Lump solutions of the 2D Toda equation

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In this research, the lump solution, which is rationally localized and decays along the directions of space variables, of a 2D Toda equation is studied. The effective method of constructing the lump solutions of this 2D Toda equation is derived, and the constraint conditions that make the lump solutions analytical and positive are obtained as well. Finally, three classes of lump solutions are constructed, 3D plots, density plots, and contour plots are given to illustrate this proposed method.

KEYWORDS
integrable equations, integrable hierarchies, soliton solutions, lump solutions, 2D Toda equation

MSC CLASSIFICATION
35Q53; 37K10; 37K20

1 INTRODUCTION

For soliton theory, how to construct the analytical solutions is one of the key researches of nonlinear partial differential equations (NPDEs).1-5 In the past years, some systematic and effective ways have been derived. For instance, the Hirota’s bilinear transformation,6,7 the generalized bilinear transformation,8 the inverse-scattering transformation,9,10 the Painlevé analysis approach,11,12 and the Darboux transformation.13 Recently, the lump-type and mixed-lump-type solutions, analytical and rationally localized, of NPDEs have been extensively studied by many researchers.14-25 However, there is no systematic study on the lump solutions of any differential-difference equations, such as the Toda equation.

In this investigation, we study the lump wave solutions of the following 2D Toda equation:

\[
\frac{1}{4} \Delta Q_n = e^{Q_{n-1}-Q_n} - e^{Q_n-Q_{n+1}}.
\]

As an important integrable equation, the Toda equation plays a vital role in solid state physics, engineering, and other areas.26-28 Since the KP-I equation can be regarded as the continuum limit of some generalized Toda lattice equation, then we can study the lump solution of Toda equation similarly.

In Section 2, an effective approach of constructing the lump solutions of Equation (1) are derived. Three applications and the corresponding graphs are proposed in Section 3. Finally, we conclude this paper with some conclusions in Section 4.

2 METHOD

In order to establish an effective method for constructing the lump solutions of Equation (1), we first transform Equation (1) into
by using the logarithmic transformation

\[ Q_n = \ln(u_n). \]  

Moreover, we have to set

\[ u_n = \frac{f_{n-1}}{f_n}, \quad f_n = (p_1 n + p_2 x + p_3 y)^2 + (p_3 n + p_4 x + p_5 y)^2 + p_6, \]  

where the parameters \( p_i, 1 \leq 6 \) are real to be determined. Plugging Equation (4) into Equation (2), we get all the values of the parameters \( p_i, 1 \leq 6 \), which yield the conditions and the lump solutions of Equation (1). Based on the aforementioned method, three specific classes of lump solutions and the corresponding graphs of Equation (1) are presented in the oncoming section.

3 | APPLICATIONS

In this section, we will employ the results of Section 2 to construct the lump solutions of the 2D Toda equation.

3.1 | Lump solution: Case I

First, we take the following quadratic function which is used to generate the first class of the lump solutions:

\[ f_n = (n + p_1 x + p_2 y)^2 + (p_3 y)^2 + k. \]  

By the complex computations in Maple, the values of \( p_1, p_2, \) and \( p_3 \) are obtained as

\[ p_1 = \pm \sqrt{\frac{4k + 1}{k}}, \quad p_2 = 0, \quad p_3 = \pm \sqrt{\frac{1}{k}}, \]  

with the condition \( k > 0 \) which guarantees that the lump solutions are analytical and positive. Therefore, we have

\[ u_n = \frac{f_{n-1}}{f_n} = \frac{(n - 1) + p_1 x)^2 + (p_3 y)^2 + k}{(n + p_1 x)^2 + (p_3 y)^2 + k}, \]  

which implies that the function

\[ Q_n = \ln(u_n) = \ln \frac{(n - 1) + p_1 x)^2 + (p_3 y)^2 + k}{(n + p_1 x)^2 + (p_3 y)^2 + k}, \]  

decay along spacial directions and \( Q_n = \ln(u_n) \) is a lump solution of Equation (1). Figure 1 can illustrate the dynamics of this obtained solution by choosing specific parameters.

3.2 | Lump solution: Case II

Now, we use another quadratic function to generate the second class of the lump solutions of the 2D Toda equation,

\[ f_n = (n + p_1 x + p_2 y)^2 + (p_3 x - p_5 y)^2 + k. \]  

Using the computer algebra system Maple, we get the parameters \( p_1, p_2, \) and \( p_3 \) as

\[ p_1 = \pm \sqrt{\frac{4k + 1}{2k}}, \quad p_2 = \pm \sqrt{\frac{4k + 1}{2k}}, \quad p_3 = \pm \sqrt{\frac{1}{2k}}, \]  

FIGURE 1 3D evolution profiles, density plots, and contour plots of 1-lump wave solution (16) with the specific parameters $k = 1$. (a), (d), and (g) $n = 1$, (b), (e), and (h) for $n = 6$ and (c), (f), and (i) are for $n = 12$ [Colour figure can be viewed at wileyonlinelibrary.com]

where $k > 0$. Thus,

$$u_n = \frac{f_{n-1}}{f_n} = \frac{(n-1) + p_3 x}{(n + p_1 x)^2 + (p_3 x - p_3 y)^2 + k}. \quad (11)$$

Additionally, we get

$$Q_n = \ln(u_n) = \ln \frac{(n-1) + p_3 x}{(n + p_1 x)^2 + (p_3 x - p_3 y)^2 + k}. \quad (12)$$
It is obvious that $Q_n = \ln(u_n)$ is a lump solution of Equation (1) since it decays along space directions. The dynamical properties of $Q_n$ can be seen from Figure 2 by taking specific parameters.

3.3 Lump solution: Case III

Finally, we take the quadratic function as follows:

$$f_n = (n + p_1x + p_2y)^2 + (n + p_3y)^2 + k.$$ (13)
Similarly as the procedure above, All the values of $p_1$, $p_2$, and $p_3$ are given by

\[ p_1 = \pm \sqrt{\frac{8k + 4}{k^2 + k}}, p_2 = \pm \sqrt{\frac{k + 1}{k}}, p_3 = \pm \sqrt{\frac{(k + 1)\sqrt{k+1}}{k}}, \tag{14} \]

with the condition $k > 0$. Then, we get

\[ u_n = \frac{f_{n-1}}{f_n} = \frac{(n - 1 + p_1 x + p_2 y)^2 + ((n - 1) + p_3 y)^2 + k}{(n + p_1 x + p_2 y)^2 + (n + p_3 y)^2 + k}, \tag{15} \]
which generates the lump solution of Equation (1) as

\[ Q_n = \ln(u_n) = \ln \left( \frac{(n - 1) + p_1 x + p_2 y)^2 + ((n - 1) + p_3 y)^2 + k}{(n + p_1 x + p_2 y)^2 + (n + p_3 y)^2 + k} \right) \]  

since it decays in all space directions, rationally localized and analytical. The corresponding dynamical properties are illustrated by Figure 3 vividly.

4 | CONCLUSIONS

In this research, an effective method for constructing the lump solutions of Equation (1) is shown. Three classes of the lump solutions are constructed and illustrated by the corresponding graphs vividly. In the near future, we will discuss the interaction solutions of Equation (1), such as lump-cosine solutions.

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CONFLICT OF INTEREST

The authors declare that they have no conflict of interest.

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REFERENCES
