, Z. Y. Qin and X. Lü, *Nonlinear Dynam.* **84** (2016) 923.
17. Y. N. Tang, S. Q. Tao and Q. Guan, *Comput. Math. Appl.* **72** (2016) 2334.
18. X. E. Zhang and Y. Chen, *Commun. Nonlinear Sci.* **52** (2017) 24.
19. H. Q. Zhan and W. X. Ma, *Comput. Math. Appl.* **74** (2017) 1399.
20. W. X. Ma and Y. Zhou, *J. Differ. Equations* **264** (2018) 2633.
21. J. B. Zhang and W. X. Ma, *Comput. Math. Appl.* **74** (2017) 591.
22. W. X. Ma, X. L. Yong and H. Q. Zhang, *Comput. Math. Appl.* **75** (2018) 289.
23. J. Y. Yang, W. X. Ma and Z. Y. Qin, *Anal. Math. Phys.* **17** (2017) 1.
24. M. Jimbo and T. Miwa, *Publ. Res. Inst. Math. Sci.* **19** (1983) 943.
25. R. Hirota, *The Direct Method in Soliton Theory* (Cambridge University Press, 2004).
26. A. M. Wazwaz, *Appl. Math. Lett.* **64** (2017) 21.
27. G. Q. Xu, *Chaos Soliton. Fract.* **30** (2006) 71.
28. A. M. Wazwaz, *Appl. Math. Comput.* **203** (2008) 592.
29. S. H. Ma, J. P. Fang and C. L. Zheng, *Chaos Soliton. Fract.* **40** (2009) 1352.
30. W. X. Ma and J. H. Lee, *Chaos Soliton. Fract.* **42** (2009) 1356.
31. Y. Zhang, L. G. Jin and Y. L. Kang, *Appl. Math. Comput.* **219** (2012) 2601.
32. Y. N. Tang, *Appl. Math. Model.* **37** (2013) 6631.
33. M. S. Osman, H. I. Abdel-Gawad and M. A. El Mahdy, *Res. Phys.* **8** (2018) 1054.
34. M. S. Osman, *Optik* **156** (2018) 169.
35. M. S. Osman, A. Korkmaz, H. Rezazadeh, M. Mirzazadeh, M. Eslami and Q. Zhou, *Chinese J. Phys.* **56** (2018) 2500.
36. H. Rezazadeh, M. S. Osman, M. Eslami, M. Ekici, A. Sonmezoglu, M. Asma, W. A. M. Othman, B. R. Wong, M. Mirzazadeh, Q. Zhou, A. Biswas and M. Belick, *Optik* **164** (2018) 84.
37. H. I. Abdel-Gawad and M. Osman, *Indian J. Pure Appl. Math.* **45** (2014) 1.
38. A. M. Wazwaz and M. S. Osman, *Comput. Math. Appl.* **76** (2018) 276.
39. M. S. Osman and J. A. T. Machado, *Nonlinear Dynam.* **93** (2018) 733.
40. M. S. Osman, *Open Phys.* **14** (2016) 26.

41. M. S. Osman and A. T. Machado, *J. Electromagnet. Waves* **32** (2018) 1457.
42. M. S. Osman, *Nonlinear Dynam.* **89** (2017) 2283.
43. N. Singh and Y. Stepanyants, *Wave Motion* **64** (2016) 92.
44. A. A. Minzoni and N. F. Smyth, *Wave Motion* **24** (1996) 291.
45. H. Leblond, D. Kremer and D. Mihalache, *Phys. Rev. A* **80** (2009) 053812.
46. P. Müller, C. Garrett and A. Osborne, *Oceanography* **18** (2005) 66.
47. D. R. Solli, C. Ropers, P. Koonath and B. Jalali, *Nature* **450** (2007) 1054.