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Study of travelling wave solutions for some special-type nonlinear evolution equations

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Abstract

The tanh-function expansion method has been improved and used to construct travelling wave solutions of the form $U = \sum_{j=0}^{n} a_j \tanh^j \xi$ for some special-type nonlinear evolution equations, which have a variety of physical applications. The positive integer n can be determined by balancing the highest order linear term with the nonlinear term in the evolution equations. We improve the tanh-function expansion method with $n = 0$ by introducing a new transform $U = -W'(\xi)/W^2$. A nonlinear wave equation with source terms, and mKdV-type equations, are considered in order to show the effectiveness of the improved scheme. We also propose the tanh-function expansion method of implicit function form, and apply it to a Harry Dym-type equation as an example.

Keywords: travelling wave solution, nonlinear evolution equation, function expansion method, symbolic computation

1. Introduction

It is very important in mathematical physics to seek exact solutions of nonlinear evolution equations (NLEEs). For example, the bell-shaped sech and kink-shaped tanh solutions can be used to analyse and model the wave phenomena observed in fluid dynamics and in plasma and elastic media. There are several approaches to constructing exact solutions, including Lie symmetry group reduction, the inverse scattering, Darboux and Bäcklund transformations, and the Hirota bilinear operator, homogeneous balance and tanh-function expansion methods, which are all incorporated into the transformed rational function method. The tanh-function expansion method is a feasible and particularly powerful algorithm, because one can write solitary travelling wave solutions of a given NLEE as polynomials of hyperbolic functions and transform the evolution equation under consideration into a system of algebraic equations to solve. Several extensions of the tanh-function expansion method have also been proposed for solving various kinds of NLEEs. These expansion methods include the Jacobi elliptic function, exponential function, Riccati equation, tanh–coth and G'/G expansion methods [1–21]. Many well-known NLEEs have been solved using these methods, such as the KdV, mKdV, Camassa–Holm and the Kaup–Kupershmidt-type equations, and the Jaulent–Miodek, (2 + 1)-dimensional dispersive long wave and (2 + 1)-dimensional Hirota systems. In addition, some special-type NLEEs, including the double sine-Gordon equation, the coupled Schrödinger−KdV system and the (2 + 1)-dimensional Davey–Stewartson equation, require some kinds of pre-possessing techniques because such equations cannot be directly dealt with by these methods.

Let us recall how the tanh-function expansion method works for a given NLEE

$$H(u, u_t, u_x, u_{xx}, \cdots) = 0. \quad (1.1)$$

By means of a travelling wave reduction

$$u(x, t) = U(\xi), \quad \xi = x + ct, \tag{1.2}$$

we can transform (1.1) into the following ordinary differential equation (ODE)

$$H(U, U', U'', \cdots) = 0. \tag{1.3}$$

The next crucial step is that the ansatz of exact solutions is expressed as a polynomial in the tanh function,

$$U(\xi) = \sum_{j=0}^{n} a_j \tanh^j \xi \tag{1.4}$$

because a derivative of $\tanh\xi$ is a polynomial in $\tanh\xi$, e.g., $(\tanh\xi)' = 1 - \tanh^2\xi$. The positive integer n in the expansion can be determined by balancing the highest order linear term with the nonlinear term in the equation under consideration. By substituting (1.4) into (1.3) and equating all coefficients of the powers of $\tanh\xi$ to zero, we can obtain a system of algebraic equations on the parameters, wave numbers and frequencies. Thus, we can explicitly obtain the parameters a_j and c, with the help of symbolic computation systems such as Maple and Mathematica.

In order to construct diverse exact solutions, (1.4) can be modified to

$$U(\xi) = \sum_{j=0}^{n} a_j \varphi^j(\xi), \qquad \varphi'(\xi) = b + \varphi^2(\xi). \tag{1.5}$$

Considering that the Riccati equation $\varphi'(\xi) = b + \varphi^2(\xi)$ admits several types of exact solutions [21],

$$\varphi(\xi) = \begin{cases} -\sqrt{-b}\,\tanh(\sqrt{-b}\,\xi), \quad -\sqrt{-b}\,\coth(\sqrt{-b}\,\xi), & b < 0,\\ -\frac{1}{\xi}, & b = 0,\\ \sqrt{b}\,\tan(\sqrt{b}\,\xi), \quad -\sqrt{b}\,\cot(\sqrt{b}\,\xi), & b > 0, \end{cases} \tag{1.6}$$

we can then obtain various types of exact solution, and (1.4) is only a special case in (1.6). Of course, we can consider more generalized expansion forms [3–5, 8] for constructing more exact solutions, such as the Weierstrass and Jacobi doubly periodic wave solutions.

The paper is structured as follows. In section 2, we study a special-type case that has not been considered, namely $n = 0$ in (1.4) (or (1.5)), by balancing the highest order linear term with the nonlinear term. In section 3, we propose the tanh-function expansion method of implicit function form and apply it to a Harry Dym-type equation as an example. In section 4, we give a short summary.

2. The tanh-function expansion method with $n = 0$

In this section, we illustrate our approach for dealing with the case of $n = 0$ with two examples.

Example 1. We consider the nonlinear wave equation with source terms [22]

$$u_{tt} + 2u_x^2 - \frac{4}{3}uu_{xx} + \alpha u + \beta u^2 = 0. \tag{2.1}$$

Let $u(x, t) = U(\xi)$, $\xi = x + ct$, then (2.1) reduces to

$$c^2U'' + 2U'^2 - \frac{4}{3}UU'' + \alpha U + \beta U^2 = 0. \tag{2.2}$$

Balancing between U'' and UU'' yields $n = 0$, which is not a positive integer. So if we take $U = V'$, then (2.2) becomes

$$c^2V''' + 2V''^2 - \frac{4}{3}V'V''' + \alpha V' + \beta V'^2 = 0. \tag{2.3}$$

Furthermore, let $V = W^{-1}$, then (2.3) becomes

$$\begin{aligned} &3c^2W^2W''' - 18c^2WW'W'' + 18c^2W'^3 + 3\alpha W'W^2\\ &-3\beta W'^2 + 4W'W''' - 6W''^2 = 0. \end{aligned} \tag{2.4}$$

Now W^2W''' and W''^2 give the desired balancing number $n = 1$. In this case, we can assume that

$$W = a_0 + a_1\varphi, \qquad \varphi' = b + \varphi^2. \tag{2.5}$$

Substituting (2.5) into (2.4), we have

$$\begin{aligned} &6a_1^2bc^2 + 18a_0^2c^2 + 3a_1^2\alpha + 8a_1b - 3a_1\beta = 0,\\ &ba_0(4bc^2 - \alpha) = 0,\\ &24a_1^2b^2c^2 + 24a_0^2bc^2 + 3a_1^2b\alpha + 3a_0^2\alpha\\ &\quad + 16a_1b^2 - 6a_1b\beta = 0,\\ &18a_1^2b^2c^2 + 6a_0^2bc^2 + 3a_0^2\alpha + 8a_1b^2 - 3a_1b\beta = 0. \end{aligned} \tag{2.6}$$

This system has a solution

$$\begin{aligned} &4bc^2 - \alpha = 0,\\ &18a_1^2bc^2 + 18a_0^2c^2 + 8a_1b - 3\beta a_1 = 0. \end{aligned} \tag{2.7}$$

So we can get the following three cases which correspond to exact solutions $U = \frac{-W'(\xi)}{W^2}$, $\xi = x + ct$ of (2.1),

Case 1. $\alpha = 0$.

$$W = \pm\sqrt{\frac{\beta a_1}{6c^2}} - \frac{a_1}{\xi}. \tag{2.8}$$

Case 2. $\alpha < 0$.

$$\begin{aligned} W &= \pm\sqrt{\frac{3\beta a_1 - 8a_1b}{18c^2} - a_1^2b} - a_1\sqrt{\frac{-\alpha}{4c^2}}\tanh\left(\sqrt{\frac{-\alpha}{4c^2}}\xi\right);\\ W &= \pm\sqrt{\frac{3\beta a_1 - 8a_1b}{18c^2} - a_1^2b} - a_1\sqrt{\frac{-\alpha}{4c^2}}\coth\left(\sqrt{\frac{-\alpha}{4c^2}}\xi\right). \end{aligned} \tag{2.9}$$

Case 3. $\alpha > 0$.

$$W = \pm\sqrt{\frac{3\beta a_1 - 8a_1 b}{18c^2} - a_1^2 b} + a_1\sqrt{\frac{\alpha}{4c^2}}\tan\left(\sqrt{\frac{\alpha}{4c^2}}\xi\right);$$
$$W = \pm\sqrt{\frac{3\beta a_1 - 8a_1 b}{18c^2} - a_1^2 b} - a_1\sqrt{\frac{\alpha}{4c^2}}\cot\left(\sqrt{\frac{\alpha}{4c^2}}\xi\right). \quad (2.10)$$

Example 2. We consider the mKdV-type equation [23]

$$u_t + u_{xxx} + \alpha u^2 u_x + \beta u u_x u_{xx} = 0. \quad (2.11)$$

Let $u(x, t) = U(\xi)$, $\xi = x + ct$, and then (2.11) reduces to

$$cU' + U''' + \alpha U^2 U' + \beta U U' U'' = 0. \quad (2.12)$$

Balancing between U''' and $UU'U''$ yields $n = 0$, which is not a positive integer. So let $U = V'$, and then (2.12) becomes

$$cU'' + U^{(4)} + \alpha U'^2 U'' + \beta U' U'' U''' = 0. \quad (2.13)$$

Furthermore, let $V = W^{-1}$, and then (2.13) becomes

$$\begin{aligned} &cW^7W'' - 2cW^6W'^2 + \alpha W^3W'^2W'' - 2\alpha W^2W'^4 \\ &+\beta W^3W'W''W''' - 2\beta W^2W'^3W''' \\ &-6\beta W^2W'^2W''^2 + 18\beta WW'^4W'' - 12\beta W'^2 \\ &-24W'^4W^4 + 36W'^2W''W^5 - 6W''^2W^6 \\ &-8W'W'''W^6 + W^{(4)}W^7 = 0. \end{aligned} \quad (2.14)$$

Thus we can assume that W has ansatz (2.5). Substituting (2.5) into (2.14) and equating all coefficients of the powers of $\varphi(\xi)$ to zero, we have $b = \frac{c}{4}$, $\alpha = c\beta$ and $a_0^2 + ba_1^2 = 0$. So we obtain exact solutions which correspond to $U = \frac{-W'(\xi)}{W^2}$, $\xi = x + ct$ of (2.11),

$$W = \pm\frac{a_1}{2}\sqrt{-c} - \frac{a_1}{2}\sqrt{-c}\tanh\left(\frac{1}{2}\sqrt{-c}\,\xi\right);$$
$$W = \pm\frac{a_1}{2}\sqrt{-c} - \frac{a_1}{2}\sqrt{-c}\coth\left(\frac{1}{2}\sqrt{-c}\,\xi\right). \quad (2.15)$$

3. The tanh-function expansion method of implicit function form

In this section, we firstly outline our new method which includes the following four main steps:

Step 1: Using the travelling wave reduction (1.2), namely $u(x, t) = U(\xi)$, $\xi = x + ct$, we transform (1.1) into an ODE (1.3).

Step 2: We expand an exact solution of (1.3) in the form

$$U(\xi) = \sum_{j=0}^{n} a_j\varphi^j(s), \quad \varphi'(s) = b + \varphi^2(s),$$
$$\xi = d_1 s + d_2\varphi(s), \quad d_2 \neq 0. \quad (3.1)$$

Here n can be obtained by balancing suitable nonlinear terms and the highest linear derivative term.

Step 3: Substituting the above expansion (3.1) into the ODE (1.3) and equating the coefficients of all powers of $\varphi(s)$ to zero, we can get a system of algebraic equations. Thus parameters c, $a_j\,(j = 0, 1, \cdots, n)$, b, d_1, d_2 can be determined explicitly by solving this system.

Step 4: Considering that (1.6) admits several types of exact solution, we can then obtain various types of exact solution to the evolution equation with implicit function form under consideration.

In fact, all the explicit function expansion methods [2–15, 19, 20] can be generalized to the case of implicit functions $\xi = f(s)$.

In what follows, we consider the Harry Dym-type equation [24–26]

$$v_t - 2\tau\left(v^{\frac{3}{2}}\right)_x - 2\left(v^{-\frac{1}{2}}\right)_{xxx} = 0, \quad \tau > 0, \quad (3.2)$$

by using the above new method. Under the transformation $v = u^{-2}$, $(u > 0)$, (3.2) can be written as

$$uu_t - 3\tau u_x + u^4 u_{xxx} = 0. \quad (3.3)$$

Performing the transform $u(x, t) = U(\xi)$, $\xi = x + ct$, we can obtain

$$cUU_\xi - 3\tau U_\xi + U^4 U_{\xi\xi\xi} = 0. \quad (3.4)$$

We rewrite this formula (3.4) as

$$U_{\xi\xi\xi} + cU^{-3}U_\xi - 3\tau U^{-4}U_\xi = 0.$$

Integrating with respect to ξ once, we have

$$U_{\xi\xi} - \frac{1}{2}cU^{-2} + \tau U^{-3} - \frac{h_1}{2} = 0,$$

where $h_1/2$ is a constant of integration. Multiplying the above equation by U_ξ and integrating with respect to ξ once again, we have

$$U^2U_\xi^2 - h_1U^3 - h_2U^2 + cU - \tau = 0, \quad (3.5)$$

where h_2 is another constant of integration [24].

Balancing $U^2U_\xi^2$ with U^3, $h_1 \neq 0$, we have $4n - 2 = 3n$, from which $n = 2$. Therefore we can expand an exact solution of the above equation (3.5) in the form

$$U(\xi) = a_0 + a_1\varphi(s) + a_2\varphi^2(s), \quad \varphi'(s) = b + \varphi^2(s),$$
$$\xi = d_1 s + d_2\varphi(s), \quad d_2 \neq 0, \quad (3.6)$$

according to (3.1). Substituting (3.6) into (3.5) yields a set of algebraic equations with the help of symbolic computation software Maple,

$$
\begin{aligned}
&a_0^2a_1^2b^2 + a_0cd_1^2 + 2a_0bcd_1d_2 + a_0b^2cd_2^2 - a_0^3d_1^2h_1\\
&\quad -2a_0^3bd_1d_2h_1 - a_0^3b^2d_2^2h_1 - a_0^2d_1^2h_2\\
&\quad -2a_0^2bd_1d_2h_2 - a_0^2b^2d_2^2h_2 - d_1^2\tau\\
&\quad -2bd_1d_2\tau - b^2d_2^2\tau = 0,\\
&2a_0a_1^3b^2 + 4a_0^2a_1a_2b^2 + a_1cd_1^2 + 2a_1bcd_1d_2\\
&\quad +a_1b^2cd_2^2 - 3a_0^2a_1d_1^2h_1 - 6a_0^2a_1bd_1d_2h_1\\
&\quad -3a_0^2a_1b^2d_2^2h_1 - 2a_0a_1d_1^2h_2\\
&\quad -4a_0a_1bd_1d_2h_2 - 2a_0a_1b^2d_2^2h_2 = 0,
\end{aligned}
$$

$$
\begin{aligned}
&2a_0^2a_1^2b + a_1^4b^2 + 10a_0a_1^2a_2b^2 + 4a_0^2a_2^2b^2 + a_2cd_1^2\\
&\quad +2a_0cd_1d_2 + 2a_2bcd_1d_2 + 2a_0bcd_2^2\\
&\quad +a_2b^2cd_2^2 - 3a_0a_1^2d_1^2h_1 - 3a_0^2a_2d_1^2h_1 - 2a_0^3d_1d_2h_1\\
&\quad -6a_0a_1^2bd_1d_2h_1 - 6a_0^2a_2bd_1d_2h_1\\
&\quad -2a_0^3bd_2^2h_1 - 3a_0a_1^2b^2d_2^2h_1 - 3a_0^2a_2b^2d_2^2h_1\\
&\quad -a_1^2d_1^2h_2 - 2a_0a_2d_1^2h_2 - 2a_0^2d_1d_2h_2\\
&\quad -2a_1^2bd_1d_2h_2 - 4a_0a_2bd_1d_2h_2 - 2a_0^2bd_2^2h_2\\
&\quad -a_1^2b^2d_2^2h_2 - 2a_0a_2b^2d_2^2h_2 - 2d_1d_2\tau\\
&\quad -2bd_2^2\tau = 0,
\end{aligned}
$$

$$
\begin{aligned}
&4a_0a_1^3b + 8a_0^2a_1a_2b + 6a_1^3a_2b^2 + 16a_0a_1a_2^2b^2\\
&\quad +2a_1cd_1d_2 + 2a_1bcd_2^2 - a_1^3d_1^2h_1 - 6a_0a_1a_2d_1^2h_1\\
&\quad -6a_0^2a_1d_1d_2h_1 - 2a_1^3bd_1d_2h_1 - 12a_0a_1a_2bd_1d_2h_1\\
&\quad -6a_0^2a_1bd_2^2h_1 - a_1^3b^2d_2^2h_1\\
&\quad -6a_0a_1a_2b^2d_2^2h_1 - 2a_1a_2d_1^2h_2 - 4a_0a_1d_1d_2h_2\\
&\quad -4a_1a_2bd_1d_2h_2 - 4a_0a_1bd_2^2h_2\\
&\quad -2a_1a_2b^2d_2^2h_2 = 0,
\end{aligned}
$$

$$
\begin{aligned}
&a_0^2a_1^2 + 2a_1^4b + 20a_0a_1^2a_2b + 8a_0^2a_2^2b + 13a_1^2a_2^2b^2\\
&\quad +8a_0a_2^3b^2 + 2a_2cd_1d_2 + a_0cd_2^2 + 2a_2bcd_2^2\\
&\quad -3a_1^2a_2d_1^2h_1 - 3a_0a_2^2d_1^2h_1 - 6a_0a_1^2d_1d_2h_1\\
&\quad -6a_0^2a_2d_1d_2h_1 - 6a_1^2a_2bd_1d_2h_1 - 6a_0a_2^2bd_1d_2h_1\\
&\quad -a_0^3d_2^2h_1 - 6a_0a_1^2bd_2^2h_1 - 6a_0^2a_2bd_2^2h_1\\
&\quad -3a_1^2a_2b^2d_2^2h_1 - 3a_0a_2^2b^2d_2^2h_1 - a_2^2d_1^2h_2\\
&\quad -2a_1^2d_1d_2h_2 - 4a_0a_2d_1d_2h_2 - 2a_2^2bd_1d_2h_2\\
&\quad -a_0^2d_2^2h_2 - 2a_1^2bd_2^2h_2 - 4a_0a_2bd_2^2h_2\\
&\quad -a_2^2b^2d_2^2h_2 - d_2^2\tau = 0,
\end{aligned}
$$

$$
\begin{aligned}
&2a_0a_1^3 + 4a_0^2a_1a_2 + 12a_1^3a_2b + 32a_0a_1a_2^2b + 12a_1a_2^3b^2\\
&\quad +a_1cd_2^2 - 3a_1a_2^2d_1^2h_1 - 2a_1^3d_1d_2h_1\\
&\quad -12a_0a_1a_2d_1d_2h_1 - 6a_1a_2^2bd_1d_2h_1 - 3a_0^2a_1d_2^2h_1\\
&\quad -2a_1^3bd_2^2h_1 - 12a_0a_1a_2bd_2^2h_1\\
&\quad -3a_1a_2^2b^2d_2^2h_1 - 4a_1a_2d_1d_2h_2\\
&\quad -2a_0a_1d_2^2h_2 - 4a_1a_2bd_2^2h_2 = 0,
\end{aligned}
$$

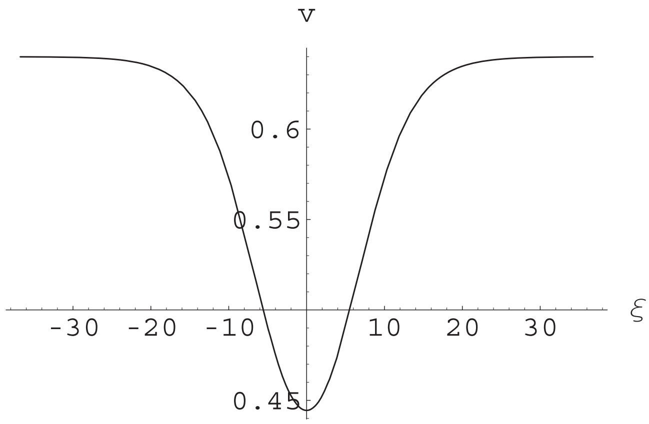

Figure 1. $(v, \xi) = (U^{-2}, \xi)$ with (3.8) has a parameterized soliton solution achieved by setting $b = -1$, $h_1 = -1/2$, $d_1 = 5\sqrt{2}$ and $d_2 = -\sqrt{2}$.

$$
\begin{aligned}
&a_1^4 + 10a_0a_1^2a_2 + 4a_0^2a_2^2 + 26a_1^2a_2^2b + 16a_0a_2^3b\\
&\quad +4a_2^4b^2 + a_2cd_2^2 - a_2^3d_1^2h_1 - 6a_1^2a_2d_1d_2h_1\\
&\quad -6a_0a_2^2d_1d_2h_1 - 2a_2^3bd_1d_2h_1 - 3a_0a_1^2d_2^2h_1\\
&\quad -3a_0^2a_2d_2^2h_1 - 6a_1^2a_2bd_2^2h_1 - 6a_0a_2^2bd_2^2h_1\\
&\quad -a_2^3b^2d_2^2h_1 - 2a_2^2d_1d_2h_2 - a_1^2d_2^2h_2\\
&\quad -2a_0a_2d_2^2h_2 - 2a_2^2bd_2^2h_2 = 0,\\
&6a_1^3a_2 + 16a_0a_1a_2^2 + 24a_1a_2^3b - 6a_1a_2^2d_1d_2h_1\\
&\quad -a_1^3d_2^2h_1 - 6a_0a_1a_2d_2^2h_1 - 6a_1a_2^2bd_2^2h_1\\
&\quad -2a_1a_2d_2^2h_2 = 0,\\
&13a_1^2a_2^2 + 8a_0a_2^3 + 8a_2^4b - 2a_2^3d_1d_2h_1 - 3a_1^2a_2d_2^2h_1\\
&\quad -3a_0a_2^2d_2^2h_1 - 2a_2^3bd_2^2h_1 - a_2^2d_2^2h_2 = 0,\\
&12a_1a_2^3 - 3a_1a_2^2d_2^2h_1 = 0,\\
&4a_2^4 - a_2^3d_2^2h_1 = 0.
\end{aligned}
$$

Solving the above system by means of symbolic computation software again, we can actually obtain several types of exact solution according to (1.6),

$$
\begin{aligned}
a_2 &= \frac{1}{4}d_2^2h_1, \quad d_2,\ h_1 \neq 0,\\
a_1 &= 0,\\
a_0 &= \frac{1}{4}d_2h_1(d_1 + bd_2),\\
\tau &= -\frac{1}{64}d_1^2d_2^3h_1^4(d_1 + bd_2),\\
c &= -\frac{1}{16}d_1d_2^2h_1^3(3d_1 + 2bd_2),\\
h_2 &= -\frac{1}{4}d_2h_1^2(3d_1 + bd_2).
\end{aligned} \tag{3.7}
$$

The sign of b can be used to exactly judge the type of travelling wave solutions. For example, when $b < 0$, (3.5) admits exact solutions of tanh-function-type (see figure 1)

$$
\begin{aligned}
U &= \frac{1}{4}d_2h_1(d_1 + bd_2) - \frac{1}{4}d_2^2h_1\sqrt{-b}\,\tanh^2(\sqrt{-b}\,s),\\
\xi &= d_1s - d_2\sqrt{-b}\,\tanh(\sqrt{-b}\,s),
\end{aligned} \tag{3.8}
$$

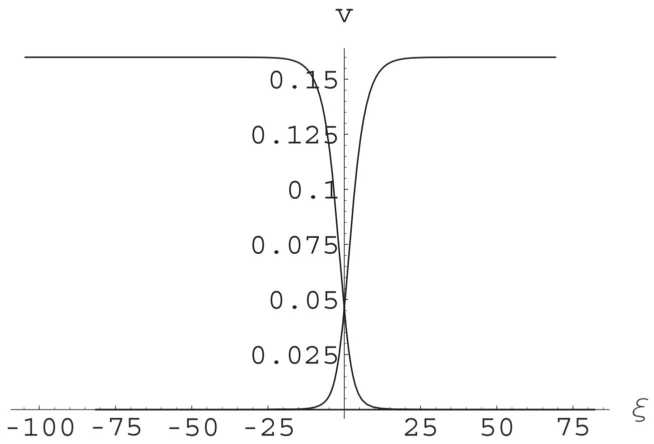

Figure 2. $(v, \xi) = (U^{-2}, \xi)$ with (3.9) has a parameterized anti-kink solution when $s > 0$, and a kink solution when $s < 0$, achieved by setting $b = -1$, $h_1 = 1$, $d_1 = 5\sqrt{2}$ and $d_2 = \sqrt{2}$.

and coth-function-type (see figure 2)

$$U = \frac{1}{4}d_2h_1(d_1 + bd_2) - \frac{1}{4}d_2^2h_1\sqrt{-b}\,\coth^2(\sqrt{-b}\,s),$$
$$\xi = d_1s - d_2\sqrt{-b}\,\coth(\sqrt{-b}\,s), \tag{3.9}$$

where τ, c, h_2 are given by (3.7). Particularly, when $b = -1$, $h_1 = -\frac{1}{2}$, $d_1 = 2\sqrt{2}$ and $d_2 = -2\sqrt{2}$, we have $a_2 = -1$, $a_1 = 0$, $a_0 = 2$, $h_2 = 2$, $\tau = 1$ and $c = \frac{5}{2}$, according to (3.7). This special result has been also obtained in [24].

4. Summary

From the above two examples, (2.1) and (2.11), it is clear to see that travelling wave solutions (1.6) to a NLEE with $n = 0$ can be obtained by the new transform $U = \frac{-W'(\xi)}{W^2}$ through simple algebraic calculations. It can also be seen that the present method may be generalized to obtain multi-travelling wave solutions and solve coupled nonlinear evolution systems, which will add to the explicit solutions found using the multiple exponential function method and the linear superposition principle [27–29].

Taking the Harry Dym-type equation (3.2) as an example, we have proposed the implicit tanh-function expansion method for constructing more exact solutions of NLEEs. Various kinds of explicit function expansion methods presented in [1–21] can also be generalized to the case of implicit functions. In fact, one can easily write the corresponding version of implicit functions such as $\xi = f(s)$ in step 2, section 3. Thus, exact solutions obtained by the implicit function expansion method include not only the polynomial, exponential, rational, triangular periodic wave, hyperbolic and Jacobi doubly periodic wave solutions, but also the loop-type soliton solutions. For example, by using this new implicit tanh-function expansion method, more exact solutions for the short pulse equation $u_{xt} = u + 1/6(u^3)_{xx}$ can be obtained, which have implicit function forms corresponding to loop-type solitons [30].

This new implicit function expansion method can also deal with the following coupled Harry Dym-type system [31]

$$\begin{cases} u_t = \frac{1}{2}\left(\frac{1}{\sqrt{v}}\right)_{xxx} - 2\alpha\left(\frac{1}{v^\beta}\right)_x, & \beta < 0, \\ v_t = u_x\left(\frac{1}{\sqrt{v}}\right) + 2u\left(\frac{1}{\sqrt{v}}\right)_x \end{cases}$$

and the (2+1)-dimensional Harry Dym-type equation [32, 33]

$$u_t + u^3u_{xxx} + \frac{3}{u}\left(u^2\partial_x^{-1}\left(\frac{u_y}{u^2}\right)\right)_y = 0.$$

However, this new method also has a disadvantage, which is that the amount of computation required is very large. Of course, modern calculating machines and the development of software systems make such computation possible. For the sake of simplicity, we omit the calculation process and results.

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