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Exact rational solutions to a Boussinesq-like equation in $(1+1)$-dimensions

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ABSTRACT

A Boussinesq-like nonlinear differential equation in $(1+1)$-dimensions is introduced by using a generalized bilinear differential equation with the generalized bilinear derivatives $D_{3,x}$ and $D_{3,t}$. A class of rational solutions, generated from polynomial solutions to the associated generalized bilinear equation, is constructed for the presented Boussinesq-like equation. It is conjectured that this class of rational solutions contain all such rational solutions to the new Boussinesq-like equation. More concretely, the conjecture says that if a polynomial $f = f(x,t)$ in x and t solves $f_{tt}f - f_t^2 + 3f_{xx}^2 = 0$, then the degree of f with respect to t must be less than or equal to 1.

1. Introduction

On one hand, Hirota bilinear equations have been generalized by involving different prime numbers [1], which generate diverse nonlinear differential equations possessing potential applications. On the other hand, there has been a growing interest in rational solutions to nonlinear differential equations (see, e.g., [2,3]). A kind of interesting rational solutions – rogue wave solutions – draws a big attention of mathematicians and physicists worldwide and such rational solutions describe significant nonlinear wave phenomena, particularly in oceanography [4,5] and nonlinear optics [6,7]. One of current interests to us is to discuss about rational solutions to a new kind of nonlinear differential equations associated with generalized bilinear equations.

Rational solutions to integrable equations (see [8,9]) have been considered systematically by using the Wronskian formulation and the Casoratian formulation. Particular examples include the KdV equation, the Boussinesq equation, and the Toda lattice equation (see, e.g., [10–12], respectively). Rational solutions to the non-integrable $(3+1)$-dimensional KP I [13,14] and KP II [15] are considered by different approaches such as the tanh-function method [16], the tanh–coth function method [17], and the $\frac{G'}{G}$-expansion method [18].

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Moreover, rational solutions to the (3 + 1)-dimensional KP II can be generated from rational solutions to the good Boussinesq equation [15].

In this paper, we would like to introduce a Boussinesq-like nonlinear differential equation by using a generalized bilinear differential equation of Boussinesq type. From polynomial solutions to the associated generalized bilinear equation, we will construct a class of rational solutions to the presented Boussinesq-like equation, which have a specific requirement on the degrees of the spatial variable and the temporal variable. A conjecture will be presented during our analysis that the resulting class of rational solutions contain all such rational solutions to the Boussinesq-like equation, and a few concluding remarks will be given at the end of the paper.

2. A Boussinesq-like equation

We consider a generalized bilinear differential equation of Boussinesq type:

$$(D_{3,t}^2 + D_{3,x}^4) f \cdot f = 2f_{tt}f - 2f_t^2 + 6f_{xx}^2 = 0. \tag{2.1}$$

This equation possesses the same bilinear type as the standard Boussinesq one [19]. The bilinear differential operators involved above are a kind of generalized bilinear differential operators presented in [1,20,21]:

$$\begin{aligned} D_{p,x}^m D_{p,t}^n f \cdot f &= \left(\frac{\partial}{\partial x} + \alpha_p \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} + \alpha_p \frac{\partial}{\partial t'}\right)^n f(x,t) f(x',t')|_{x'=x,t'=t} \\ &= \sum_{i=0}^m \sum_{j=0}^n \binom{m}{i} \binom{n}{j} \alpha_p^i \alpha_p^j \frac{\partial^{m-i}}{\partial x^{m-i}} \frac{\partial^i}{\partial x'^i} \frac{\partial^{n-j}}{\partial x^{n-j}} \frac{\partial^j}{\partial t'^j} f(x,t) f(x',t')|_{x'=x,t'=t} \\ &= \sum_{i=0}^m \sum_{j=0}^n \binom{m}{i} \binom{n}{j} \alpha_p^i \alpha_p^j \frac{\partial^{m+n-i-j} f(x,t)}{\partial x^{m-i} \partial t^{n-j}} \frac{\partial^{i+j} f(x,t)}{\partial x^i \partial t^j}, \quad m,n \geq 0, \end{aligned} \tag{2.2}$$

where α_p^s is computed as follows:

$$\alpha_p^s = (-1)^{r_p(s)}, \quad s = r_p(s) \bmod p. \tag{2.3}$$

Note that

$$\alpha_p^i \alpha_p^j \neq \alpha_p^{i+j}, \quad i,j \geq 0,$$

when p is a prime number greater than 2. If taking $p = 3$, we have

$$\alpha_3 = -1, \qquad \alpha_3^2 = 1, \qquad \alpha_3^3 = 1, \qquad \alpha_3^4 = -1, \qquad \alpha_3^5 = 1, \qquad \alpha_3^6 = 1,$$

and thus

$$D_{3,t}^2 f \cdot f = 2f_{tt}f - 2f_t^2, \qquad D_{3,x}^4 f \cdot f = 6f_{xx}^2.$$

In the case of $p = 2$, which is the Hirota case, we have

$$D_{2,t}^2 f \cdot f = 2f_{tt}f - 2f_t^2, \qquad D_{2,x}^4 f \cdot f = 2f_{xxxx}f - 8f_{xxx}f_x + 6f_{xx}^2,$$

which generates the standard bilinear Boussinesq equation [19]:

$$(D_{2,t}^2 + D_{2,x}^4) f \cdot f = 2f_{tt}f - 2f_t^2 + 2f_{xxxx}f - 8f_{xxx}f_x + 6f_{xx}^2 = 0, \tag{2.4}$$

and the Boussinesq equation:

$$u_{tt} + (u^2)_{xx} + u_{xxxx} = 0, \tag{2.5}$$

under the transformation $u = 6(\ln f)_{xx}$ [11].

Motivated by Bell polynomial theories [20,21], we use a dependent variable transformation

$$u = 2(\ln f)_x, \tag{2.6}$$

to introduce a Boussinesq-like scalar nonlinear differential equation

$$u_{tt} + 3u_x u_{xx} + 3uu_x^2 + \frac{3}{2}u^3 u_x + \frac{3}{2}u^2 u_{xx} = 0, \tag{2.7}$$

from the generalized bilinear equation (2.1). By virtue of the transformation (2.6), we actually have the following equality:

$$\left[\frac{(D_{3,t}^2 + D_{3,x}^4) f \cdot f}{f^2}\right]_x = u_{tt} + 3u_x u_{xx} + 3uu_x^2 + \frac{3}{2}u^3 u_x + \frac{3}{2}u^2 u_{xx}, \tag{2.8}$$

and thus, if f is a solution to (2.1), then $u = 2(\ln f)_x$ solves the Boussinesq-like equation (2.7). The Boussinesq-like equation (2.7) is much more complicated than the standard one (2.5), while their bilinear counterparts just have an opposite characteristic (i.e., (2.1) is simpler than (2.4)).

Resonant solutions in terms of the two kinds of transcendental functions: exponential functions and trigonometric functions, have been discussed for generalized bilinear equations [20–22]. In what follows, we would like to consider rational solutions to the Boussinesq-like equation (2.7), based on polynomial solutions to the generalized bilinear equations (2.1).

3. Rational solutions

By a Maple computation on

$$f = \sum_{i=0}^{7} \sum_{j=0}^{7} c_{ij} x^i t^j,$$

we find that any such polynomial solution f does not allow the degree of t greater than 1. We conjecture that this is true, namely, any polynomial solution f to the generalized bilinear equation (2.1) must have the degree of t not greater than 1.

We formulate a polynomial solution to the generalized bilinear equation (2.1) as

$$f = \sum_{i=0}^{n} p_i t^i = \sum_{i=0}^{n} p_i(x) t^i, \tag{3.1}$$

where n is a nonnegative integer and the p_i's are polynomials of x with $p_n \neq 0$. Assume that $n \geq 2$. We want to present a contradiction to this assumption. Since $n \geq 2$, we can compute that

$$\begin{aligned}
& f_{tt} f - f_t^2 + 3f_{xx}^2 \\
&\quad = \left[\sum_{i=0}^{n-2} (i+2)(i+1) p_{i+2} t^i\right] \left(\sum_{i=0}^{n} p_i t^i\right) - \left[\sum_{i=0}^{n-1} (i+1) p_{i+1} t^i\right]^2 + 3\left(\sum_{i=0}^{n} p_{i,xx} t^i\right)^2 \\
&\quad = \sum_{k=0}^{2n-2} \left[\sum_{i+j=k,\, 0\leq i\leq n-2,\, 0\leq j\leq n} (i+2)(i+1) p_{i+2} p_j\right] t^k \\
&\quad\quad - \sum_{k=0}^{2n-2} \left[\sum_{i+j=k,\, 0\leq i,j\leq n-1} (i+1)(j+1) p_{i+1} p_{j+1}\right] t^k \\
&\quad\quad + 3\sum_{k=0}^{2n} \left(\sum_{i+j=k,\, 0\leq i,j\leq n} p_{i,xx} p_{j,xx}\right) t^k.
\end{aligned}$$

This is a polynomial in t with coefficients being polynomials in x. Obviously, the coefficients of the first five highest orders are

$$\begin{cases} t^{2n}: & 3p_{n,xx}^2, \\ t^{2n-1}: & 6p_{n,xx}p_{n-1,xx}, \\ t^{2n-2}: & -np_n^2 + 3p_{n-1,xx}^2 + 6p_{n-2,xx}p_{n,xx}, \\ t^{2n-3}: & -2(n-1)p_{n-1}p_n + 6p_{n-3,xx}p_{n,xx} + 6p_{n-2,xx}p_{n-1,xx}, \\ t^{2n-4}: & -2(n-3)p_{n-2}p_n - (n-1)p_{n-1}^2 + 6p_{n-4,xx}p_{n,xx} + 6p_{n-3,xx}p_{n-1,xx} + 3p_{n-2,xx}^2, \end{cases} \tag{3.2}$$

where we adopt $p_i = 0$ if $i < 0$. Thus, balancing the coefficients of t^{2n} in (2.1), we have

$$p_{n,xx} = 0, \tag{3.3}$$

which guarantees that the coefficients of t^{2n-1} get balanced, and then a balance of the coefficients of t^{2n-2} tells

$$-np_n^2 + 3p_{n-1,xx}^2 = 0, \tag{3.4}$$

which leads to

$$p_{n-1,xx} = \varepsilon\frac{\sqrt{3n}}{3}p_n, \quad \varepsilon = \pm 1. \tag{3.5}$$

The above two equalities present a specific requirement on the structure of polynomial solutions to (2.1).

Assume that $n = 2$. We would like to derive a contradiction to this assumption. From the above sums of coefficients of powers t^i with $i = 4, 2, 1$ in (3.2), we obtain

$$p_{2,xx} = 0, \qquad p_{1,xx} = \varepsilon\frac{\sqrt{6}}{3}p_2, \qquad p_{0,xx} = \varepsilon\frac{\sqrt{6}}{6}p_1, \tag{3.6}$$

and the sum of coefficients of t^3 is automatically zero, based on (3.2). Then, upon setting

$$p_2 = 6ax + 2b, \quad a, b = \text{consts.},$$

the sum of coefficients of the constant terms in (2.1) becomes

$$2(2p_0p_2 - p_1^2 + 3p_{0,xx}^2) = 4p_0p_2 - p_1^2 = -\frac{4}{15}a^2x^6 - \frac{8}{15}abx^5 - \frac{2}{9}b^2x^4 + q(x),$$

where q is a third-order polynomial in x. It should be equal to zero, which implies that $a = b = 0$. This fact contradicts that $p_2 \neq 0$, i.e., the assumption of $n = 2$. Therefore, it is impossible that $n = 2$.

Let us now set $n = 1$. At this moment, we have

$$p_{0,xx} = \varepsilon\frac{\sqrt{3}}{3}p_1, \tag{3.7}$$

which is guaranteed by a balance of the constant terms in (2.1). Since we have (3.3), i.e., $p_{1,xx} = 0$, we can set

$$p_1 = \varepsilon\sqrt{3}(6ax + 2b), \tag{3.8}$$

which neatly generates

$$p_0 = ax^3 + bx^2 + cx + d, \tag{3.9}$$

where a, b, c, d are arbitrary constants. This way, we obtain a class of polynomial solutions to (2.1):

$$f = \varepsilon\sqrt{3}(6ax + 2b)t + (ax^3 + bx^2 + cx + d), \tag{3.10}$$

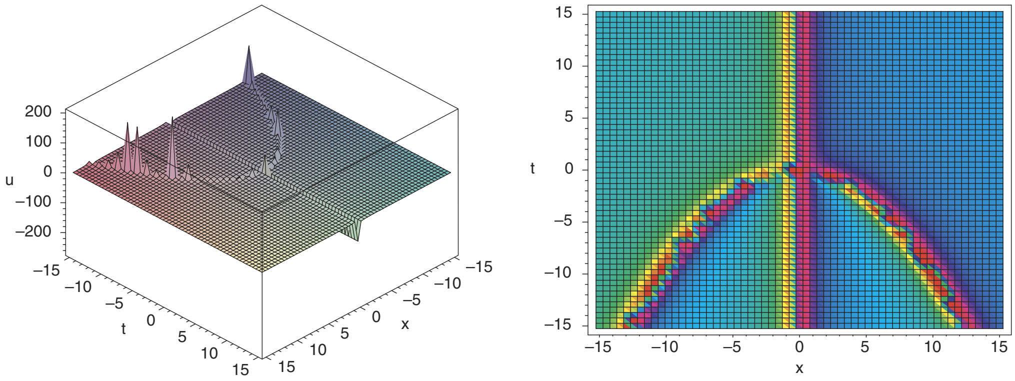

Fig. 1. Pictures of (3.12) with $\varepsilon = 1$: 3d plot (left) and density plot (right).

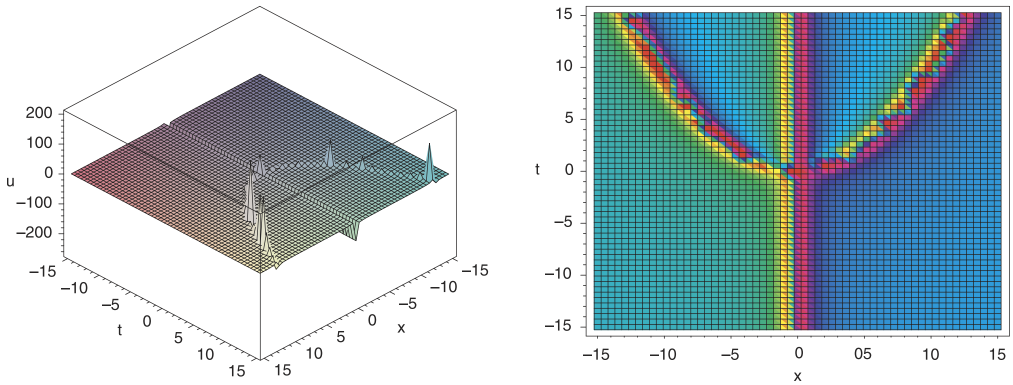

Fig. 2. Pictures of (3.12) with $\varepsilon = -1$: 3d plot (left) and density plot (right).

with a, b, c, d being arbitrary constants. Further, the corresponding rational solutions to the Boussinesq-like equation (2.7) read

$$u = \frac{2(\varepsilon 6\sqrt{3}at + 3ax^2 + 2bx + c)}{\varepsilon\sqrt{3}(6ax + 2b)t + (ax^3 + bx^2 + cx + d)}, \tag{3.11}$$

where $\varepsilon = \pm 1$ and a, b, c, d are arbitrary constants.

We conjecture that the class of rational solutions in (3.11) exhaust all rational solutions to the Boussinesq-like equation (2.7), which are generated from polynomial solutions to the generalized bilinear equation (2.1). More concretely, the conjecture is equivalent to the following statement.

Conjecture. *If a polynomial $f = f(x, t)$ in x and t solves the bilinear equation $f_{tt}f - f_t^2 + 3f_{xx}^2 = 0$, then the degree of f with respect to t must be less than or equal to* 1.

Two special solutions of (3.11) with $a = 1$, $b = 2$, $c = 2$ and $d = 3$ are

$$u = \frac{2(\varepsilon 6\sqrt{3}t + 3x^2 + 4x + 2)}{\varepsilon\sqrt{3}(6x + 4)t + (x^3 + 2x^2 + 2x + 3)}. \tag{3.12}$$

Two pictures of the solution (3.12) with $\varepsilon = 1$ are given in Fig. 1, and two pictures of the solution (3.12) with $\varepsilon = -1$ are given in Fig. 2.

4. Concluding remarks

Based on the standard Boussinesq equation, we introduced a Boussinesq-like nonlinear differential equation through a generalized bilinear equation of Boussinesq type, and constructed one class of rational solutions to the resulting Boussinesq-like equation. The basic starting point is the generalized bilinear differential operators $D_{3,x}$ and $D_{3,t}$ presented in [1,20,21].

We point out that it is worth checking if there are any Wronskian solutions and multiple soliton type solutions to the Boussinesq-like nonlinear equation (2.7), which even solve its Cauchy problem (see, e.g., [23] for analysis on the Burgers type equations). A conjecture is that the presented class of rational solutions in (3.11) contains all rational solutions to the Boussinesq-like nonlinear equation (2.7), generated from polynomial solutions to the generalized bilinear equation (2.1) through the transformation (2.6). Interestingly, the standard Boussinesq equation has infinitely many classes of rational solutions [11].

Similarly, a kind of generalized tri-linear differential equations was discussed in [24], together with resonant solutions. Rational solutions to generalized tri-linear differential equations, which can always be viewed as continuous functions of the extended complex variables, particularly rogue wave solutions, will be of considerable interest. Higher-order rogue wave solutions should be connected with generalized Wronskian solutions [25] and generalized Darboux transformations [26]. Exact periodic wave solutions to generalized bilinear equations would be another interesting topic [27].

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