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Exact solutions of the Rosenau–Hyman equation, coupled KdV system and Burgers–Huxley equation using modified transformed rational function method

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In this research, we study the exact solutions of the Rosenau–Hyman equation, the coupled KdV system and the Burgers–Huxley equation using modified transformed rational function method. In this paper, the simplest equation is the Bernoulli equation. We are not only obtain the exact solutions of the aforementioned equations and system but also give some geometric descriptions of obtained solutions. All can be illustrated vividly by the given graphs.

Keywords: Bernoulli equation; homogeneous balance method; modified transformed rational function method; exact solutions; geometric property.

1. Introduction

In the last years, the nonlinear partial differential equations have been widely applied to many natural systems, for instance, the biology, chaos and ecology. Moreover, the exact analytical solutions of the nonlinear partial differential equations (NPDE) play a key role in several research directions, for example, descriptions of different kinds of waves, as initial condition for simulation process. Thus, people

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pay a lot of attention to this important research area.[1–18] A lot of excellent methods for finding exact solutions of the nonlinear partial differential equations have been established, such as, homogeneous balance method,[19–22] hyperbolic function method,[23] F-expansion method[24] and variable-detached method,[25–28] the Bäcklund transform,[29,30] the Jacobi elliptic function method,[31] the extended tanh-function method,[32] Hirota bilinear operator method[33] and generalized bilinear operator method,[34] etc. One of the direct methods is the method of simplest equation, which was presented by Kudryashov in 1988.[35] This method and modified method of simplest equation have been used to find exact solutions of NPDE,[36–40] such as Fisher equation and Fisher-like equations,[41] generalized Kuramoto–Sivashinsky equation.[42] In Ref. 43, a more general direct approach-transformed rational function method was investigated.

In this study, based on the results of Ref. 43, we presented a modified transformed rational function method, and study the exact solutions of the Rosenau–Hyman equation, the coupled KdV system and the Burgers–Huxley equation using modified transformed rational function method. Moreover, some geometric descriptions of obtained solutions will be studied and illustrated vividly by the graphs.

This paper is organized as follows. In Sec. 2, we introduce modified transformed rational function method; then we construct new exact analytical solutions of the Rosenau–Hyman equation, the coupled KdV system and the Burgers–Huxley equation in Secs. 3–5. Meanwhile, some geometric properties of obtained results are given therein. Finally, we summarize and discuss the results briefly.

2. Modified Transformed Rational Function Method

We begin this section with introducing the homogeneous balance method. Now, let us take the Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \alpha\frac{\partial^2 u}{\partial x^2} = 0\,, \tag{1}$$

as an example. Suppose that

$$\begin{aligned} u &= \frac{\partial^{m+n} f(\omega)}{\partial x^m \partial t^n} + \text{partial derivative terms with lower than } m+n \text{ order of } f(\omega) \\ &= f^{(m+n)}\omega_x^m \omega_t^n + \text{terms with lower than } (m+n) \text{ degree in derivatives of } \omega(x,t)\,, \end{aligned} \tag{2}$$

where $\omega = \omega(x,t)$, $f = f(\omega)$ and $m \geq 0, n \geq 0$ are integers to be determined.

The nonlinear term in Eq. (1) is transformed into

$$\begin{aligned} uu_x = {} & f^{(m+n)} f^{(m+n+1)} \omega_x^{(2m+1)} \omega_t^{2n} \\ & + \text{terms with lower than } 2(m+n)+1 \\ & \text{degree in various derivatives of } \omega(x,t)\,. \end{aligned} \tag{3}$$

The highest order partial derivative term in Eq. (1) is converted into

$$u_{xx} = f^{(m+n+2)}\omega_x^{(m+2)}\omega_t^n + \text{terms with lower than } (m+n+2) \text{ degree in various derivatives of } \omega(x,t)\,. \tag{4}$$

Requiring the highest degrees in partial derivatives of $\omega(x,t)$ in (3) and (4) are equal yields

$$2m+1 = m+2, \quad 2n = n, \tag{5}$$

which has a non-negative integer solution: $m = 1$, $n = 0$. Thus

$$u = \frac{\partial f}{\partial \omega}\frac{\partial \omega}{\partial x} + a_1 f + a_2\,. \tag{6}$$

Now, in this research, we always choose $r = 0$, but it is the least order of derivatives in the given equation in Ref. 43, which implies that r may be $0, 1, 2$ or other non-negative integers. We also let η satisfy the Bernoulli equation $\phi'(\xi) = \phi^{b+1}(\xi) - \phi(\xi)$. Using the direct computation in Wsolve, we get its exact solution as $\eta = \phi(\xi) = \frac{1}{(1+ce^{b\xi+\xi_0})^{\frac{1}{b}}}$, where c is a parameter. Throughout this paper, we assume that $c = 1$, that is, $\phi(\xi) = \frac{1}{(1+e^{b\xi+\xi_0})^{\frac{1}{b}}}$. We let ν be a polynomial with respect to η, the degree of which can be determined by the homogeneous method. Thus the algorithm of the modified transformed rational function method is:

Algorithm 2.1.

Step 1. For any given nonlinear partial differential equation with respect to x, t,

$$H(u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \ldots) = 0\,, \tag{7}$$

where H is a polynomial with respect to $u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \ldots$. By the traveling wave transform

$$u(x,t) = u(\xi), \quad \xi = kx - \lambda t\,, \tag{8}$$

Equation (7) can be transformed into an ordinary differential equation

$$L(u, u', u'', u''', \ldots) = 0, \tag{9}$$

where k and λ are constants to be determined, and $u' = \frac{du}{d\xi}$.

Step 2. Let

$$u(\xi) = \sum_{i=0}^{n} a_i(\phi(\xi))^i, \quad a_n \neq 0\,, \tag{10}$$

be a solution of Eq. (9), where n is a positive integer, which can be obtained by the homogeneous balance method, a_i are constants to be determined, and $\phi(\xi)$ has the following form:

$$\phi(\xi) = \left(\frac{1}{1+e^{b\xi+\xi_0}}\right)^{\frac{1}{b}} = \frac{1}{2^b}\left(1 - \tanh\left(\frac{b\xi+\xi_0}{2}\right)\right)^{\frac{1}{b}}, \tag{11}$$

where ξ_0 is a constant, and $\phi(\xi)$ is a solution of the following ordinary differential equation:

$$\frac{d\phi(\xi)}{d\xi} = \phi(\xi)^{b+1} - \phi(\xi)\,, \tag{12}$$

where b is a positive integer.

Step 3. Substitute Eqs. (10) and (12) into Eq. (9), then collect all the coefficients of $\phi^i(\xi)$ and let all of them be zero, thus we can get a system of algebraic equations with respect to the parameters $k, \lambda, a_i (i = 0, 1, 2, \ldots, n)$.

Step 4. Applying characteristic set technique to solve the system of algebraic equations obtained in Step 3, we can have the values of $k, \lambda, a_i (i = 0, 1, 2, \ldots, n)$.

Step 5. Substitute the obtained $k, \lambda, a_i (i = 0, 1, 2, \ldots, n)$ into (10), then we obtain all the new exact analytical solutions of Eq. (7).

Remark. In this study, we use homogeneous balance method to determine the value of n in formula (10), by which we can get several types of the solitary wave solutions of Eq. (7).

In order to get some geometric properties of solutions of Eq. (7), we will applied the associated Monge formula $M = (x, t, u(x, t))$, to studying the Gaussian curvature K and the mean curvature H, which are defined by

$$K = \frac{L_{11}L_{22} - L_{12}^2}{g_{11}g_{22} - g_{12}^2}, \quad H = \frac{L_{11}g_{22} + L_{22}g_{11} - 2L_{12}g_{12}}{2(g_{11}g_{22} - g_{12}^2)}\,,$$

$$g_{11}g_{22} - g_{12}^2 \neq 0\,,$$

where

$$g_{11} = M_x \cdot M_x, \quad g_{12} = M_x \cdot M_t, \quad g_{22} = M_t \cdot M_t\,,$$

$$L_{11} = M_{xx} \cdot N, \quad L_{12} = M_{xt} \cdot N, \quad L_{22} = M_{tt} \cdot N$$

and

$$N = \frac{M_x \times M_t}{\|M_x \times M_t\|}\,.$$

In the following sections, we will apply this method to study some important partial differential equations.

3. Exact Solutions of the Rosenau–Hyman Equation

The Rosenau–Hyman equation is studied in this section, which is

$$u_t + \alpha u u_x + \beta(u^2)_{xxx} = 0\,, \tag{13}$$

where α is the parameter of the convection term, and β is the parameter of the nonlinear dispersion term. We will use the proposed method of simplest equation to search the exact analytical solutions of it. First, making use of the traveling wave transform (8), then (13) becomes

$$2\beta k^3(uu''' + 3u'u'') + \alpha k u^2 u' - \lambda u' = 0\,. \tag{14}$$

We will give the detailed results of the following three cases.

Case 1. Taking the parameter $b = 1$, one obtains

$$\phi(\xi) = \frac{1}{1 + e^{\xi+\xi_0}} \tag{15}$$

and the auxiliary ordinary differential equation

$$\frac{d\phi(\xi)}{d\xi} = \phi(\xi)^2 - \phi(\xi)\,. \tag{16}$$

From the homogeneous balance method, it follows that $n = 2$, then the solution of Eq. (14) can be written in the following form:

$$u(\xi) = a_0 + a_1\phi(\xi) + a_2\phi^2(\xi)\,. \tag{17}$$

Now, applying (15), (16) and (17) to (14), we can obtain a system of nonlinear algebraic relations among the parameters $a_0, a_1, a_2, k, \lambda$ and the solution of the system by applying characteristic set, then the exact analytical solution of Eq. (13) can be obtained when $b = 1$

$$u = \frac{-5\beta k^2}{\alpha}(1 - 12\phi(\xi) + 12\phi^2(\xi))\,, \tag{18}$$

where $\xi = kx - \frac{15k^5\beta^2}{\alpha}t$.

Case 2. If $b = 2$, then

$$\phi(\xi) = \left(\frac{1}{1 + e^{2\xi+\xi_0}}\right)^{\frac{1}{2}}\,, \tag{19}$$

hence the simplest equation becomes

$$\frac{d\phi(\xi)}{d\xi} = \phi(\xi)^3 - \phi(\xi)\,. \tag{20}$$

From the homogeneous balance method, it follows that $n = 4$, then the solution of Eq. (14) can be written in the following form:

$$u(\xi) = \sum_{i=0}^{4} a_i\phi(\xi)^i\,. \tag{21}$$

Applying (19), (20) and (21) to (14), we can obtain a system of algebraic equations for $a_0, a_1, a_2, a_3, a_4, k, \lambda$. Employing Wsolve to this system, then

$$u = -\frac{20\beta k^2}{\alpha} + \frac{240\beta k^2}{\alpha}\phi(\xi)^2 - \frac{240\beta k^2}{\alpha}\phi(\xi)^4 \tag{22}$$

is an exact analytical solution of Eq. (13), where $\xi = kx - \frac{240k^5\beta^2}{\alpha}t$.

Case 3. If $b = 3$, then

$$\phi(\xi) = \left(\frac{1}{1 + e^{3\xi+\xi_0}}\right)^{\frac{1}{3}}, \tag{23}$$

and the auxiliary ordinary differential equation is

$$\frac{d\phi(\xi)}{d\xi} = \phi(\xi)^4 - \phi(\xi)\,. \tag{24}$$

By the homogeneous balance method, it follows that $n = 6$, then the solution of Eq. (14) can be written in the following form:

$$u(\xi) = \sum_{i=0}^{6} a_i \phi(\xi)^i\,. \tag{25}$$

Therefore, employing (23), (24) and (25) on (14), we can obtain a system of algebraic equations for $a_0, a_1, a_2, a_3, a_4, a_5, a_6, k, \lambda$. Applying characteristic set in Wsolve to this system, we can get the exact analytical solution of Eq. (13)

$$u = -\frac{45\beta k^2}{\alpha} + \frac{540\beta k^2}{\alpha}\phi(\xi)^3 - \frac{540\beta k^2}{\alpha}\phi(\xi)^6\,, \tag{26}$$

where $\xi = kx - \frac{1215k^5\beta^2}{\alpha}t$.

In addition, we can obtain several classes of the solitary wave solutions of Eq. (13) by choosing other values of the parameter b.

To discuss the geometric properties of the solution (18), we rewrite it as

$$u(\xi) = \frac{-5\beta k^2}{\alpha}\left(-2 + 3\tanh^2\left(\frac{\xi+\xi_0}{2}\right)\right), \quad \xi = kx - \frac{15k^5\beta^2}{\alpha}t\,, \tag{27}$$

then its Monge formula is described as follows:

$$M = \left(x, t, \frac{-5\beta k^2}{\alpha}\left(-2 + 3\tanh^2\left(\frac{\xi+\xi_0}{2}\right)\right)\right)\,. \tag{28}$$

Thus, the related quantities and normal vector of M are obtained:

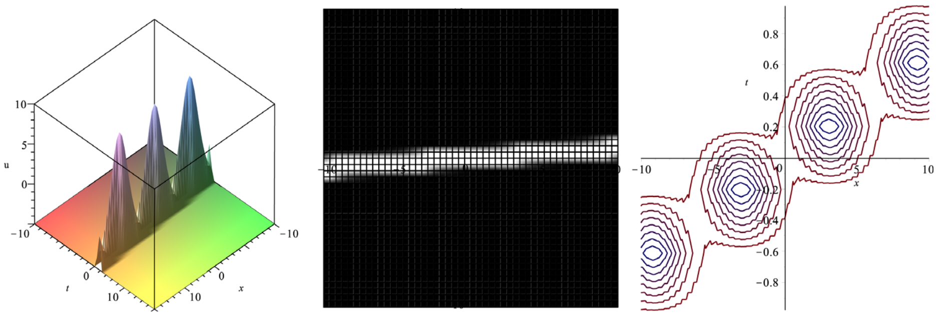
$$g_{11} = 1 + \frac{225\beta^2 k^6 \tanh^2(\frac{\xi+\xi_0}{2})\operatorname{sech}^4(\frac{\xi+\xi_0}{2})^2}{16\alpha}\,,$$

$$g_{12} = -\frac{3375\beta^6 k^{10}\lambda \tanh^2(\frac{\xi+\xi_0}{2})\operatorname{sech}^4(\frac{\xi+\xi_0}{2})}{16\alpha^3},$$

$$g_{22} = 1 + \frac{3375\beta^8 k^{14}\lambda^2 \tanh^2(\frac{\xi+\xi_0}{2})\operatorname{sech}^4(\frac{\xi+\xi_0}{2})}{1024\alpha^4}\,,$$

$$N = \frac{(-\frac{15\beta k^3}{4\alpha}\tanh(\frac{\xi+\xi_0}{2})\operatorname{sech}^2(\frac{\xi+\xi_0}{2}), \frac{225k^7\beta^3}{4\alpha^2}\tanh(\frac{\xi+\xi_0}{2})\operatorname{sech}^2(\frac{\xi+\xi_0}{2}), 1)}{\sqrt{1 + (\frac{225\beta^2 k^6}{16\alpha^2} + \frac{50625\beta^6 k^{14}}{16\alpha^2})\tanh^2(\frac{\xi+\xi_0}{2})\operatorname{sech}^4(\frac{\xi+\xi_0}{2})}}\,,$$

$$L_{11} = \frac{\frac{15\beta k^4}{8\alpha}(3\tanh^2(\frac{\xi+\xi_0}{2}) - 1)\operatorname{sech}^2(\frac{\xi+\xi_0}{2})}{\sqrt{1 + (\frac{225\beta^2 k^6}{16\alpha^2} + \frac{50625\beta^6 k^{14}}{16\alpha^2})\tanh^2(\frac{\xi+\xi_0}{2})\operatorname{sech}^4(\frac{\xi+\xi_0}{2})}}\,,$$



(a) (b) (c)

Fig. 1. (a) Shape and surface, (b) density plot, (c) contour plot of solution (18) with specific values of parameters $k = 1$, $\alpha = 1$, $\beta = 1$.

$$L_{12} = \frac{\frac{225\beta^3 k^8}{8\alpha^2}(3\tanh^2(\frac{\xi+\xi_0}{2}) - 1)\,\mathrm{sech}^2(\frac{\xi+\xi_0}{2})}{\sqrt{1 + (\frac{225\beta^2 k^6}{16\alpha^2} + \frac{50625\beta^6 k^{14}}{16\alpha^2})\tanh^2(\frac{\xi+\xi_0}{2})\,\mathrm{sech}^4(\frac{\xi+\xi_0}{2})}}\,,$$

$$L_{22} = \frac{\frac{3375\beta^5 k^{12}}{8\alpha^3}(3\tanh^2(\frac{\xi+\xi_0}{2}) - 1)\,\mathrm{sech}^2(\frac{\xi+\xi_0}{2})}{\sqrt{1 + (\frac{225\beta^2 k^6}{16\alpha^2} + \frac{50625\beta^6 k^{14}}{16\alpha^2})\tanh^2(\frac{\xi+\xi_0}{2})\,\mathrm{sech}^4(\frac{\xi+\xi_0}{2})}}\,. \tag{29}$$

The Gaussian curvature K and the mean curvature H are given, respectively, as follows:

$$K = 0, \quad H = \frac{(\frac{15\beta k^4}{2\alpha} + \frac{3375\beta^5 k^8}{2\alpha^3})(3\tanh^2(\frac{\xi+\xi_0}{2}) - 1)\,\mathrm{sech}^2(\frac{\xi+\xi_0}{2})}{(1 + (\frac{225\beta^2 k^6}{16\alpha^2} + \frac{50625\beta^6 k^{14}}{16\alpha^2})\tanh^2(\frac{\xi+\xi_0}{2})\,\mathrm{sech}^4(\frac{\xi+\xi_0}{2}))^{\frac{3}{2}}}\,.$$

Therefore, a family of parabolic surfaces $(K = 0, H \neq 0)$ is represented by the solution (18); on the cuspidal edge $x = \frac{15\beta^2 k^4}{\alpha}t - \frac{\xi_0}{k} + \frac{2}{k}\tanh^{-1}(\pm\frac{1}{\sqrt{3}})$, a family of planes $(K = 0, H = 0)$ is represented by (18), which can be given by $(x, t, \frac{5\beta k^2}{\alpha})$. Furthermore, we can obtain the singular points of (18) as $x = \frac{15\beta^2 k^4}{\alpha}t - \frac{\xi_0}{k}$.

The following Fig. 1 give us the graph, density plot and contour plot of solution (18) vividly.

4. Exact Solutions of the Coupled KdV System

In this section, we study the exact solutions and their geometric properties of the coupled KdV system, which is written in the following form:

$$u_t + 6\alpha u u_x - 6 v v_x + \alpha u_{xxx} = 0, \quad v_t + 3\alpha u v_x + \alpha v_{xxx} = 0\,, \tag{30}$$

To find the solitary wave solutions to the system (27), we let

$$u(x,t) = u(\xi), \quad v(x,t) = v(\xi), \quad \xi = kx - \lambda t\,, \tag{31}$$

then the system (27) becomes

$$-\lambda u' + 6k\alpha u u' - 6k v v' + \alpha k^3 u''' = 0, \quad -\lambda v' + 3k\alpha u v' + \alpha k^3 v''' = 0\,. \tag{32}$$

Similarly, as in the previous example, if we choose $b = 1$, then $\phi(\xi) = \frac{1}{1+e^{\xi+\xi_0}}$.

The homogeneous balance method implies $n = 2$ and $m = 1, 2$. Therefore, we can choose

$$u(\xi) = a_0 + a_1\phi(\xi) + a_2\phi(\xi)^2, \quad v(\xi) = c_0 + c_1\phi(\xi)\,. \tag{33}$$

$$u(\xi) = a_0 + a_1\phi(\xi) + a_2\phi(\xi)^2, \quad v(\xi) = c_0 + c_1\phi(\xi) + c_2\phi(\xi)^2\,. \tag{34}$$

Substituting (32) into (31), we can get a system of algebraic equations for $a_0, a_1, a_2, b_0, b_1, k, \lambda$. Similarly, substituting (33) into (31), we can also get a system of algebraic equations for $a_0, a_1, a_2, b_0, b_1, b_2, k, \lambda$. Using characteristic set in Wsolve, we can obtain the solutions to the coupled KdV system as

$$\begin{aligned} u_1(\xi) &= -\frac{\alpha k^3 - \lambda}{6k\alpha} + 2k^2\phi(\xi) - 2k^2\phi(\xi)^2\,, \\ v_1(\xi) &= -\frac{\alpha k^4 - 2k\lambda}{3\alpha} + 2\frac{\alpha k^4 - k\lambda}{3\alpha}\phi(\xi)\,. \end{aligned} \tag{35}$$

$$\begin{aligned} u_2(\xi) &= \frac{-k^3\alpha + \lambda}{3k\alpha} + 4k^2\phi(\xi) - 4k^2\phi(\xi)^2\,, \\ v_2(\xi) &= -\frac{-\sqrt{2}k^2\alpha + 4k\lambda}{6\alpha} - 2\sqrt{2}k^2\phi(\xi) + 2\sqrt{2}k^2\phi(\xi)^2\,, \end{aligned} \tag{36}$$

and the Gaussian curvature K and mean curvature H of the surfaces defined by the solutions (34) and (35), respectively:

$K_{1,2} = 0,$

$$H_1 = \pm\frac{(2\cosh^2(\frac{\xi+\xi_0}{2})) - 3\cosh^5(\frac{\xi+\xi_0}{2})k^2(k^2+\lambda^2)}{(k^6\cosh^2(\frac{\xi+\xi_0}{2}) + k^4\lambda^2\cosh^2(\frac{\xi+\xi_0}{2}) + 4\cosh^6(\frac{\xi+\xi_0}{2}) - k^6 - k^4\lambda^2)^{\frac{3}{2}}}\,,$$

$$H_2 = \pm\frac{18\sinh(\frac{\xi+\xi_0}{2})\cosh^3(\frac{\xi+\xi_0}{2})k(\alpha k^3 - \lambda)\alpha^2(k^2+\lambda^2)}{(\alpha^2k^{10} + \alpha^2k^8\lambda^2 - 2\alpha k^7\lambda - 2\alpha k^5\lambda^3 + 36\alpha^2\cosh^4(\frac{\xi+\xi_0}{2}) + k^4\lambda^2 + k^2\lambda^4)^{\frac{3}{2}}}\,,$$

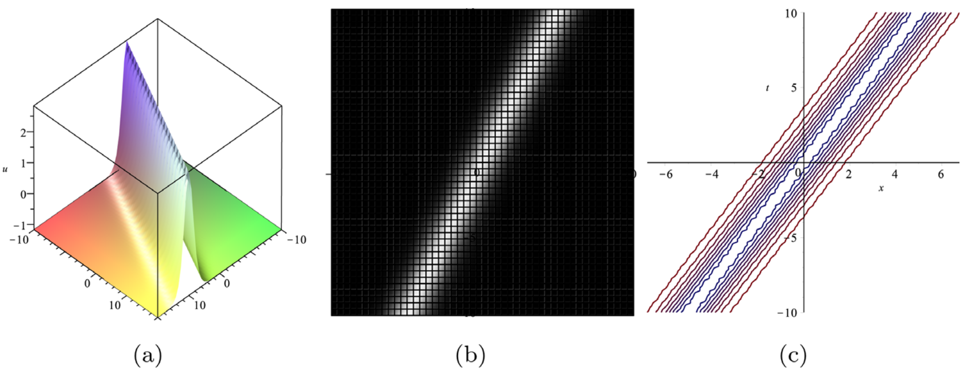
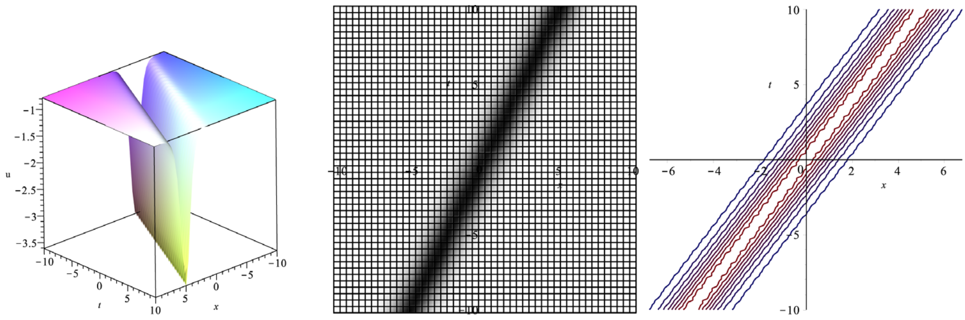


Fig. 2. (a) Shape and surface, (b) density plot, (c) contour plot of u_2 of solution (35) with specific values of parameters $k = 2$, $\alpha = 1$, $\lambda = 1$.



(a) (b) (c)

Fig. 3. (a) Shape and surface, (b) density plot, (c) contour plot of v_2 of solution (35) with specific values of parameters $k = 2$, $\alpha = 1$, $\lambda = 1$.

(37)

where $\xi = kx - \lambda t$. It follows that the solutions $\dot{u}_1$ and v_1 represent a family of parabolic surfaces and a family of planes.

Figures 2 and 3 are the graphs, density plot and contour plot of solution (35).

5. Exact Solutions of the Burgers–Huxley Equation

Now, let us discuss the Burgers–Huxley equation described by

$$u_t + \alpha u u_x - u_{xx} = \beta u(1-u)(u-s)\,, \tag{38}$$

where α, β and s are constants. This equation is a typical model for describing the interaction between reaction mechanism, convection effect and diffusion transport. In 1987, Satsuma firstly obtained two solitary wave solutions by using Hirota method. When $\beta = 0$ and $\alpha = 1$, (37) is reduced to the Burgers equation; when $\alpha = 0$, (37) becomes the Hexley equation. There are many researchers who used various techniques to obtain the solutions of the Burgers–Huxley equation. To do this, similar to Example 3.2, we let

$$u(x,t) = u(\xi), \quad v(x,t) = v(\xi), \quad \xi = kx - \lambda t\,, \tag{39}$$

making use of (35), Eq. (34) will become

$$-\lambda u' + \alpha k u u' - k^2 u'' = \beta u(1-u)(u-s)\,. \tag{40}$$

Case 1. If we choose $b = 1$, then $\phi(\xi) = \frac{1}{1+e^{\xi+\xi_0}}$. By the homogeneous balance method, the parameter $n = 1$, so that we can take

$$u(\xi) = a_0 + a_1\phi(\xi)\,. \tag{41}$$

Substituting (40) into (39) in Wsolve, we can get a system of algebraic equations for parameters a_0, a_1, k, λ. Solving this system by using characteristic set in Wsolve, the solitary wave solutions to (37) are obtained as follows:

$$u(\xi) = \phi(\xi)\,, \tag{42}$$

where

$$\xi = kx - \frac{\alpha k + \beta(1-2s)}{2}t, \quad k = \frac{\alpha \pm \sqrt{\alpha^2 + 8\beta}}{4}$$

and

$$u(\xi) = s\phi(\xi)\,, \tag{43}$$

where

$$\xi = kx - \frac{s(\alpha k + \beta(s-2))}{2}t, \quad k = \frac{(\alpha \pm \sqrt{\alpha^2 + 8\beta})s}{4}\,.$$

Case 2. Let $b = 2$, then $\phi(\xi) = \frac{1}{1+e^{2\xi+\xi_0}}$. It follows $n = 2$ by the homogeneous balance method. So that we can take

$$u(\xi) = a_0 + a_1\phi(\xi) + a_1\phi(\xi)^2 \tag{44}$$

with the help of Wsolve, substituting (43) into (39), we can get a system of algebraic equations for parameters a_0, a_1, a_2, k, λ. Using characteristic set in Wsolve, the solitary wave solutions of (37) are obtained as follows:

$$u(\xi) = 1 - \phi(\xi)\,, \tag{45}$$

where

$$\xi = kx - \frac{2\alpha k + \beta(2s-1)}{4}t, \quad k = \frac{-\alpha \pm \sqrt{\alpha^2 + 8\beta}}{8}\,,$$

and

$$u(\xi) = s\phi(\xi)\,, \tag{46}$$

where

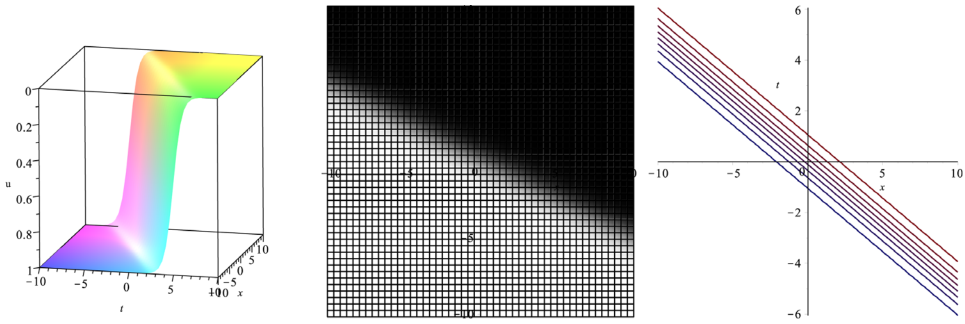
$$\xi = kx - \frac{2\alpha k + \beta s(2-s)}{4}t, \quad k = \frac{(-\alpha \pm \sqrt{\alpha^2 + 8\beta})s}{8}\,.$$

By taking different values of the parameter b, we can obtain other types of the solitary wave solutions of Eq. (37). Furthermore, we can perform the computations of the Gaussian curvature K and mean curvature H of the solution (40), and obtain them as follows:

$$K = 0, \quad H = \frac{\lambda \tanh(\frac{\xi+\xi_0}{2})\,\mathrm{sech}^2(\frac{\xi+\xi_0}{2})}{(64 + \mathrm{sech}^2(\frac{\xi+\xi_0}{2}))^{\frac{3}{2}}}\,. \tag{47}$$

It follows that the solution (40) represents hyperbolic surfaces and planes on the cuspidal edge $x = \frac{\alpha k + \beta(1-2s)}{2k}t - \frac{\xi_0}{k}$.

The 3D plot, density plot and contour plot of solution (40) with specific value of parameters have been shown in Fig. 4.



(a) (b) (c)

Fig. 4. (a) Shape and surface, (b) density plot, (c) contour plot of solution (40) with $s = 3$, $\alpha = 1$, $\beta = 1$.

6. Conclusion

This research mainly deals with the Rosenau–Hyman equation, the coupled KdV system and the Burgers–Huxley equation using modified transformed rational function method. To the best knowledge of us, some new exact solutions of the aforementioned equations, moreover a some geometric descriptions of obtained solutions. All can be illustrated vividly by the given graphs. The results will play some important roles in the studying of some nonlinear phenomena in other research areas, for instance, nonlinear physics.

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