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RESEARCH ARTICLE

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Lump solutions of the 2D Toda equation

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In this research, the lump solution, which is rationally localized and decays along the directions of space variables, of a 2D Toda equation is studied. The effective method of constructing the lump solutions of this 2D Toda equation is derived, and the constraint conditions that make the lump solutions analytical and positive are obtained as well. Finally, three classes of lump solutions are constructed, 3D plots, density plots, and contour plots are given to illustrate this proposed method.

KEYWORDS
integrable equations, integrable hierarchies, soliton solutions, lump solutions, 2D Toda equation

MSC CLASSIFICATION
35Q53; 37K10; 37K20

1 | INTRODUCTION

For soliton theory, how to construct the analytical solutions is one of the key researches of of nonlinear partial differential equations(NPDEs).[1-5] In the past years, some systematic and effective ways have been derived. For instance, the Hirota's bilinear transformation,[6,7] the generalized bilinear transformation,[8] the inverse-scattering transformation,[9,10] the Painlevé analysis approach,[11,12] and the Darboux transformation.[13] Recently, the lump-type and mixed-lump-type solutions, analytical and rationally localized, of NPDEs have been extensively studied by many researchers.[14-25] However, there is no systematic study on the lump solutions of any differential-difference equations, such as the Toda equation.

In this investigation, we study the lump wave solutions of the following 2D Toda equation:

$$\frac{1}{4}\Delta Q_n = e^{Q_{n-1}-Q_n} - e^{Q_n-Q_{n+1}}. \tag{1}$$

As an important integrable equation, the Toda equation plays a vital role in solid state physics, engineering, and other areas.[26-28] Since the KP-I equation can be regarded as the continuum limit of some generalized Toda lattice equation, then we can study the lump solution of Toda equation similarly.

In Section 2, an effective approach of constructing the lump solutions of Equation (1) are derived. Three applications and the corresponding graphs are proposed in Section 3. Finally, we conclude this paper with some conclusions in Section 4.

2 | METHOD

In order to establish an effective method for constructing the lump solutions of Equation (1), we first transform Equation (1) into

$$\frac{1}{4}\Delta \ln(u_n) = \frac{u_{n-1}}{u_n} - \frac{u_n}{u_{n+1}}, \tag{2}$$

by using the logarithmic transformation

$$Q_n = \ln(u_n). \tag{3}$$

Moreover, we have to set

$$u_n = \frac{f_{n-1}}{f_n}, f_n = (p_1 n + p_2 x + p_3 y)^2 + (p_3 n + p_4 x + p_5 y)^2 + p_6, \tag{4}$$

where the parameters $p_i, 1 \leq 6$ are real to be determined. Plugging Equation (4) into Equation (2), we get all the values of the parameters $p_i, 1 \leq 6$, which yield the conditions and the lump solutions of Equation (1). Based on the aforementioned method, three specific classes of lump solutions and the corresponding graphs of Equation (1) are presented in the oncoming section.

3 | APPLICATIONS

In this section, we will employ the results of Section 2 to construct the lump solutions of the 2D Toda equation.

3.1 | Lump solution: Case I

First, we take the following quadratic function which is used to generate the first class of the lump solutions:

$$f_n = (n + p_1 x + p_2 y)^2 + (p_3 y)^2 + k. \tag{5}$$

By the complex computations in Maple, the values of p_1, p_2, and p_3 are obtained as

$$p_1 = \pm\sqrt{\frac{4k+1}{k}}, p_2 = 0, p_3 = \pm\sqrt{\frac{1}{k}}, \tag{6}$$

with the condition $k > 0$ which guarantees that the lump solutions are analytical and positive. Therefore, we have

$$u_n = \frac{f_{n-1}}{f_n} = \frac{((n-1) + p_1 x)^2 + (p_3 y)^2 + k}{(n + p_1 x)^2 + (p_3 y)^2 + k}, \tag{7}$$

which implies that the function

$$Q_n = \ln(u_n) = \ln\frac{((n-1) + p_1 x)^2 + (p_3 y)^2 + k}{(n + p_1 x)^2 + (p_3 y)^2 + k} \tag{8}$$

decay along spacial directions and $Q_n = \ln(u_n)$ is a lump solution of Equation (1). Figure 1 can illustrate the dynamics of this obtained solution by choosing specific parameters.

3.2 | Lump solution: Case II

Now, we use another quadratic function to generate the second class of the lump solutions of the 2D Toda equation,

$$f_n = (n + p_1 x + p_2 y)^2 + (p_3 x - p_3 y)^2 + k. \tag{9}$$

Using the computer algebra system Maple, we get the parameters p_1, p_2, and p_3 as

$$p_1 = \pm\sqrt{\frac{4k+1}{2k}}, p_2 = \pm\sqrt{\frac{4k+1}{2k}}, p_3 = \pm\sqrt{\frac{1}{2k}}, \tag{10}$$

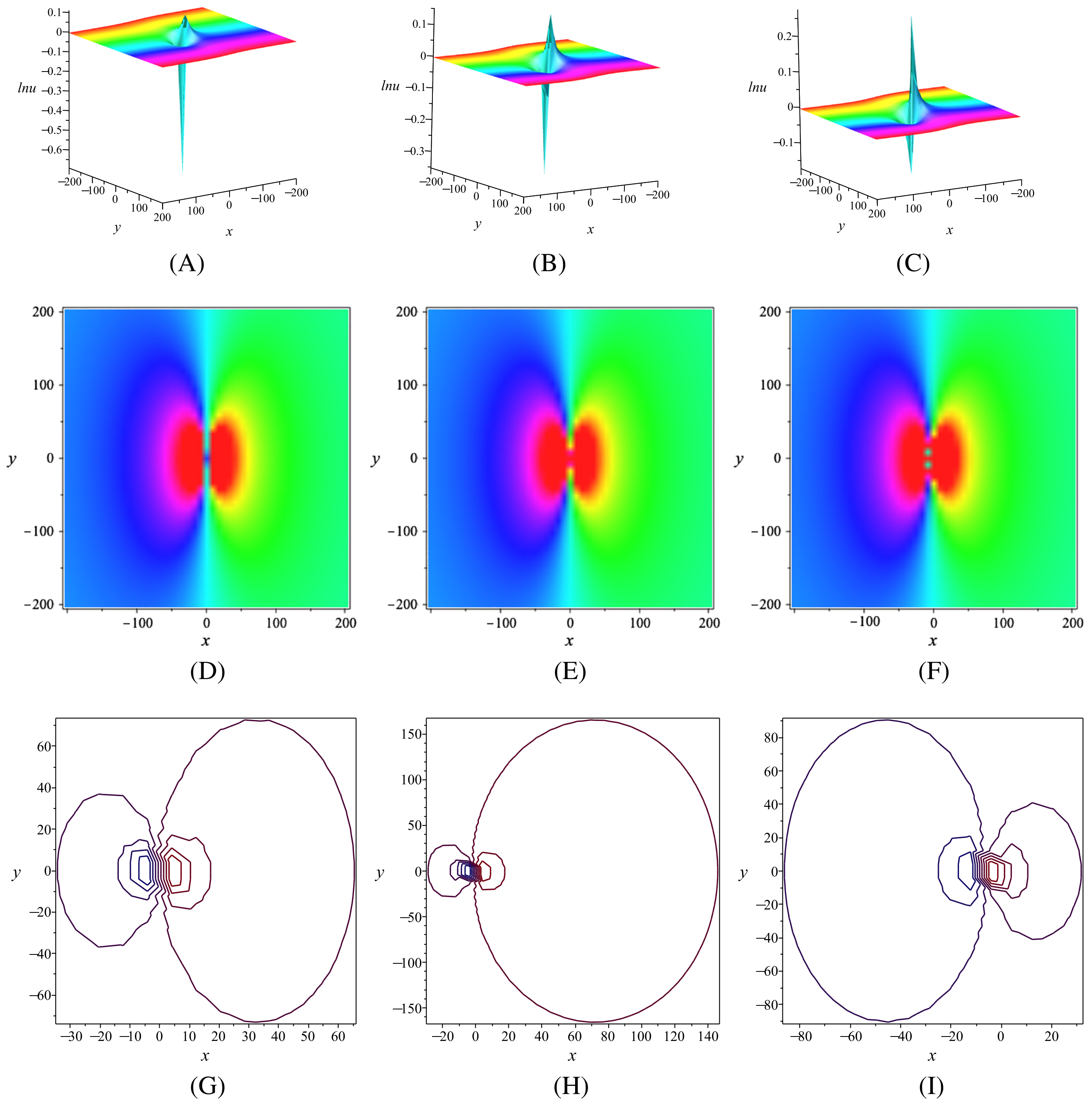

FIGURE 1 3D evolution profiles, density plots, and contour plots of 1-lump wave solution (16) with the specific parameters $k = 1$. (a), (d), and (g) $n = 1$, (b), (e), and (h) for $n = 6$ and (c), (f), and (i) are for $n = 12$ [Colour figure can be viewed at wileyonlinelibrary.com]

where $k > 0$. Thus,

$$u_n = \frac{f_{n-1}}{f_n} = \frac{((n-1) + p_1 x)^2 + (p_3 x - p_3 y)^2 + k}{(n + p_1 x)^2 + (p_3 x - p_3 y)^2 + k}. \tag{11}$$

Additionally, we get

$$Q_n = \ln(u_n) = \ln \frac{((n-1) + p_1 x)^2 + (p_3 x - p_3 y)^2 + k}{(n + p_1 x)^2 + (p_3 x - p_3 y)^2 + k}. \tag{12}$$

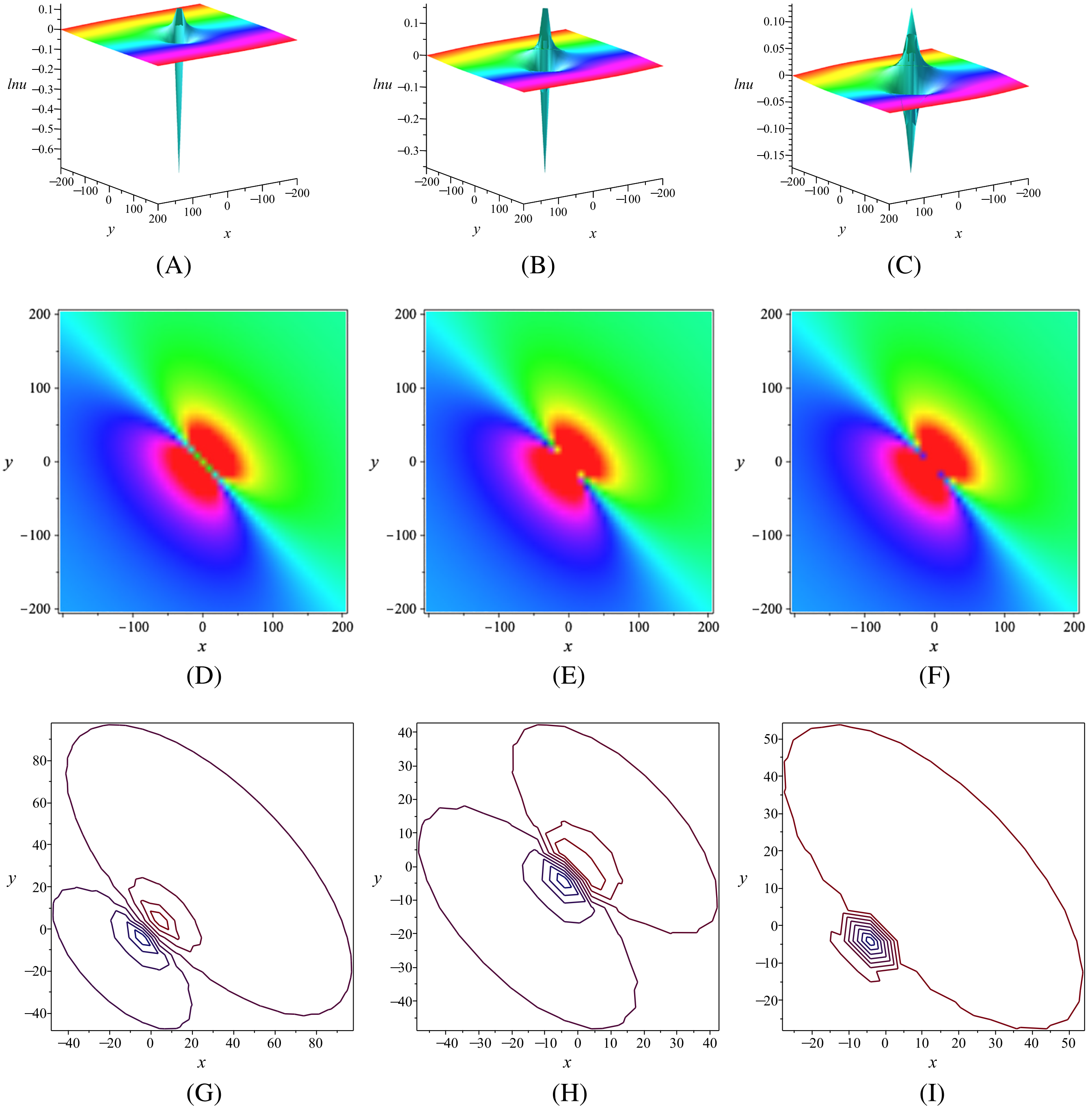

FIGURE 2 3D evolution profiles, density plots and contour plots of 1-lump wave solution (16) with the specific parameters $k = 1$. (a), (b), and (g) $n = 1$; (b), (e), and (h) for $n = 6$; (c), (f), and (i) are for $n = 12$ [Colour figure can be viewed at wileyonlinelibrary.com]

It is obvious that $Q_n = \ln(u_n)$ is a lump solution of Equation (1) since it decays along space directions. The dynamical properties of Q_n can be seen from Figure 2 by taking specific parameters.

3.3 | Lump solution: Case III

Finally, we take the quadratic function as follows:

$$f_n = (n + p_1x + p_2y)^2 + (n + p_3y)^2 + k. \tag{13}$$

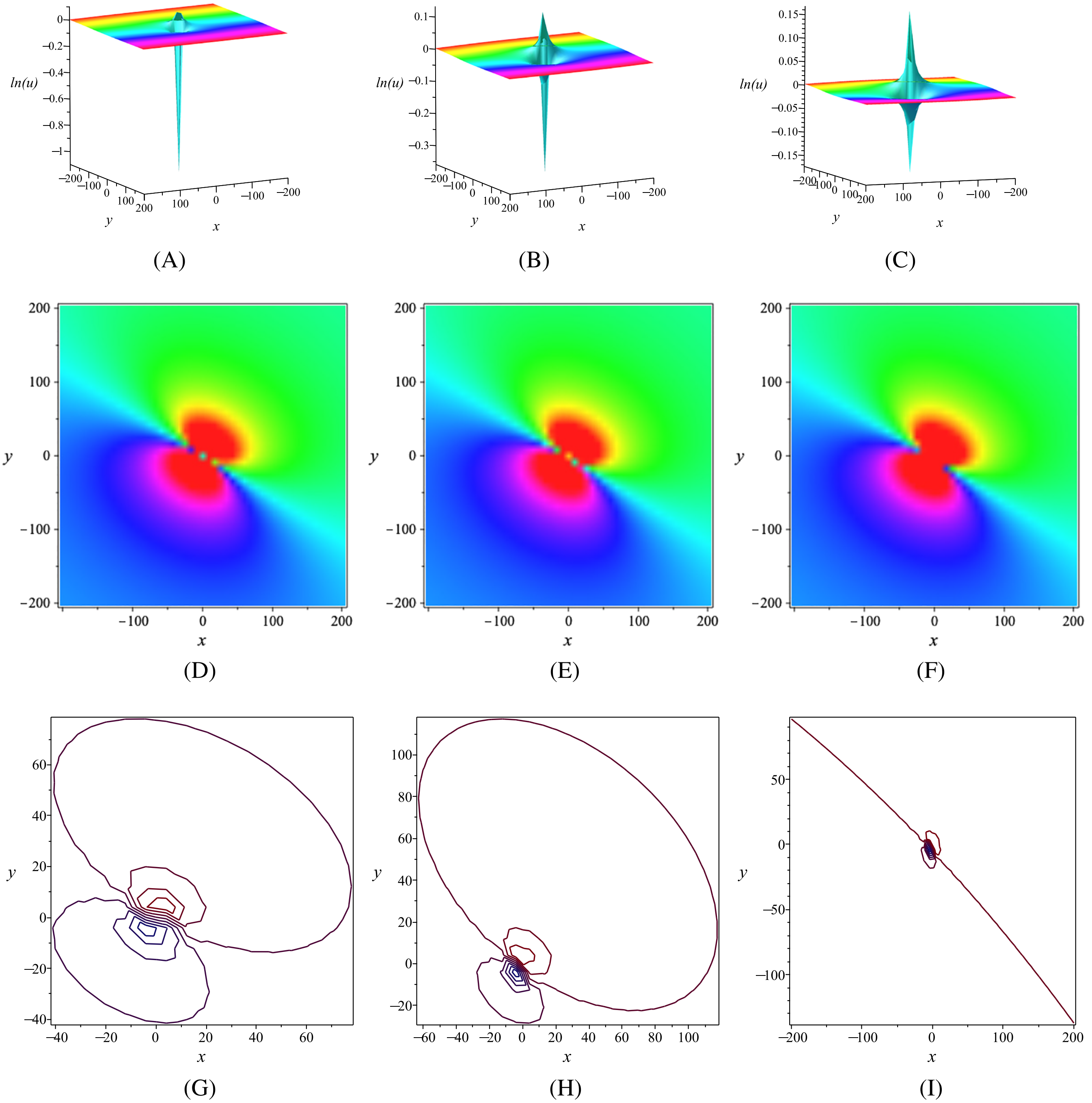

FIGURE 3 3D evolution profiles, density plots and contour plots of 1-lump wave solution (16) with the specific parameters $k = 1$. (a), (d), and (g) $n = 1$; (b), (e), and (h) for $n = 6$; (c), (f), and (i) are for $n = 12$ [Colour figure can be viewed at wileyonlinelibrary.com]

Similarly as the procedure above, All the values of p_1, p_2, and p_3 are given by

$$p_1 = \pm\sqrt{\frac{8k+4}{k^2+k}}, p_2 = \pm\sqrt{\frac{k+1}{k}}, p_3 = \pm\sqrt{\frac{(k+1)\sqrt{\frac{k+1}{k}}}{k}}, \tag{14}$$

with the condition $k > 0$. Then, we get

$$u_n = \frac{f_{n-1}}{f_n} = \frac{((n-1)+p_1x+p_2y)^2 + ((n-1)+p_3y)^2 + k}{(n+p_1x+p_2y)^2 + (n+p_3y)^2 + k}, \tag{15}$$

which generates the lump solution of Equation (1) as

$$Q_n = \ln(u_n) = \ln \frac{((n-1)+p_1x+p_2y)^2+((n-1)+p_3y)^2+k}{(n+p_1x+p_2y)^2+(n+p_3y)^2+k} \tag{16}$$

since it decays in all space directions, rationally localized and analytical. The corresponding dynamical properties are illustrated by Figure 3 vividly.

4 | CONCLUSIONS

In this research, an effective method for constructing the lump solutions of Equation (1) is shown. Three classes of the lump solutions are constructed and illustrated by the corresponding graphs vividly. In the near future, we will discuss the interaction solutions of Equation (1), such as lump-cosine solutions.

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CONFLICT OF INTEREST

The authors declare that they have no conflict of interest.

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REFERENCES

1. Wang DS, Li HB, Wang J. The novel solutions of auxiliary equation and their application to the (2+1)-dimensional Burgers equations. *Chaos Soliton Fract*. 2008;38:74-382.
2. Yan ZY. Vector financial rogue waves. *Phys Lett A*. 2011;375:4274-4279.
3. Wazwaz AM, Osman MS. Analyzing the combined multi-waves polynomial solutions in a two-layer-liquid medium. *Comput Math Appl*. 2018;04:018. https://doi.org/10.1016/j.camwa.
4. Wang DS, Zeng X, Ma YQ. Exact vortex solitons in a quasi-two- dimensional Bose-Einstein condensate with spatially inhomogeneous cubic-quintic nonlinearity. *Phys Lett A*. 2012;376:3067-3070.
5. Wang H, Wang YH. CRE solvability and soliton-cnoidal wave interaction solutions of the dissipative (2+1)-dimensional AKNS equation. *Appl Math Lett*. 2017;69:161-167.
6. Hirota R. Exact soliton of the Korteweg-deVries equation for multiple collisions of solitons. *Phys Rev Lett*. 1971;27:1192-1194.
7. Hirota R. *The Direct Method in Soliton Theory*. Cambridge: Cambridge University Press; 2004.
8. Ma WX, Bilinear equations and resonant solutions characterized by Bell polynomials. *Rep Math Phys*. 2013;72:41-56.
9. Ablowitz MJ, Clarkson PA. *Solitons, Nonlinear Evolution Equations and Inverse Scattering*. NewYork: Cambridge University Press; 1991.
10. Johnson RS, Thompson S. A solution of the inverse scattering problem for the Kadomtsev-Petviashvili equation by the method of separation of variables. *Phys Lett A*. 1978;66:278-281.
11. Weiss J, Tabor M, Carnevale G. The Painlevé property for partial differential equations. *J Math Phys*. 1983;24:522-526.
12. Wazwaz AM. Painlevé analysis for a new integrable equation combining the modified Calogero-Bogoyavlenskii-Schiff (MCBS) equation with its negative-order form. *Nonlinear Dyn*; 91:877-883. https://doi.org/10.1007/s11071-017-3916-0
13. Matveev VB, Salle MA. *Darboux Transformations and Solitons*. Berlin: Springer-Verlag; 1991.
14. Ma WX, Zhou Y. Lump solutions to nonlinear partial differential equations via Hirota bilinear forms. *J Diff Eqs*. 2018;264:2633-2659.
15. Lü X, Wang JP, Lin FH, Zhou XW. Lump dynamics of a generalized two-dimensional Boussinesq equation in shallow water. *Nonlinear Dyn*. 2018;91:1249-1259.
16. Lü X, Ma WX. Study of lump dynamics based on a dimensionally reduced Hirota bilinear equation. *Nonlinear Dyn*. 2016;85:1217-1222.
17. Manukure S, Zhou Y, Ma WX. Lump solutions to a (2 + 1)-dimensional extended KP. *Comput Math Appl*. 2018;75:2414-2419.
18. Lü X, Chen ST, Ma WX. Constructing lump solutions to a generalized Kadomtsev-Petviashvili-Boussinesq equation. *Nonlinear Dyn*. 2016;86:523-534.

19. Satsuma J, Ablowitz MJ. Two-dimensional lumps in nonlinear dispersive systems. *J Math Phys*. 1979;20:1496-1503.
20. Zhang XE, Chen Y. Deformation rogue wave to the (2+1)-dimensiona lKdV equation. *Nonlinear Dyn*. 2017;90:755-763.
21. Zhao ZL, Chen Y, Han B. Lump soliton, mixed lump stripe and periodic lump solutions of a (2+1)-dimensional asymmetrical Nizhnik-Novikov-Veselov equation. *Mod Phys Lett B*. 2017;31:1750157.
22. Yang JY, Ma WX, Qin ZY. Lump and lump-soliton solutions to the (2+1)-dimensional Ito equation. *Anal Math Phys*. 2018;8:427-436. https://doi.org/10.1007/s13324-017-0181-9
23. Lü JQ, Bilige S, Chaolu T. The study of lump solution and interaction phenomenon to (2+1)-dimensional generalized fifth-order KdV equation. *Nonlinear Dyn*. 2018;91:1669-1676.
24. Liu JG, He Y. Abundant lump and lump-kink solutions for the new (3+1)-dimensional generalized Kadomtsev-Petviashvili equation. *Nonlinear Dyn*. 2018;92:1103-1108.
25. Kofane TC, Fokou M, Mohamadou A, Yomba E. Lump solutions and interaction phenomenon to the third-order nonlinear evolution equation. *Eur Phys J Plus*. 2017;132:465.
26. Narita K. Rational and N-breather solutions for the 2D Toda lattice equation. *J Math Anal Appl*. 2003;281:757-760.
27. Nakamura A. Exact Bessel type solution of the two-dimensional Toda lattice equation. *J Phys Soc Japan*. 1983;52:380-387.
28. Hirota R. Discrete analogue of a generalized Toda equation. *J Phys Soc Japan*. 1981;50:3785-3791.

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