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Localized waves solutions for the fifth-order coupled extended modified KdV equation

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ABSTRACT

The fifth-order coupled extended modified Korteweg–de-Vries (KdV) equation is studied. Based on seed solutions and Lax pairs, the Nth-order iterative expression of the localized wave solutions of the equation are obtained by the generalized Darboux transformation. Then, through numerical simulation, the evolution plots of the interaction of rogue waves with dark–bright solitons and the Kuznetsov–Ma breathers are derived. The results demonstrate the abundant dynamical patterns of localized waves in the fifth-order coupled systems.

1. Introduction

In recent years, the nonlinear Schrödinger equation has been used by scholars to describe naturally occurring nonlinear phenomena [1–6]; these descriptions have an extremely wide range of application in important disciplines such as optics [2], fluid mechanics [3], biomedicine [4], and oceanography [5]. Considering the increasing understanding of the complexity of natural phenomena, scholars increasingly sought integrable generalizable models of nonlinear Schrödinger equations [6]. The research developed, for example, the Kadomtsev–Petviashvili (KP) equations [7], the Kundu–Eckhaus (KE) equations [8,9], the Gerdjikov–Ivanov (GI) equations [10], and the Korteweg–de-Vries (KdV) equations [11–13]. Further, these integrable generalizable equations can be used to study localized waves [14], which comprise solitons [15], breathers [16], and rogue waves [17]. There are many methods for investigating localized wave solutions, such as the Darboux transformation [18,19], the Bäcklund transformation [20], the inverse scattering transformation [21], the Hirota bilinear method [22], and the Riemann–Hilbert method [23]. The study of localized wave solutions of nonlinear integrable equations is the theoretical basis for understanding practical problems in the objective world.

This paper mainly considers the fifth-order coupled extended modified KdV equations,

$$u_1^2(30u_1^{*2}u_{1,x} + 20u_1^*u_2^*u_{2,x}) + u_1(40|u_2|^2u_1^*u_{1,x} + 10(u_{1,x}u_{1,xx}^* + u_{1,x}^*u_{1,xx}$$
$$+u_1^*u_{1,xxx}) + 20|u_2|^2u_2^*u_{2,x} + 5(u_2^*u_{2,xxx} + u_{2,x}u_{2,xx}^* + u_{2,x}^*u_{2,xx})) + u_{1,x}(10$$
$$(u_{1,x}u_{1,x}^* + |u_2|^4 + u_2^*u_{2,xx} + u_{2,x}u_{2,x}^*) + 20u_1^*u_{1,xx} + 5u_2u_{2,xx}^*) + 5(|u_2|^2u_{1,xxx}$$

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$$
\begin{aligned}
&+u_2u_{2,x}^*u_{1,xx})+10u_2^*u_{1,xx}u_{2,x}-u_{1,t}+u_{1,xxxxx}=0,\\
&u_2^2(30u_2^{*2}u_{2,x}+20u_1^*u_2^*u_{1,x})+u_2(40|u_1|^2u_2^*u_{2,x}+10(u_{2,x}u_{2,xx}^*+u_{2,x}^*u_{2,xx}\\
&+u_2^*u_{2,xxx})+20|u_1|^2u_1^*u_{1,x}+5(u_1^*u_{1,xxx}+u_{1,x}u_{1,xx}^*+u_{1,x}^*u_{1,xx}))+u_{2,x}(10\\
&(u_{2,x}u_{2,x}^*+|u_1|^4+u_1^*u_{1,xx}+u_{1,x}u_{1,x}^*)+20u_2^*u_{2,xx}+5u_1u_{1,xx}^*)+5(|u_1|^2u_{2,xxx}\\
&+u_1u_{1,x}^*u_{2,xx})+10u_1^*u_{2,xx}u_{1,x}-u_{2,t}+u_{2,xxxxx}=0,
\end{aligned}
\tag{1}
$$

In Eq. (1), x and t are distance and time variables, respectively, and the symbol $*$ denotes complex conjugate. $u_1(x,t)$ and $u_2(x,t)$ are potential functions with zero boundary condition.

Some research results have been obtained for Eq. (1). Using Riemann–Hilbert method, Fan and Lin [6] obtained the N-soliton solutions of Eq. (1), and the structure diagram of the solitons was drawn by selecting appropriate parameters. Huang et al. [24] gave the Lax pair corresponding to Eq. (1) at $u_1=u_2$, and the first-order to the fourth-order rogue waves were displayed through the Darboux transformation; and the influence of parameters on the rogue waves' behaviors was discussed. At $u_1=u_2$, Huang et al. [25] studied the breather solutions of Eq. (1) in a non-zero background using the Darboux transformation. Liu [26] reported soliton and breather solutions to Eq. (1) at $u_1=u_2$ by the Riemann–Hilbert method in a non-zero constants context. Ren and Lin [27] generated the interaction between one-soliton and cnoidal waves of Eq. (1) by the consistent Riccati expansion (CRE) method. Wang and Zhang [28] derived the rational solutions from the first to the second orders via the generalized Darboux transformation of Eq. (1) at $u_1=u_2$; doubly periodic lattice-like and doubly localized high-peak waves were shown. At present, there are few studies on the dynamics of the higher-order localized waves of Eq. (1). Therefore, this paper will display the third-order localized waves through the generalized Darboux transformation, and describe the dynamical characteristics of localized waves via numerical simulation.

In Section 2, the generalized Darboux transformation is derived and an iterative formula for the Nth-order localized wave solutions is given. In Section 3, the evolution plots of third-order localized waves is obtained via numerical simulation, and their dynamical characteristics are analyzed. Finally, in Section 4, results are discussed and several conclusions are provided.

2. Generalized Darboux transformation

The Lax pair corresponding to Eq. (1) is

$$\Phi_x=U(\lambda)\Phi, \tag{2a}$$

$$\Phi_t=V(\lambda)\Phi, \tag{2b}$$

where

$$
\begin{aligned}
&U(\lambda)=\frac{1}{2}i\lambda J+iJP, V(\lambda)=\sum_{j=0}^{5}\lambda^jV_j,\\
&P=\begin{pmatrix}0&u_1&u_2\\-u_1^*&0&0\\-u_2^*&0&0\end{pmatrix}, J=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix},\\
&V_5=\frac{i}{2}J, V_4=iJP, V_3=P_x+iJP^2, V_2=P_xP-PP_x+iJ(2P^3-P_{xx}),\\
&V_1=-P_{xxx}+3(P_xP^2+P^2P_x)+iJ(3P^4-PP_{xx}-P_{xx}P+P_x{}^2),\\
&V_0=(PP_{xxx}-P_{xxx}P+P_{xx}P_x-P_xP_{xx}+4(P_xP^3-P^3P_x)+2(P^2P_xP-PP_xP^2))\\
&+iJ(P_{xxxx}-4(P_{xx}P^2+P^2P_{xx})-2(P_x{}^2P+PP_x{}^2+PP_{xx}P)+6(P^5-P_xPP_x)).
\end{aligned}
$$

The Darboux matrix T is constructed as

$$T=\lambda I-H\Lambda H^{-1}, \tag{3}$$

where

$$H=\begin{pmatrix}\phi_1&\varphi_1^*&\chi_1^*\\\varphi_1&-\phi_1^*&0\\\chi_1&0&-\phi_1^*\end{pmatrix}, \Lambda=\begin{pmatrix}\lambda_1&0&0\\0&\lambda_1^*&0\\0&0&\lambda_1^*\end{pmatrix}.$$

$\Phi_k=\left(\phi_k,\varphi_k,\chi_k\right)^T$ is the eigenfunction of Eq. (2) corresponding to the spectral parameters $\lambda=\lambda_1$ and seed solutions $u_1=u_1[0]$ and $u_2=u_2[0]$.

Assuming $\Phi_1=\Phi_1\left(\lambda_1,\eta\right)$ is a solution of Eq. (2) and η is a small parameter, the Taylor expansion of Φ_1 at $\eta=0$ is obtained:

$$\Phi_1=\Phi_1^{[0]}+\Phi_1^{[1]}\eta+\Phi_1^{[2]}\eta^2+\cdots+\Phi_1^{[N]}\eta^N+o\left(\eta^N\right), \tag{4}$$

where

$$\Phi_1^{[k]}=\frac{1}{k!}\frac{\partial^k}{\partial\lambda^k}\Phi_1(\lambda)\Big|_{\lambda=\lambda_1}=\left(\phi_1^{[k]},\varphi_1^{[k]},\chi_1^{[k]}\right)^T,(k=0,1,2,\ldots,N).$$

It is easy to verify that $\Phi_1^{[0]} = \Phi_1[0]$ is a special solution of Eq. (2) with $\lambda = \lambda_1$, $u_1 = u_1[0]$ and $u_2 = u_2[0]$. Therefore, the generalized Darboux transformation of the Nth-order equation is defined as

$$\begin{aligned}\Phi_1[N-1] &= \Phi_1^{[0]} + \left[\sum_{l=1}^{N-1} T_1[l]\right]\Phi_1^{[1]} + \left[\sum_{l=1}^{N-1}\sum_{h>l}^{N-1} T_1[h]T_1[l]\right]\Phi_1^{[2]} \\ &\quad + \cdots + \left[T_1[N-1]\cdots T_1[2]T_1[1]\right]\Phi_1^{[N-1]},\end{aligned} \tag{5}$$

$$u_1[N] = u_1[N-1] + \left(\lambda_1 - \lambda_1^*\right)\frac{\phi_k[k-1]\varphi_k^*[k-1]}{\left|\phi_k[k-1]\right|^2 + \left|\varphi_k[k-1]\right|^2 + \left|\chi_k[k-1]\right|^2}, \tag{6a}$$

$$u_2[N] = u_2[N-1] + \left(\lambda_1 - \lambda_1^*\right)\frac{\phi_k[k-1]\chi_k^*[k-1]}{\left|\phi_k[k-1]\right|^2 + \left|\varphi_k[k-1]\right|^2 + \left|\chi_k[k-1]\right|^2}. \tag{6b}$$

3. Dynamics of the third-order localized waves

Assume $u_1[0] = d_1 e^{i\theta}$ and $u_2[0] = d_2 e^{i\theta}$ are seed solutions of the localized waves, where

$$\theta = \mu x + \omega t,$$

$$\omega = -\mu^5 + 30\mu^3 {d_1}^4 - 20\mu^3 {d_1}^2 + 60\mu {d_1}^2 {d_2}^2 - 20\mu^3 {d_2}^2 + 30\mu {d_2}^4,$$

μ, d_1 and d_2 are arbitrary real constants. The corresponding basic vector solution at $\lambda = \left(\mu + 2i\sqrt{{d_1}^2 + {d_2}^2}\right)(1+\eta^2)$ is

$$\Phi_1(\eta) = \begin{pmatrix} \left(C_1 e^{\kappa_1+\kappa_2} - C_2 e^{\kappa_1-\kappa_2}\right) e^{\frac{i\theta}{2}} \\ \rho_1\left(C_1 e^{\kappa_1-\kappa_2} - C_2 e^{\kappa_1+\kappa_2}\right) e^{-\frac{i\theta}{2}} + \alpha d_2 e^{\kappa_3} \\ \rho_2\left(C_1 e^{\kappa_1-\kappa_2} - C_2 e^{\kappa_1+\kappa_2}\right) e^{\frac{i\theta}{2}} - \alpha d_1 e^{\kappa_3} \end{pmatrix}, \tag{7}$$

where

$$C_1 = \frac{\sqrt{\lambda - \mu + \sqrt{(\lambda-\mu)^2 + 4d_1^2 + 4d_2^2}}}{\sqrt{(\lambda-\mu)^2 + 4d_1^2 + 4d_2^2}}, C_2 = \frac{\sqrt{\lambda - \mu - \sqrt{(\lambda-\mu)^2 + 4d_1^2 + 4d_2^2}}}{\sqrt{(\lambda-\mu)^2 + 4d_1^2 + 4d_2^2}},$$

$$\kappa_1 = i\mu^5 t, \kappa_2 = \frac{1}{2}i\sqrt{(\lambda-\mu)^2 + 4d_1^2 + 4d_2^2}\,(x + ht + \Omega(\eta)), \kappa_3 = -\frac{1}{2}i\lambda x - \frac{1}{2}i\lambda^5 t,$$

$$h = \lambda^4 + \mu\lambda^3 + (\mu^2 - 2d_1^2 - 2d_2^2)\lambda^2 + \mu(\mu^2 - 6d_1^2 - 6d_2^2)\lambda + \mu^4 + \mu^2(-12d_1^2 - 12d_2^2) + 6\left(d_1^2 + d_2^2\right)^2,$$

$$\rho_1 = \frac{id_1}{\sqrt{d_1^2 + d_2^2}}, \rho_2 = \frac{id_2}{\sqrt{d_1^2 + d_2^2}},$$

$$\Omega(\eta) = \sum_{j=1}^{N}\left(m_j + in_j\right)\eta^{2j}, \left(m_j, n_j \in R\right),$$

α, m_j and n_j are arbitrary real constants. Expand $\Phi_1(\eta)$ at $\eta = 0$ using the Taylor series,

$$\Phi_1(\eta) = \Phi_1^{[0]} + \Phi_1^{[1]}\eta^2 + \Phi_1^{[2]}\eta^4 + \Phi_1^{[3]}\eta^6 + \cdots, \tag{8}$$

where

$$\Phi_1(\eta) = \left(\phi_1^{[k]}, \varphi_1^{[k]}, \chi_1^{[k]}\right)^T = \frac{1}{(2k)!}\frac{\partial^{2k}\Phi_1}{\partial\eta^{2k}}\Big|_{\eta=0}\ (k = 0, 1, 2, \ldots).$$

The expression $\Phi_1^{[j]} = \left(\phi_1^{[j]}, \varphi_1^{[j]}, \chi_1^{[j]}\right)^T (j = 1, 2)$ is complicated, its specific form is omitted. Fan et al. [6] studied the dynamical behaviors of the first-order localized waves in Eq. (1), thus, it is not repeated here. Rather, the dynamics of the third-order localized waves in Eq. (1) are of primary interest.

Based on the following limit formula,

$$\begin{aligned}\Phi_1[2] &= \lim_{\eta\to 0}\frac{T[2]\big|_{\lambda=\lambda_1(1+\eta^2)}T[1]\big|_{\lambda=\lambda_1(1+\eta^2)}\Phi_1}{\eta^4} \\ &= \lim_{\eta\to 0}\frac{\left(\lambda_1\eta^2 + T_1[2]\big|_{\lambda=\lambda_1}\right)\left(\lambda_1\eta^2 + T_1[1]\big|_{\lambda=\lambda_1}\right)\Phi_1}{\eta^4} \\ &= \lambda_1^2\Phi_1^{[0]} + \lambda_1\left(T_1[2] + T_1[1]\right)\Phi_1^{[1]} + \left(T_1[2]T_1[1]\right)\Phi_1^{[2]},\end{aligned} \tag{9}$$

and Eqs. (5) and (6), the third-order localized wave solutions can be obtained,

$$u_1[3] = u_1[2] + \left(\lambda_1 - \lambda_1^*\right) \frac{\phi_1[2]\varphi_1^*[2]}{|\phi_1[2]|^2 + |\varphi_1[2]|^2 + |\chi_1[2]|^2}, \tag{10a}$$

$$u_2[3] = u_2[2] + \left(\lambda_1 - \lambda_1^*\right) \frac{\phi_1[2]\chi_1^*[2]}{|\phi_1[2]|^2 + |\varphi_1[2]|^2 + |\chi_1[2]|^2}, \tag{10b}$$

where

$$\Phi_1[2] = \left(\phi_1^{[2]}, \varphi_1^{[2]}, \chi_1^{[2]}\right),$$

$$T_1[1] = \lambda_1 I - H_1[0]\Lambda_1 H_1[0]^{-1}, T_1[2] = \lambda_1 I - H_1[1]\Lambda_1 H_1[1]^{-1},$$

$$H_1[0] = \begin{pmatrix} \phi_1[0] & \varphi_1^*[0] & \chi_1^*[0] \\ \varphi_1[0] & -\phi_1^*[0] & 0 \\ \chi_1[0] & 0 & -\phi_1^*[0] \end{pmatrix}, H_1[1] = \begin{pmatrix} \phi_1[1] & \varphi_1^*[1] & \chi_1^*[1] \\ \varphi_1[1] & -\phi_1^*[1] & 0 \\ \chi_1[1] & 0 & -\phi_1^*[1] \end{pmatrix}, \Lambda_1 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1^* & 0 \\ 0 & 0 & \lambda_1^* \end{pmatrix}.$$

The specific expression of $u_1[3]$ and $u_2[3]$ involves eight free parameters $\mu, d_1, d_2, \alpha, m_1, n_1, m_2, n_2$ and different third-order localized waves evolution plots are obtained by varying the values of the free parameters. Then the dynamical characteristics of localized waves in different cases are discussed.

(1) If $d_1 \neq 0$ and $d_2 = 0$, there are two cases.

When parameters $m_1 = 0, n_1 = 0, m_2 = 0$ and $n_2 = 0$, it is the interaction between the third-order rogue waves and three dark–bright solitons. Under the influence of zero-amplitude background, the third-order rogue waves in the component $u_2[3]$ are difficult to identify. If $\mu = \frac{1}{5}$, the propagation direction of three solitons are parallel to the t-axis in Fig. 1. If other parameters remain unchanged and $\mu = \frac{1}{3}$, the propagation direction of bright–dark solitons changes, and the angle between the propagation direction of three-dark–bright solitons and the positive direction of t-axis is an obtuse angle, as shown in Fig. 2.

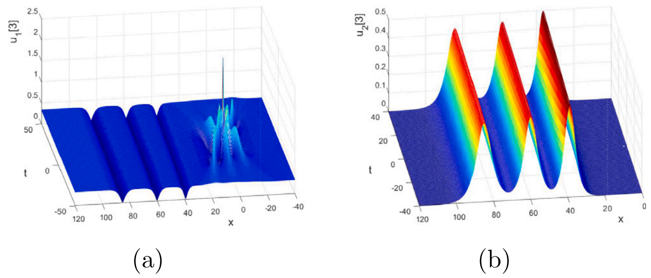


Fig. 1. The third-order localized waves with $d_1 = \frac{1}{3}, d_2 = 0, \mu = \frac{1}{5}, \alpha = \frac{1}{100000}, m_1 = 0, n_1 = 0, m_2 = 0, n_2 = 0$.

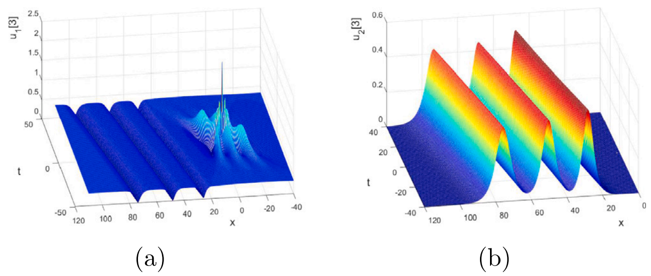
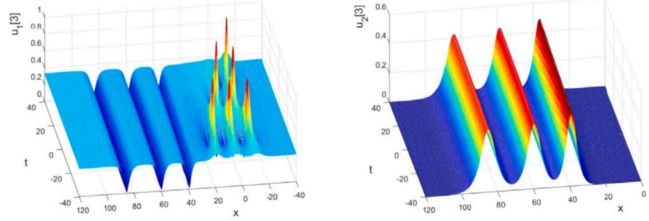


Fig. 2. The third-order localized waves with parameters the same as in Fig. 1 except for $\mu = \frac{1}{3}$.

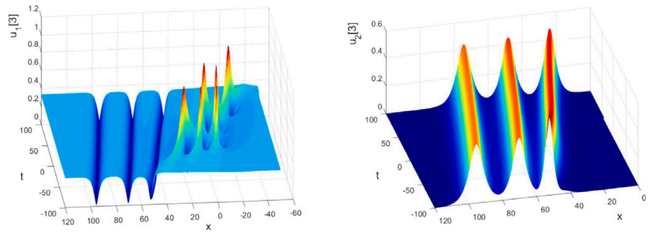
When $m_1 \neq 0, n_1 \neq 0, m_2 \neq 0$ and $n_2 \neq 0$, due to the separation function, the third-order rogue waves in the component $u_1[3]$ are separated into six first-order rogue waves in Fig. 3. Similarly, it is difficult to observe rogue waves in the component $u_2[3]$ at the zero-amplitude background. If other parameters remain unchanged and $\mu = \frac{1}{8}$, the angle between the propagation direction of bright–dark solitons and the positive direction of t-axis is an acute angle, as seen in Fig. 4.

(2)If $d_1 \neq 0$ and $d_2 \neq 0$, there are two cases.



(a) (b)

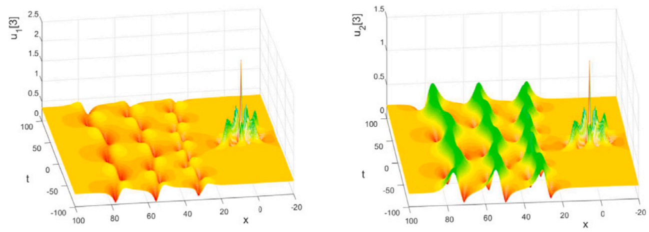
Fig. 3. The third-order localized waves with $d_1 = \frac{1}{3}, d_2 = 0, \mu = \frac{1}{5}, \alpha = \frac{1}{100000}, m_1 = 20, n_1 = 20, m_2 = 30, n_2 = 30$.



(a) (b)

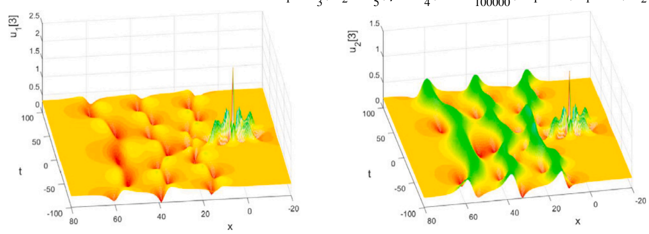
Fig. 4. The third-order localized waves with parameters the same as in Fig. 3 except for $\mu = \frac{1}{8}$.

When $m_1 = 0, n_1 = 0, m_2 = 0$ and $n_2 = 0$, the third-order rogue waves interact with three breathers (Kuznetsov–Ma breathers), and the propagation direction of breathers is parallel with the positive direction of t-axis. The third-order rogue waves suddenly appear from nowhere at time $t = 0$ in Fig. 5. When $d_1 > d_2$, the breather amplitude of the component $u_2[3]$ is higher than the one in component $u_1[3]$. If other parameters remain unchanged, the third-order rogue waves merge with three Kuznetsov–Ma breathers by increasing the value of α in Fig. 6.



(a) (b)

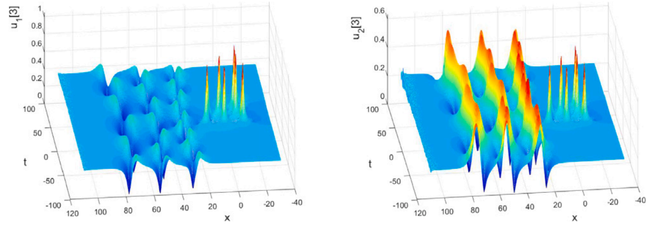
Fig. 5. The third-order localized waves with $d_1 = \frac{1}{3}, d_2 = \frac{1}{5}, \mu = \frac{1}{4}, \alpha = \frac{1}{100000}, m_1 = 0, n_1 = 0, m_2 = 0, n_2 = 0$.



(a) (b)

Fig. 6. The third-order localized waves with parameters the same as in Fig. 5 except for $\alpha = \frac{1}{200}$.

When $m_1 \neq 0, n_1 \neq 0, m_2 \neq 0$ and $n_2 \neq 0$, the third-order rogue waves interact with the third-order Kuznetsov–Ma breathers. Under the influence of the separation function $\Omega(\eta)$, the third-order rogue waves are separated into six first-order rogue waves, and arrange themselves in triangles as shown in Fig. 7.



(a) (b)

Fig. 7. The third-order localized waves with $d_1 = \frac{1}{3}, d_2 = \frac{1}{5}, \mu = \frac{1}{4}, \alpha = \frac{1}{100000}, m_1 = 30, n_1 = 30, m_2 = 40, n_2 = 40.$

4. Conclusions

In this paper, the localized wave solutions and the dynamical characteristics of the fifth-order coupled extended modified KdV equation are studied. The Nth-order generalized Darboux transformation is derived by using the Taylor expansion and the limit formula, and the third-order localized wave solutions of the equation are obtained. Among them, there are several free parameters $\mu, d_1, d_2, \alpha, m_1, n_1, m_2, n_2$ that have an important influence on the dynamical characteristics of localized waves. The parameter μ plays an important role in the propagation direction of three dark–bright solitons and Kuznetsov–Ma breathers, along the positive direction of the t-axis. Parameters d_1, d_2 affect the type of localized waves. If $d_1 \neq 0$ and $d_2 = 0$, the effect is the interaction between three dark–bright solitons and the third-order rogue waves; If $d_1 \neq 0$ and $d_2 \neq 0$, the effect is the interaction between the third-order Kuznetsov–Ma breathers and the third-order rogue waves. The parameter α determines the space of the two localized waves. As the parameter α increases, the dark–bright solitons or Kuznetsov–Ma breathers gradually fuse with the rogue waves. The parameters $m_i\,(i = 1, 2)$ and $n_i\,(i = 1, 2)$ lead to the separation of the third-order rogue waves. When $m_i \neq 0\,(i = 1, 2)$ and $n_i \neq 0\,(i = 1, 2)$, the third-order rogue waves can be separated into six first-order rogue waves. These results and the methods used contribute to the understanding of the dynamical characteristics of the third-order localized waves, contribute significantly to the body of knowledge concerning solutions of the basic equations, and provide a firm basis for future investigation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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References

[1] A. Takafumi, I. Slim, K. Hiroaki, Global dynamics above the ground state energy for the combined power-type nonlinear Schrödinger equations with energy-critical growth at low frequencies, Mem. Amer. Math. Soc. 272 (1331) (2021).
[2] A.H. Kara, B. Alqurashi, Invariance and approximate conservation laws of some nonlinear Schrödinger equation with PT-symmetric potentials and inhomogeneous nonlinearity, J. Anal. Appl. 17 (2) (2019).
[3] M. Thilagarajah, Optical solitons and stability analysis for the new (3+1)-dimensional nonlinear Schrödinger equation, J. Nonlinear Opt. Phys. 32 (02) (2023).
[4] Y.L. Yang, T. Wang, H. Guo, Existence of sign-changing solutions for a gauged nonlinear Schrödinger equation with a quintic term, J. Math. Anal. Appl. 520 (1) (2023).
[5] H.F. Wang, Y.F. Zhang, Application of Riemann-Hilbert method to an extended coupled nonlinear Schrödinger equations, J. Comput. Appl. Math. 420 (2023).
[6] W. Fan, H. Lin, N-soliton solutions for the coupled extended modified KdV equations via Riemann-Hilbert approach, Appl. Math. Lett. 134 (2022).
[7] W.X. Chen, R. Guan, L.X. Tian, Interaction phenomenon and breather wave to the extend (3+1)-dimensional Kadomtsev–Petviashvili equation, J. Math. Anal. Appl. 518 (2023).
[8] E. Parasuraman, Effect of inter modal dispersion on modulational instability of optical soliton in Kundu-Eckhaus equation with the presence of SPM and XPM, Optik 270 (2022).
[9] Y.H. Deng, X.H. Meng, G.M. Yue, Soliton solutions for the nonlocal reverse space Kundu-Eckhaus equation via symbolic calculation, Optik 252 (2022).
[10] D.N. Yang, Bright and dark solitons and periodic wave solutions for the Gerdjikov-Ivanov equation in nonlinear fiber optics, J. Math. Anal. Appl. 519 (1) (2023).

[11] Z. Zhang, X.Y. Yang, B. Li, Soliton molecules and novel smooth positons for the complex modified KdV equation, Appl. Math. Lett. 103 (2020) 106168.
[12] W.X. Ma, Riemann-Hilbert problems and N-soliton solutions for a coupled mKdV system, J. Geom. Phys. 132 (2018) 45–54.
[13] Z. Zhang, X.Y. Yang, B. Li, Novel soliton molecules and breather-positon on zero background for the complex modified KdV equation, Nonlinear Dyn. Int. J. Nonlinear Dyn. Chaos Eng. Syst. 100 (2) (2020) 1551–1557.
[14] X.Y. Li, G.F. Han, Q.L. Zhao, Interactions of localized wave and dynamics analysis in generalized derivative nonlinear Schrödinger equation, Commun. Nonlinear Sci. Numer. Simul. 114 (2022).
[15] R. Fan, Z. Zhang, Multisoliton solutions with even numbers and its generated solutions for nonlocal Fokas-Lenells equation, Commun. Theor. Phys. 72 (12) (2020) 125007.
[16] N. Liu, Soliton and breather solutions for a fifth-order modified KdV equation with a nonzero background, Appl. Math. Lett. 104 (2020) 106256.
[17] L.H. Wang, J.S. He, H. Xu, Generation of higher-order rogue waves from multi-breathers by double degeneracy in an optical fiber, Phys. Rev. E 95 (41) (2017) 042217.
[18] D.Y. Liu, H.M. Yu, Mixed localized wave solutions of the Hirota equation, Appl. Math. Lett. 118 (2021) 107154.
[19] D.Y. Yang, B. Tian, Q.X. Qu, et al., Lax pair, breathers and modulation instability of a three-component coupled derivative nonlinear Schrödinger system for a plasma, Eur. Phys. J. Plus 137 (2) (2022).
[20] M.H.M. Moussa, Z.M. Al-Halim, Painlev analysis, Bäcklund transformation and exact solutions for the (3+1)-dimensional nonlinear partial differential equation represented by Burgers equation, Ex. Counter. 2 (2022).
[21] Y. Zhang, H.H. Dong, Robust inverse scattering method to the complex modified Korteweg-de-Vries equation with nonzero background condition, Phys. Lett. A 449 (2022).
[22] S. Wael, S.M. Maowad, O.H. El Kalaawy, Conservation laws and exact solutions for coupled Schrödinger-KdV dynamical models arising in plasma, Pramana 96 (4) (2022).
[23] H.F. Wang, Y.F. Zhang, Application of Riemann-Hilbert method to an extended coupled nonlinear Schrödinger equations, J. Comput. Appl. Math. 420 (2023).
[24] L. Huang, N.N. Lv, Soliton molecules, rational positons and rogue waves for the extended complex modified KdV equation, Nonlinear Dynam. (2021).
[25] N.N. Lv, L. Huang, Breather-soliton molecules and breather-positons for the extended complex modified KdV equation, Commun. Nonlinear Sci. Numer. Simul. 107 (2022).
[26] N. Liu, Soliton and breather solutions for a fifth-order modified KdV equation with a nonzero background, Appl. Math. Lett. 104 (2020) 106256.
[27] B. Ren, J. Lin, P. Liu, Soliton molecules and the CRE method in the extended mKdV equation, Commun. Theor. Phys. 72 (5) (2020) 055005.
[28] X. Wang, J.L. Zhang, L. Wang, Conservation laws, periodic and rational solutions for an extended modified Korteweg-de-Vries equation, Nonlinear Dynam. 92 (4) (2018) 1507–1516.