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Research paper

Further study of the localized solutions of the (2+1)-dimensional B-Kadomtsev–Petviashvili equation

Yong-Li Sun [a], Jing Chen [b], Wen-Xiu Ma [c,d,e,f], Jian-Ping Yu [g,*], Chaudry Masood Khalique [e]

[a] Department of Mathematics, Beijing University of Chemical Technology, Beijing 100029, PR China
[b] School of Statistics and Mathematics, Central University of Finance and Economics, PR China
[c] Department of Mathematics, Zhejiang Normal University, Jinhua 321004, Zhejiang, PR China
[d] Department of Mathematics, King Abdulaziz University, Jeddah, Saudi Arabia
[e] International Institute for Symmetry Analysis and Mathematical Modelling, Department of Mathematical Sciences, North-West University, Mafikeng Campus, Private Bag X 2046, Mmabatho 2735, South Africa
[f] Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620, USA
[g] Department of Applied Mathematics, University of Science and Technology Beijing, Beijing 100083, PR China

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ABSTRACT

In this paper, the localized solutions of the (2+1)-dimensional B-Kadomtsev–Petviashvili (BKP) equation, which is a useful physical model, are further studied. Firstly, by using the theory of Hirota bilinear operator, the corresponding N-soliton solutions are obtained. Then the localized solutions, which are the M-lump solutions, higher-order breathers and hybrid solutions, are also constructed by taking a long-wave limit and introducing some conjugation conditions. In the meanwhile, the dynamic behaviors of these obtained solutions are analyzed and shown graphically by the corresponding numerical simulations with specific parameters, which can greatly affect the solutions, such as the propagation properties.

1. Introduction

In this paper, we consider the following (2+1)-dimensional B-Kadomtsev–Petviashvili (BKP)equation:

$$(u_t + 15uu_{xxx} + 15u_x^3 - 15u_xu_y + u_{5x})_x - 5u_{xxxy} - 5u_{yy} = 0, \tag{1}$$

which belongs to the BKP soliton hierarchy and is a generalization of the famous Caudrey–Dodd–Gibbon–Sawada–Kotera (CDGSK) equation taking the form:

$$v_t + 15vv_{xxx} + 15v_xv_{xx} + 45v^2v_x + v_{5x} = 0, \tag{2}$$

where u, v are functions of x, t. As a matter of fact, if we take $v = u_x$, then the CDGSK Eq. (2) is turned into the BKP Eq. (1). The (2+1)-dimensional BKP Eq. (1) has attracted more attention in the last decades. For instance, the lumps were studied in [1], the Wronskian and Grammian solutions were constructed in [2], the transformation groups were discussed in [3], the Adler–Shiota–van Moerbeke formula for the BKP hierarchy was presented in [4], the interaction solutions of the first BKP equation were obtained in [5], the dynamics of poles of elliptic solutions to the BKP equation was discussed

* Corresponding author.
E-mail address: jpyu@ustb.edu.cn (J.-P. Yu).

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in [6], the dKP and BKP equations with self-consistent sources were constructed in [7], the full symmetry groups, painlevé integrability and exact solutions of the nonisospectral BKP equation were investigated in [8]. Furthermore, a lot of other key contributions have been made by many other researchers.

It is noted that lump solutions are rationally localized along all directions in space, which are presented for many other integrable equations, such as the (2+1)-dimensional KP equation [9–11] and Sawada–Kotera (SK) equation [12], and so on [13–18]. Since almost all integrable equations possess Hirota bilinear forms, we can make use of Hirota bilinear forms for constructing M-lumps, higher-order breathers and interaction solutions of nonlinear evolutionary equations. The authors in [1] made use of the positive quadratic functions to construct the lump solutions of the BKP equation, which are first-order lump solutions. However, they did not discuss the M-lumps, higher-order breathers and hybrid solutions therein. Moreover, to the best of our knowledge, no people have studied the M-lumps, the higher-order breathers and the hybrid solutions between M-lumps or higher-order breathers and other solutions, such as the stripe solitons of the BKP Eq. (1). Actually, M-lumps, higher-order breathers and hybrid solutions play a more important role in studying the nonlinear phenomenon than the first-order lump and breather solutions. So that we will study the M-lumps, higher-order breathers and hybrid solutions of the (2+1)-dimensional BKP Eq. (1). The techniques used here are the Hirota direct method and the "long-wave" limits, which relate the N-soliton solutions to M-lumps and breathers. These two techniques are totally different from the positive quadratic function method.

As different kinds of localized waves, M-lumps, higher-order breathers and rogue waves and soltions play a key role in studying complex nonlinear physical systems. For example, nonlinear optics, cold atoms, plasmas and Bose–Einstein condensates. As special rational solutions, lumps are localized in all directions in space. Rogue waves and breathers are localized on a background with unstable characteristics. Specifically, breathers are viewed as the crucial prototypes to explain rogue wave phenomenon and are the localized breathing waves with a periodic structure along a certain direction [19]. As of now, a lot of effective methods have been proposed for constructing above-mentioned solutions. For instance, the inverse scattering transformation method [20], the Hirota direct method and the generalized bilinear approach [21–23], the Darboux transformation method [24–26], the Bäcklund transformation method [27,28], and so on [29–42]. As a direct and effective method, the Hirota direct method will be applied in our research.

The task of this research is to study the M-lumps, higher-order breathers and hybrid solutions of the BKP Eq. (1) by using the "long-wave" limits and Hirota direct method with the help of Maple. In Section 2, the general N-soliton solutions of (2+1)-dimensional BKP Eq. (1) are given and the analyses of their dynamic behaviors are performed. In Section 3, based on the discussion in Section 2 and by using the Hirota direct method and taking "long- wave" limits, M-lump solutions are constructed. From the numerical simulations, the dynamic behaviors are analyzed, which show that M-lump solutions are actually multiple collisions. In Section 4, using the obtained results in Section 3, the higher-order breathers are constructed by imposing some conjugate conditions on the parameters. In the meanwhile, the numerical simulations tell us that the dynamic behaviors are complicated since it consists of many breathers. In Section 5, some kinds of hybrid solutions are presented: one is the interaction between a lump wave and a stripe soliton wave, the other is the interaction between a breather and a stripe soliton. The numerical simulations further show that these two kinds of collisions are elastic. Finally, some discussions are given briefly.

2. N-soliton solutions

By using the Hirota direct method, the N-soliton solutions of the BKP Eq. (1) are studied in this part since it is pretty useful for constructing the M-lumps, higher-order breathers and hybrid solutions. As the following is the Hirota bilinear BKP equation of (1)

$$(D_x^6 - 5D_x^3 D_y + D_x D_t - 5D_y^2) f \cdot f = 0, \tag{3}$$

which is obtained by substituting the following logarithmic transformation into (1)

$$u = 2(\ln f)_x, \tag{4}$$

with $f = f(x, y, t)$. The notations D_x, D_y, D_t are the Hirota bilinear derivative operators defined by

$$\begin{aligned} &D_x^k D_y^m D_t^n f \cdot g = (\partial_x - \partial_{x'})^k (\partial_y - \partial_{y'})^m (\partial_t - \partial_{t'})^n \\ &f(x, y, t) g(x', y', t')|_{x'=x, y'=y, t'=t}, \end{aligned} \tag{5}$$

with $k, m, n \in \mathbb{Z}^+ \cup \{0\}$. It is obvious that if f solves the bilinear BKP Eq. (3), then $u = 2(\ln f)_x$ solves the BKP Eq. (1). In order to find N-soltion solutions of the BKP Eq. (1), we begin with the N-soliton solutions of the bilinear BKP Eq. (3), which reads

$$f = f_N = \sum_{\mu=0,1} \exp\left(\sum_{i=1}^{N} \mu_i \xi_i + \sum_{1 \le i < j \le N} \mu_i \mu_j A_{ij}\right), \tag{6}$$

where

$$\begin{aligned}&\xi_i = k_i(x + p_i y + \omega_i t + \xi_i^0),\ \omega_i = k_i^4 - 5k_i^2 p_i - 5p_i^2,\\ &\exp(A_{ij}) = \frac{k_i^4 - 3k_i^3 k_j + (4k_j^2 - 2p_i - p_j)k_i^2 - 3k_j(k_j^2 - p_i - p_j)k_i + k_j^4 - (p_i + 2p_j)k_j^2 + (p_i - p_j)^2}{k_i^4 + 3k_i^3 k_j + (4k_j^2 - 2p_i - p_j)k_i^2 + 3k_j(k_j^2 - p_i - p_j)k_i + k_j^4 - (p_i + 2p_j)k_j^2 + (p_i - p_j)^2},\\ &k_i, p_i, \omega_i, \xi_i^0 \in \mathbb{R}, i, j = 1, \ldots, N.\end{aligned} \tag{7}$$

In the expression (6), $\sum_{\mu=0,1}$ means the summation over all possible combinations of $\mu_i = 0, 1$, and the notation $\sum_{1\leq i<j\leq N}$ means the summation over all possible pairs (i, j) chosen from the set $\{1, 2, \ldots, N\}$ with the condition $i < j$. For example, the choice $N = 3$ gives the 3-soliton solution of the bilinear BKP equation (3) as the following

$$\begin{aligned} f = f_3 \quad = \quad & 1 + \exp(\xi_1) + \exp(\xi_2) + \exp(\xi_3)\\ & + \exp(\xi_1 + \xi_2 + A_{12}) + \exp(\xi_1 + \xi_3 + A_{13}) + \exp(\xi_2 + \xi_3 + A_{23})\\ & + \exp(\xi_1 + \xi_2 + \xi_3 + A_{12} + A_{13} + A_{23}).\end{aligned} \tag{8}$$

Substituting (8) into (4) yields a 3-soliton solution of the BKP Eq. (1).

3. M-lump solution

Based on the N-soliton solutions obtained by using the Hirota bilinear method in Section 2, we will construct M-lump solutions of the BKP Eq. (1) by taking a "long wave" limit of the corresponding N-soliton solutions given in (6). In order to understand the dynamic behaviors of M-lump solutions, some examples and corresponding graphs are given, from which we can see that multiple collisions exist in M-lump solutions. According to N-soliton solutions (6), we take $exp(\xi_i^0) = -1, i = 1, \ldots, N$. Now N-soliton solution of the bilinear BKP equation f_N can be readily rewritten in the following form:

$$f = f_N = \sum_{\mu=0,1} \Pi_{i=1}^{N}(-1)^{\mu_i}\exp(\mu_i\xi_i)\Pi_{1\leq i<j\leq N}\exp(\mu_i\mu_j A_{ij}), \tag{9}$$

where

$$\xi_i = k_i(x + p_i y - (k_i^4 - 5k_i^2 p_i - 5p_i^2)t), (i = 1, \ldots, N). \tag{10}$$

Then, taking a "long-wave" limit of $k_i \to 0$ and letting the k_i's have the same asymptotic order, we have

$$f = f_N = \sum_{\mu=0,1} \Pi_{i=1}^{N}(-1)^{\mu_i}(1 + \mu_i k_i\theta_i)\Pi_{1\leq i<j\leq N}(1 + \mu_i\mu_j k_i k_j B_{ij}) + O(k^{N+1}), \tag{11}$$

with

$$\theta_i = x + p_i y + 5p_i^2 t, B_{ij} = \frac{6(p_i + p_j)}{(p_i - p_j)^2}, (i, j = 1, \ldots, N; i \neq j). \tag{12}$$

According to the symmetric property of f_N with respect to parameters k_i's, it is found that f_N can be factorized by the product $\Pi_{i=1}^N k_i$. Using the logarithmic transformation (4), we get the rational solution of (1). It is readily proved that

$$u = 2(\ln(\frac{f_N}{\Pi_{i=1}^N k_i}))_x \tag{13}$$

is a rational solution of the BKP Eq. (1). Therefore, the constant factor $\Pi_{i=1}^N k_i$ can be omitted from f_N, and still denote the result $\frac{f_N}{\Pi_{i=1}^N k_i}$ as f_N. The simplified form of M-lump solution of the bilinear BKP Eq. (3) is given in the form of

$$\begin{aligned} f = f_N \quad = \quad & \Pi_{i=1}^N\theta_i + \tfrac{1}{2}\textstyle\sum_{i,j} B_{ij}\Pi_{l\neq i,j}^N\theta_l + \cdots\\ & + \quad \tfrac{1}{M!2^M}\textstyle\sum_{i,j,k,l,\ldots,m,n} \overbrace{B_{ij}B_{kl}\cdots B_{mn}}^{M}\,\Pi_{l\neq i,j,k,l,\ldots,m,n}^N\theta_l + \cdots,\end{aligned} \tag{14}$$

where θ_l's are defined by (12). (14) usually yields a singular rational solution, but if we take

$$p_{M+i} = p_i^*, (i = 1, \ldots, M), \tag{15}$$

where $*$ denotes the conjugate of a complex function and $N = 2M$, then (14) leads to M-lump solutions of the bilinear BKP Eq. (3), which implies that the M-lump solutions can be obtained from (4). The above discussions have been confirmed in [9]. In the following subsections, some numerical simulations are given to illustrate the above method and results.

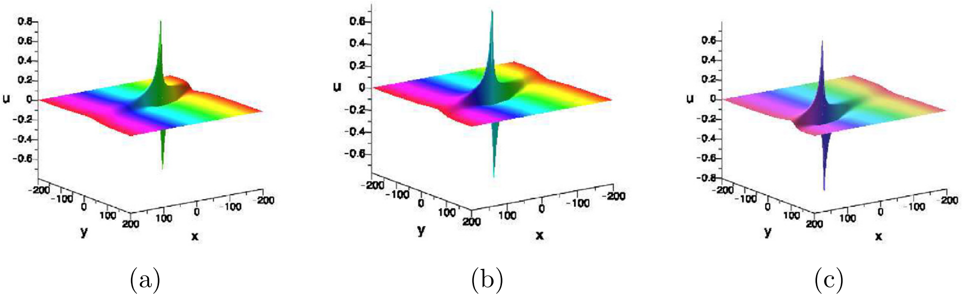


Fig. 1. Evolution profile of the 1-lump solution u expressed in (23) at different time. The specific parameters are $a=-2, b=1$: (a) $t=-3$; (b) $t=0$; (c) $t=3$.

3.1. 1-lump solution

In order to construct 1-lump solution of the BKP Eq. (1), we have to take $N=2$ which implies $M=1$ according to the aforementioned discussion. From (9), we have

$$f_2 = 1 - e^{\xi_1} - e^{\xi_2} + e^{\xi_1+\xi_2+A_{12}}, \tag{16}$$

where

$$\xi_i = k_i(x + p_i y - (k_i^4 - 5k_i^2 p_i - 5p_i^2)t), (i=1,2). \tag{17}$$

Taking the "long wave" limit, $k_i \to 0, (i=1,2)$ and removing the constant factor k_1k_2 leads to f_2 in (14) as follows

$$f_2 = \theta_1\theta_2 + B_{12}, \tag{18}$$

with

$$\theta_i = x + p_i y + 5p_i^2 t, B_{12} = \frac{6(p_1+p_2)}{(p_1-p_2)^2}, (i=1,2). \tag{19}$$

Taking $p_2 = p_1^*$ and $B_{12} > 0$ yields the 1-lump solution of the blinear BKP Eq. (3) as follows

$$f_2 = \theta_1\theta_1^* + \frac{6(p_1+p_2)}{(p_1-p_2)^2} > 0. \tag{20}$$

Plugging $p_1 = a + bI$ into (4) with $I^2=-1$, we obtain

$$\begin{aligned} f_1 &= (5a^2t + ay + x)^2 + b^2(50a^2t^2 + (10ay - 10x)t + y^2) + 25b^4t^2 \\ &= (x' + ay')^2 + b^2y'^2 - \tfrac{3a}{b^2}, \end{aligned} \tag{21}$$

where

$$x' = x - 5(a^2+b^2)t, y' = y + 10at. \tag{22}$$

From (21) and the logarithmic transformation (4), we have the 1-lump of the BKP Eq. (1)

$$u = \frac{4(x'+ay')}{(x'+ay')^2 + b^2y'^2 - \frac{3a}{b^2}}. \tag{23}$$

where the condition $a < 0$ since $B_{12} = -\frac{3a}{b^2} > 0$. It is obvious that this nonsingular rational solution is decaying as $O(1/x^2, 1/y^2)$ for $|x|, |y| \to \infty$ and moving with the velocity according to (22)

$$v_x = 5(a^2+b^2), v_y = -10a. \tag{24}$$

Numerical simulations are presented to illustrate the dynamic behaviors of 1-lump solution (23) of the BKP Eq. (1).

3.2. M-lump solutions

In this part, M-lump solutions will be studied. The first thing to consider is 2-lump solution. To construct 2-lump solution, we have to take $N = 2M = 4$ which implies $M=2$ and $\exp(\xi_i^0) = -1, i = 1, \ldots, 4)$, then the 2-lump solution of the bilinear BKP Eq. (3) can be expressed as the following by virtue of (14)

$$\begin{aligned} f_2 = {} & \theta_1\theta_2\theta_3\theta_4 + B_{12}\theta_3\theta_4 + B_{13}\theta_2\theta_4 + B_{14}\theta_2\theta_3 \\ & + B_{23}\theta_1\theta_4 + B_{24}\theta_1\theta_3 + B_{34}\theta_1\theta_2 + B_{12}B_{34} + B_{13}B_{24} + B_{14}B_{23}, \end{aligned} \tag{25}$$

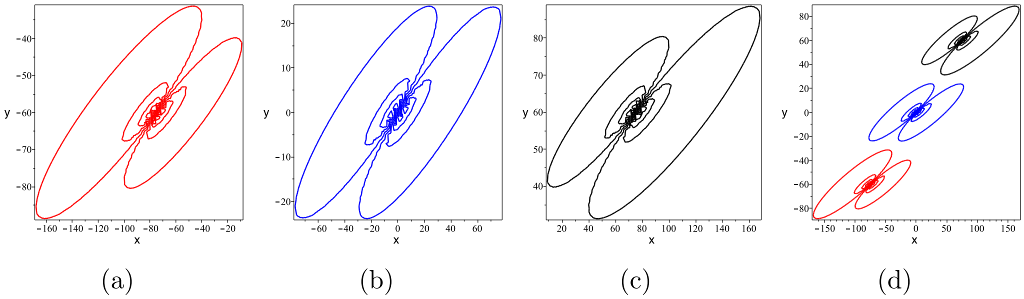


Fig. 2. Contour profiles of the 1-lump solution u in (23). The specific parameters are $a = -2, b = 1$: (a) $t = -3$; (b) $t = 0$; (c) $t = 3$; (d) shows that this 1-lump wave travels along the line $y = 4/5(x - \sqrt{6})$. The red, blue and black graphs represent the contours of 1-lump wave at time $t = -3, 0, 3$, respectively.

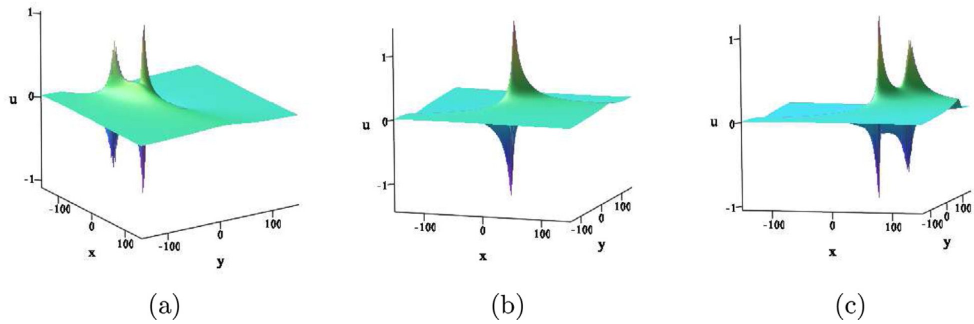


Fig. 3. Evolution profile of the 2-lump solution u at different time. The specific parameters are $a = -2, b = 1; c = -1, d = 1$: (a) $t = -3$; (b) $t = 0$; (c) $t = 3$.

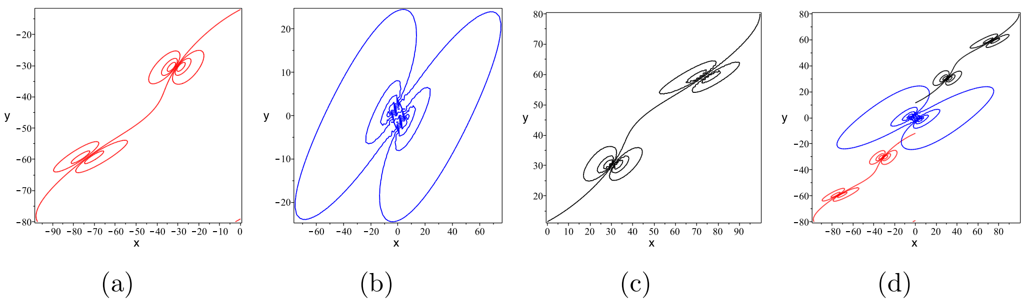


Fig. 4. Contour profiles of the 2-lump solution u. The specific parameters are $a = -2, b = 1; c = -1, d = 1$: (a) $t = -3$; (b) $t = 0$; (c) $t = 3$; (d) shows that this 2-lump wave travels along diagonally in the xOy plane and almost interacts at the moment $t = 0$, and then they separate from each other. The red, blue and black graphs represent the contours of 2-lump wave at time $t = -3, 0, 3$, respectively.

where

$$\theta_i = x + p_i y + 5p_i^2 t, (i = 1, \dots, 4), B_{ij} = \frac{6(p_i + p_j)}{(p_i - p_j)^2}, (1 \le i < j \le 4). \tag{26}$$

The required conditions for construct 2-lump solution is obtained as follows

$$p_3 = p_1^* = a - bI, p_4 = p_2^* = c - dI, I^2 = -1. \tag{27}$$

The dynamic properties of 2-lump solution is graphically illustrated by Figs. 3 and 4 with specific parameters. Similarly, if we take $N = 2M = 6$, which means $M = 3$, and $\exp(\xi_i^0 = -1)$, and $p_{3+i} = p_i^*, (i = 1, 2, 3)$, then, after a tedious computations, we obtain the 3-lump solution f_3 of the bilinear BKP Eq. (3), the expression of the 3-lump solution is omitted here since it contains a lot of terms and is very complicated. the 3-lump solution of the BKP Eq. (1) is obtained after plugging f_3 into the logarithmic transformation (4). The evolution profiles of the 3-lump solution with specific parameters are depicted in the following figures, which show us that there exist collisions among these kind of 3-lump solutions.

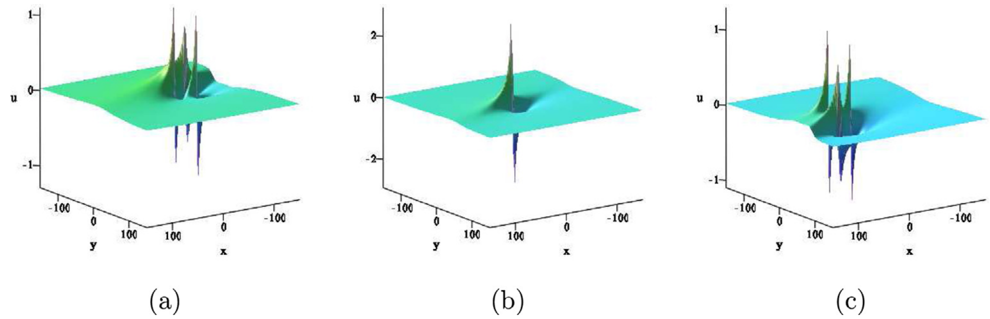


Fig. 5. Evolution profile of the 3-lump solution u at different time. The specific parameters are $a=-2, b=1; c=-1, d=1; r=-1, s=2$: (a) $t=-3$; (b) $t=0$; (c) $t=3$.

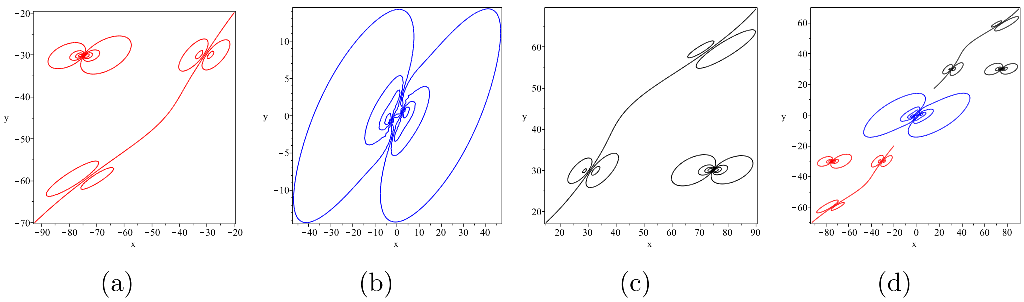


Fig. 6. Contour profiles of the 3-lump solution u. The specific parameters are $a=-2, b=1; c=-1, d=1, r=-1, s=2$: (a) $t=-3$; (b) $t=0$; (c) $t=3$; (d) shows that this 3-lump wave travels along diagonally in the xOy plane and collide with each other at around the moment $t=0$, and then they separate from each other. The red, blue and black graphs represent the contours of 3-lump wave at time $t=-3, 0, 3$, respectively.

Remark. (1) In a word, when choosing $N=2M$ and $p_{M+i}=p_i^*, (i=1,\ldots,M)$ with the condition $B_{ij}>0$, then we can get M-lump solutions by taking the "long-wave" limit of the corresponding N-solitons.

(2) If we impose extra conditions on the parameters in f_N, then we can get higher-order breathers, which will be studied in the next section.

4. Higher-order breathers

In this section, we will construct the higher-order breathers of the BKP Eq. (1) from the higher-order breathers of the bilinear BKP Eq. (3). Therefore, we have to impose extra conditions on the related parameters of the N-soliton solutions (6)–(7) as follows:

$$\begin{aligned} &N=2M, k_{i+1}=k_i^*, p_{i+1}=p_i^*, \omega_{i+1}=\omega_i^*,\\ &\xi_{i+1}^0=\xi_i^{0*}, (i=1,3,5,\ldots). \end{aligned} \tag{28}$$

4.1. 1st-order breather

According to the requirements for the parameters in (28), we have to select the parameters as follows:

$$\begin{aligned} &N=2M=2, k_2=k_1^*=\alpha_1-I\gamma_1, p_2=p_1^*=\beta_1-I\delta_1,\\ &\omega_2=\omega_1^*=\rho_1-I\sigma_1, \xi_2^0=\xi_1^{0*}=0, \end{aligned} \tag{29}$$

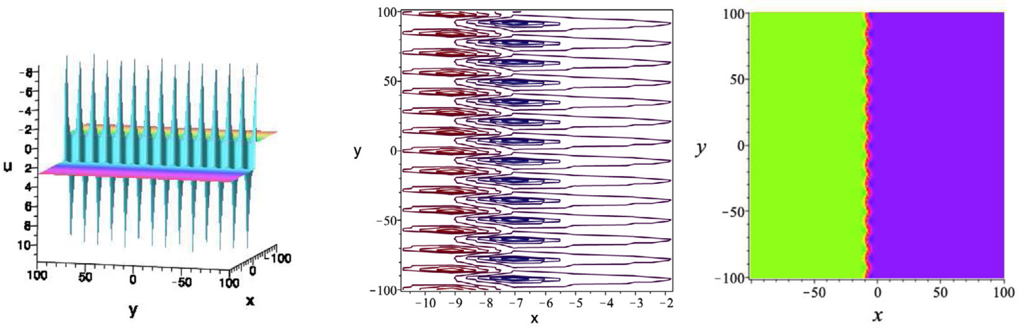
which implies $M=1$. Then, after tedious symbolic computations, the 1st-order breather solution of the bilinear BKP Eq. (3) obtained as the following is

$$f_1=1+2e^{\Re(\xi_1)}\cos(\Im(\xi_1))+e^{A_{12}}e^{2\Re(\xi_1)}, \tag{30}$$

where $\Re$, $\Im$ denote the real part and imaginary part of a complex number, respectively. And

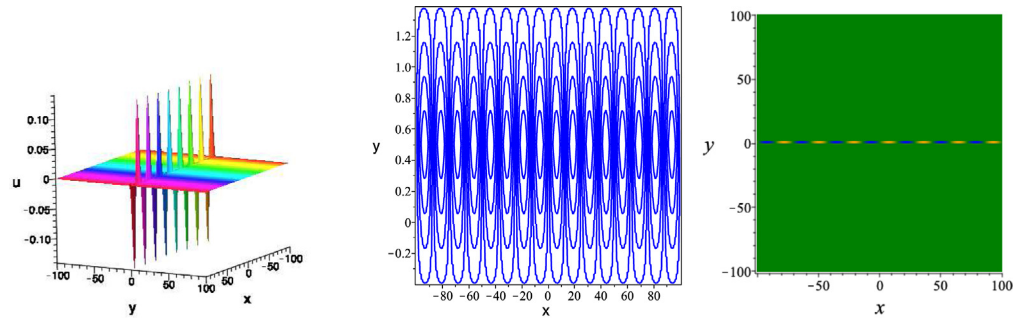
$$e^{A12}=\frac{(\alpha_1\gamma_1)^2-3\gamma_1^4-\alpha_1\gamma_1\delta_1-3\beta_1\gamma_1^2+\delta_1^2}{-3\alpha_1^4+(\gamma_1^2+3\beta_1)\alpha_1^2-\alpha_1\gamma_1\delta_1+\delta_1^2}. \tag{31}$$

The dynamic behaviors of 1st-order breather are shown in Fig. 7 and (8).



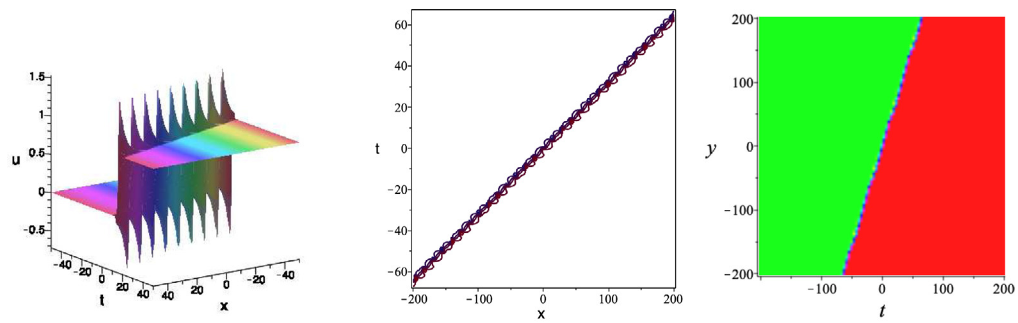
(a) (b) (c)

Fig. 7. Evolution profile of the 1-breather solution u at time $t = -3$. The specific parameters are $\alpha_1 = 2/3, \gamma_1 = 0, \beta_1 = 0, \delta_1 = 2/3$: (a) 3D plot; (b) Contour plot; (c) Density plot.



(a) (b) (c)

Fig. 8. Contour profiles of the 1-lump solution u at time $t = 0$. The specific parameters are $\alpha_1 = 2/3, \gamma_1 = 0, \beta_1 = 0, \delta_1 = 2/3$: (a) 3D plot; (b) Contour plot; (c) Density plot.



(a) (b) (c)

Fig. 9. Evolution profile of the 1-breather solution u at $x = 0$, with specific parameters $\alpha_1 = 1/5, \gamma_1 = 1/10, \beta_1 = -1, \delta_1 = 1$: (a) 3D plot; (b) Contour plot; (c) Density plot.

Remark. (1) From Fig. 7 while k_i's and p_i's are real numbers and pure imaginary numbers, respectively, the 1st-order breather obtained is only y-periodic and localized in the x-axis.

(2) From Fig. 8, if k_i's and p_i's are pure imaginary numbers, the 1st-order breather obtained is only x-periodic and localized in the y-axis.

(3) Form Fig. 9, if k_i's and p_i's are complex numbers, the 1-order breather obtained is (x, t)-periodic and localized in xOt-plane. What should be pointed out is that the periodicity and dynamic behaviors of the breathers in (x, t)-plane and (y, t)-plane depend on the coefficients of x, t, and of y, t. Therefore, the numerical simulations of the breathers in yOt are omitted.

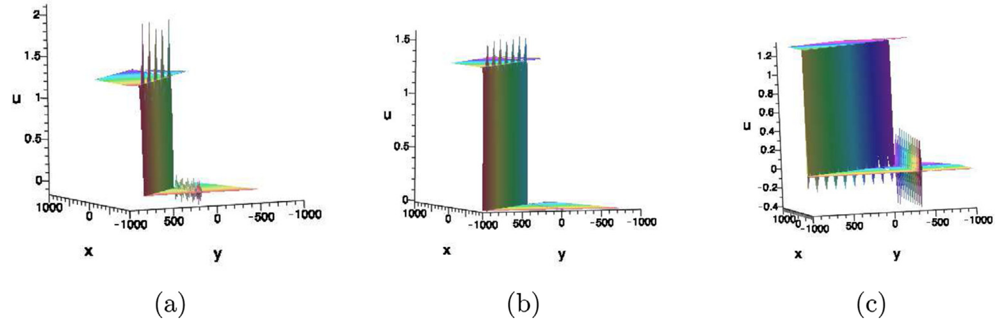


Fig. 10. Evolution profile of the 2-breather solution u at different time: (a) $t = -10$; (b) $t = 1$; (c) $t = 20$.

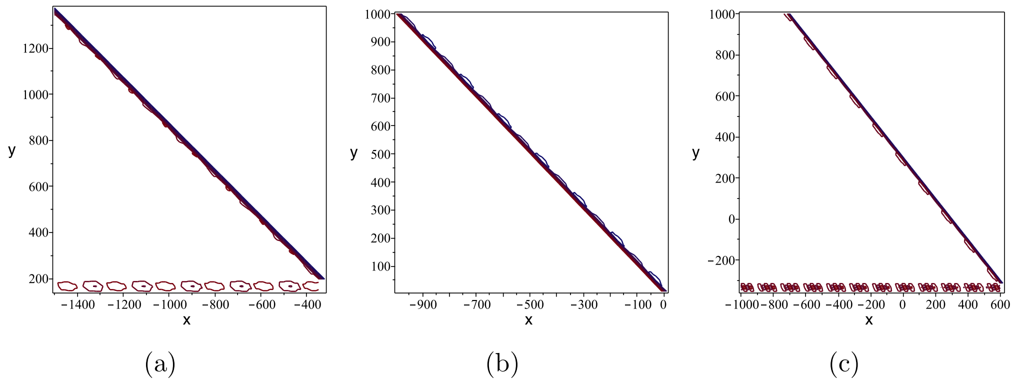


Fig. 11. Contour profiles of the 2-breather solution u at different time: (a) $t = -10$; (b) $t = 1$; (c) $t = 20$.

4.2. 2nd-order breather

According to (28) and proceeding similarly as in the previous subsection, it is readily to construct the 2nd-order breather solutions by taking $N = 2M = 4$ and parameters as follows:

$$\begin{aligned} &k_2 = k_1^* = \alpha_1 - I\gamma_1, p_2 = p_1^* = \beta_1 - I\delta_1, \xi_1^0 = \xi_2^0 = 0;\\ &k_4 = k_3^* = \alpha_2 - I\gamma_2, p_4 = p_3^* = \beta_2 - I\delta_2, \xi_3^0 = \xi_4^0 = 0. \end{aligned} \tag{32}$$

In order to show the dynamic properties of the 2nd-order breather of the BKP Eq. (1), we take the particular parameters: $\alpha_1 = 1/3, \gamma_1 = 0, \beta_1 = 1, \delta_1 = 2; \alpha_2 = 0, \gamma_2 = 4/5, \beta_2 = 2, \delta_2 = 1$. And the dynamic behaviors of the 2-order breather solutions u are shown in Figs. 10 and 11 at different time. It is easily observed that these 2nd-order breather solutions consist of two 1st-order breathers, which implies that there exist collisions in the 2nd-order breather. Therefore, the dynamic behaviors of the 2nd-order breather wave will often change during its traveling.

5. Hybrid solutions

The aim of this section is to study the hybrid solutions of the BKP Eq. (1). Since the computations are so complex and hard, we only discuss the following two kinds of hybrid solutions: one is the interaction solutions between 1-lump and 1-soliton solutions; and the other is the interactions between 1-breather and 1-soliton solutions. The hybrid solutions play an important role in studying nonlinear physical phenomenon. Thus, it is of interest to do some research on interaction solutions.

5.1. The hybrid solution between 1-lump and 1-soliton solutions

In order to obtain the hybrid solutions between 1-lump and 1-soliton solutions, the requirements as the following are

$$N = 3, \xi_1^0 = \xi_2^0 = -1. \tag{33}$$

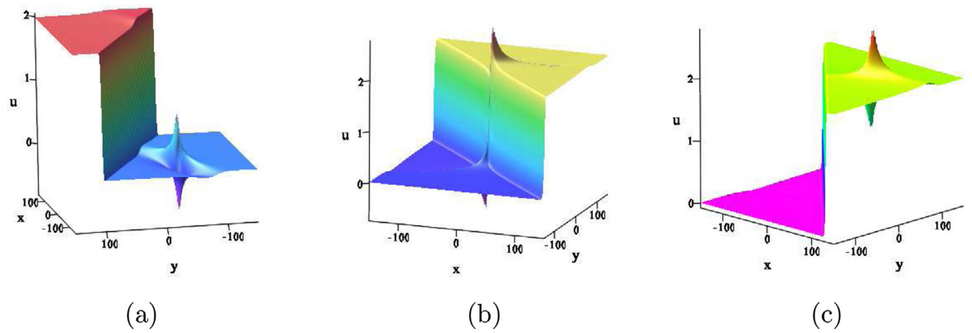


Fig. 12. Evolution profile of the hybrid solution u at different time: (a) $t = -2$; (b) $t = 0$; (c) $t = 2$.

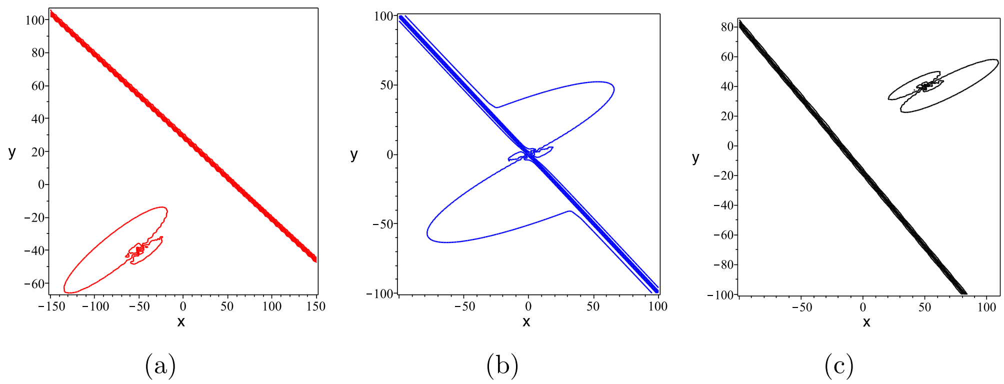


Fig. 13. Contour profiles of the hybrid solution u at different time: (a) $t = -2$; (b) $t = 0$; (c) $t = 2$.

Then, plugging (33) into f_3 and taking "long-wave" limit of $k_1, k_2 \to 0$ yields the hybrid solution of the bilinear BKP Eq. (3) as follows:

$$f_3 = c_0(1 + e^{\xi_3})\theta_1\theta_2 + (c_1\theta_1 + c_2\theta_2 + c_3)e^{\xi_3} + c_4, \tag{34}$$

where c_i's are complex and complicated polynomials of p_i's and k_3. Therefore, we omit their detailed expressions herein. And θ_i's are defined in (19). Then, the hybrid solution of the BKP Eq. (1) is derived by using the logarithmic transformation (4). Fig. 12 can illustrate the dynamic properties of this hybrid solution with specific parameters.

The parameters taken in the above-mentioned graphs are: $p_2 = p_1^* = -2 - I$; $k_3 = 1$, $p_3 = 2$. Investigating Figs. 12 and 13, we can readily find that they travel along the opposite directions, and then they collide with each other around the moment $t = 0$. During the collisions, the lump wave is swallowed by the line soliton. After the collision, they keep propagating forward changing the velocity and amplitude, only are the phases changed. Thus, we conclude that this collision is elastic.

5.2. The hybrid solution between 1st-breather and 1-soliton solutions

We will derive the hybrid solution between the 1-order breather and 1-soliton waves. To achieve this, we have to take the parameters as follows:

$$\begin{aligned} &N = 2M = 3;\ \xi_i^0 = 0, (i = 1, 2, 3);\\ &k_2 = k_1^* = \alpha_1 - I\gamma_1, p_2 = p_1^* = \beta_1 - I\delta_1;\\ &k_3 = \alpha_3, p_3 = \beta_3. \end{aligned} \tag{35}$$

Substituting (35) into f_3 leads to the second hybrid solution of the bilinear BKP Eq. (3). The second hybrid solution of the BKP Eq. (1) can be obtained by taking the logarithmic transformation (4). Since the expression is so complicated, we omit them herein. But the dynamic properties can be displayed by the following figures with specific parameters: $\alpha_1 = 2/3$, $\gamma_1 = 0$, $\beta_1 = 0$, $\delta_1 = 3/2$, $k_3 = 1$, $p_3 = 1$.

From Figs. 14 and 15, we found that the 1st-order breather in the hybrid solution is y-periodic and localized in x-axis, and it is parallel to the line soliton. In addition, we find that the breather travels along the positive direction of x-axis, but the 1-soliton propagates along the negative direction of x-axis. Then, they collide with each other. The breather

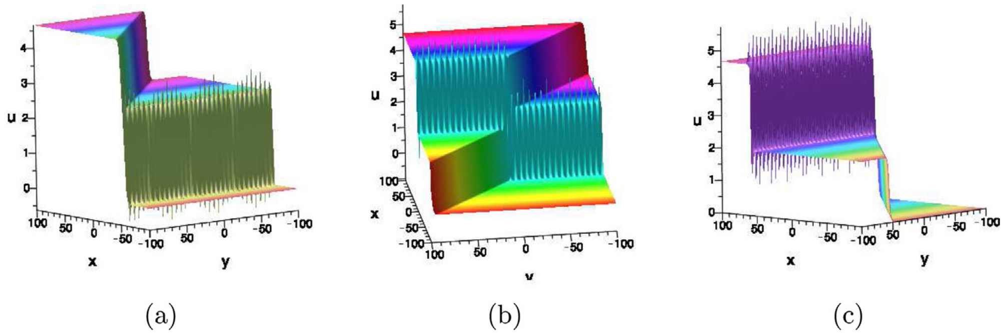


Fig. 14. Evolution profile of the hybrid solution u at different time: (a) $t = -2$; (b) $t = 0$; (c) $t = 2$.

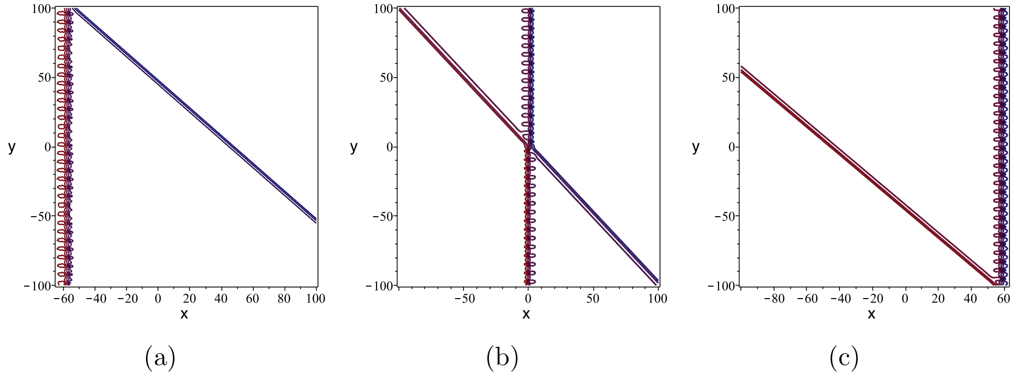


Fig. 15. Contour profiles of the hybrid solution u at different time: (a) $t = -2$; (b) $t = 0$; (c) $t = 2$.

wave moves much faster than the 1-soliton. After the collision, they keep the amplitudes, velocity, shape and energies unchanging while propagating . Thus, we conclude that the collision is elastic.

6. Conclusions

The (2+1)-dimensional BKP Eq. (1) is an integrable model of great importance, and has been applied in many research fields. For this work, by using the Hirota direct method, multiple soliton solutions are firstly obtained. Secondly, by taking "long wave" limit to the corresponding multiple soliton solutions, the permanent M-lump solutions are obtained. Then, we impose some conditions, such as the conjugation condition, to the related parameters in the multiple soliton solutions, then the higher-order breathers are readily derived. For the aforementioned discussions, the multiplicities of solitons are even. However, if they are taken as odd numbers, some important and localized hybrid solutions can be constructed. Some numerical simulations of the dynamic properties of the above solutions are performed. We conclude that there exist collisions among M-lump waves, higher-order breather waves and hybrid solutions. Especially, since the dynamic behaviors of all the solutions usually are affected by the parameters, then they have richer dynamic behaviors. As of the hybrid solutions, we only discussed: the interaction solution between 1-lump and 1-soliton solutions, and the interaction solution between 1st-order breather and a soliton solution. The numerical simulations show that the lump wave or breather wave are swallowed by the soliton wave during the collisions. Particularly, these collisions are elastic, which implies that there is no loss of energy.

For the future work. Firstly, we will study more complicated hybrid solutions, such as the interactions between M-lump waves or higher-order breather waves and stripe soliton waves since these sort of interaction solutions are of importance and worth studying. Secondly, we will apply the obtained results and method to study nonlinear physical models and other localized waves.

CRediT authorship contribution statement

Yong-Li Sun: Conceptualization, Methodology, Formal analysis, Investigation, Supervision. **Jing Chen:** Data curation, Writing – original draft, Software, Visualization. **Wen-Xiu Ma:** Validation, Resources, Writing – review & editing. **Jian-Ping Yu:** Methodology, Formal analysis, Investigation, Writing – review & editing. **Chaudry Masood Khalique:** Validation, Investigation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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