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Integrable counterparts of the D-Kaup–Newell soliton hierarchy

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ABSTRACT

Two integrable counterparts of the D-Kaup–Newell soliton hierarchy are constructed from a matrix spectral problem associated with the three dimensional special orthogonal Lie algebra $\mathrm{so}(3,\mathbb{R})$. An application of the trace identity presents Hamiltonian or quasi-Hamiltonian structures of the resulting counterpart soliton hierarchies, thereby showing their Liouville integrability, i.e., the existence of infinitely many commuting symmetries and conserved densities. The involved Hamiltonian and quasi-Hamiltonian properties are shown by computer algebra systems.

1. Introduction

Soliton equations generated from zero curvature equations provide concrete examples of integrable equations possessing Hamiltonian structures (see, e.g., [1–5]). It is a starting point of generating soliton equations to formulate matrix spectral problems (or Lax pairs) associated with given matrix loop algebras. The trace identity [6] and the variational identity [7] offer powerful techniques for furnishing their Hamiltonian structures.

Soliton equations usually come in hierarchies. The zero curvature formulation (see, e.g., [6,8]) provides a practical approach for building soliton hierarchies. Typical examples of soliton hierarchies which fit into this formulation include the Korteweg-de Vries hierarchy [9], the Ablowitz–Kaup–Newell–Segur hierarchy [10], the Dirac hierarchy [11], the Kaup–Newell hierarchy [12] and the Wadati–Konno–Ichikawa hierarchy [13]. Note that those hierarchies only involve one or two dependent variables. The case of three or more dependent variables is highly complicated and needs a considerable investment of time. Integrable couplings are such examples with a large number of dependent variables, possessing triangular forms.

Very recently, the three-dimensional special orthogonal Lie algebra $\mathrm{so}(3,\mathbb{R})$ has been used in constructing soliton hierarchies (see, e.g., [14,15]). This simple Lie algebra can be realized through 3×3 skew-symmetric matrices, and thus, it has the following basis

$$e_1=\begin{bmatrix}0&0&-1\\0&0&0\\1&0&0\end{bmatrix},\ e_2=\begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix},\ e_3=\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix},\tag{1.1}$$

whose commutation relations read

$$[e_1,e_2]=e_3,\quad [e_2,e_3]=e_1,\quad [e_3,e_1]=e_2.\tag{1.2}$$

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Its derived algebra is the algebra itself, and thus, it is also 3-dimensional. The only other three-dimensional real Lie algebra with a three-dimensional derived algebra is the special linear algebra sl(2,$\mathbb{R}$), which has been widely used in studying soliton equations in soliton theory (see, e.g., [9–13]).

The matrix loop algebra we shall adopt in this paper is

$$\widetilde{\mathrm{so}}(3,\mathbb{R}) = \left\{ \sum_{i\geqslant 0} M_i \lambda^{n-i} | M_i \in \mathrm{so}(3,\mathbb{R}),\ i \geqslant 0,\ n \in \mathbb{Z} \right\}. \tag{1.3}$$

It is the space of all Laurent series in λ with coefficients in so$(3,\mathbb{R})$ and a finite regular part. Particular examples of this loop algebra $\widetilde{\mathrm{so}}(3,\mathbb{R})$ are linear combinations of the form:

$$p_1(\lambda)e_1 + p_2(\lambda)e_2 + p_3(\lambda)e_3$$

with arbitrary Laurent polynomials p_1, p_2, p_3 in λ, which constitute a Lie subalgebra. Due to the circular commutation relations (1.2), the loop algebra $\widetilde{\mathrm{so}}(3,\mathbb{R})$ provides a good structural basis for our study of soliton equations possessing Hamiltonian structures and quasi-Hamiltonian structures, and several new soliton hierarchies have been already worked out from $\widetilde{\mathrm{so}}(3,\mathbb{R})$, indeed (see, e.g., [14–19]).

In this paper, we would like to use $\widetilde{\mathrm{so}}(3,\mathbb{R})$ to introduce a counterpart matrix spectral problem for the D-Kaup–Newell spectral problem (see [20]), and compute two integrable counterpart hierarchies of the D-Kaup–Newell soliton hierarchy [20] by zero curvature equations. An application of the trace identity will engender Hamiltonian or quasi-Hamiltonian structures for every member in the resulting two counterpart soliton hierarchies, and thus, the counterpart soliton hierarchies are Liouville integrable. The new counterpart hierarchies provide two other interesting examples of soliton hierarchies associated with the matrix loop algebra $\widetilde{\mathrm{so}}(3,\mathbb{R})$. A few concluding remarks and comments will finish the paper.

2. Integrable counterparts of the D-Kaup–Newell hierarchy

To generate integrable counterparts, associated with so(3,$\mathbb{R}$), of the D-Kaup–Newell soliton hierarchy [20], let us introduce a new 3×3 matrix spectral problem:

$$\phi_x = U\phi = U(u,\lambda)\phi, \quad u = \begin{bmatrix} p \\ q \\ r \end{bmatrix}, \quad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}, \tag{2.1}$$

where the spectral matrix U is chosen as

$$U = U(u,\lambda) = (\lambda^2 + r)e_1 + \lambda p e_2 + \lambda q e_3 = \begin{bmatrix} 0 & -\lambda q & -\lambda^2 - r \\ \lambda q & 0 & -\lambda p \\ \lambda^2 + r & \lambda p & 0 \end{bmatrix} \in \widetilde{\mathrm{so}}(3,\mathbb{R}). \tag{2.2}$$

This linear combination is the selfsame as the D-Kaup–Newell one associated with sl(2,$\mathbb{R}$) [20].

Once a matrix spectral problem is chosen, it becomes a standard routine to compute soliton hierarchies from the matrix spectral problem by the zero curvature formulation (see [6,8]). First, we solve the stationary zero curvature equation

$$W_x = [U, W], \quad W \in \widetilde{\mathrm{so}}(3,\mathbb{R}). \tag{2.3}$$

If we assume W to be

$$W = ae_1 + be_2 + ce_3 = \begin{bmatrix} 0 & -c & -a \\ c & 0 & -b \\ a & b & 0 \end{bmatrix}, \tag{2.4}$$

then the Eq. (2.3) becomes

$$\begin{cases} a_x = \lambda pc - \lambda qb, \\ b_x = -\lambda^2 c - rc + \lambda qa, \\ c_x = -\lambda pa + \lambda^2 b + rb. \end{cases} \tag{2.5}$$

Further, let a, b and c possess the following Laurent expansions in λ:

$$a = \sum_{i\geqslant 0} a_i \lambda^{-2i}, \quad b = \sum_{i\geqslant 0} b_i \lambda^{-2i-1}, \quad c = \sum_{i\geqslant 0} c_i \lambda^{-2i-1}, \tag{2.6}$$

and take the initial data

$$a_0 = 1, \quad b_0 = p, \quad c_0 = q, \tag{2.7}$$

which are required by the equations on the highest powers of λ in (2.5):

$$a_{0,x} = pc_0 - qb_0, \quad b_0 = pa_0, \quad c_0 = qa_0.$$

Then, the system (2.5) leads equivalently to

$$\begin{cases} a_{i+1,x} = -pb_{i,x} - qc_{i,x} - r(pc_i - qb_i), \\ b_{i+1} = c_{i,x} + pa_{i+1} - rb_i, \qquad\qquad i \geqslant 0, \\ c_{i+1} = -b_{i,x} + qa_{i+1} - rc_i, \end{cases} \tag{2.8}$$

the first of which is because from (2.5), we have

$$-\lambda a_x = -\lambda^2(pc - qb) = p(b_x + rc - \lambda qa) + q(c_x + \lambda pa - rb) = pb_x + qc_x + r(pc - qb).$$

While using the above recursion relations (2.8), we impose the condition that the constants of integration take the value of zero:

$$a_i|_{u=0} = b_i|_{u=0} = c_i|_{u=0} = 0, \quad i \geqslant 1, \tag{2.9}$$

to determine the sequence of $\{a_i, b_i, c_i | i \geqslant 1\}$ uniquely. This way, the first two sets can be computed as follows:

$$a_1 = \frac{1}{2}(p^2 + q^2), \; b_1 = q_x - \frac{1}{2}p(p^2 + q^2) - pr, \; c_1 = -p_x - \frac{1}{2}(p^2 + q^2)q - qr;$$

$$a_2 = p_x q - pq_x + \frac{3}{8}(p^2 + q^2)^2 + (p^2 + q^2)r,$$

$$b_2 = -p_{xx} - \frac{3}{2}(p^2 + q^2)q_x - 2q_x r - qr_x + \frac{3}{8}p(p^2 + q^2)^2 + \frac{3}{2}p(p^2 + q^2)r + pr^2,$$

$$c_2 = -q_{xx} + \frac{3}{2}p_x(p^2 + q^2) + 2p_x r + pr_x + qr^2 + \frac{3}{2}(p^2 + q^2)qr + \frac{3}{8}(p^2 + q^2)^2 q.$$

We point out that the localness of the first two sets of $\{a_i, b_i, c_i | i \geqslant 1\}$ is not an accident, and actually, the functions $a_i, b_i, c_i, \; i \geqslant 1$, are all local. We state and prove this localness property as follows.

Proposition 1. *The functions $a_i, b_i, c_i, \; i \geqslant 1$, recursively defined by (2.8) from the initial values of (2.7), are all local.*

Proof. Since the condition (2.9) does not create any nonlocalness problem, we can assume that this condition (2.9) holds, that is, the constants of integration take the value of zero, as we need in our discussion.

First from $W_x = [U, W]$, where W is defined by (2.4) with (2.6), we can have

$$\frac{d}{dx}\mathrm{tr}(W^2) = 2\mathrm{tr}(WW_x) = 2\mathrm{tr}(W[U, W]) = 0,$$

and so, due to $\mathrm{tr}(W^2) = -2(a^2 + b^2 + c^2)$, we arrive at

$$a^2 + b^2 + c^2 = (a^2 + b^2 + c^2)|_{u=0} = 1,$$

the last step of which follows from the initial data (2.7) and the condition (2.9). Then, this implies, by using (2.6), that

$$a_i = -\frac{1}{2}\sum_{k+l=i,\, k,l \geqslant 1} a_k a_l - \frac{1}{2}\sum_{k+l=i-1,\, k,l \geqslant 0} b_k b_l - \frac{1}{2}\sum_{k+l=i-1,\, k,l \geqslant 0} c_k c_l, \quad i \geqslant 1.$$

Based on this recursion relation and the last two recursion relations in (2.8), an application of the mathematical induction finally shows that all the functions $a_i, b_i, c_i, \; i \geqslant 1$, are differential functions in u, and so, they are all local. □

Now as usual, let us calculate

$$(\lambda(\lambda^{2m+1}W)_+)_x - [U, \lambda(\lambda^{2m+1}W)_+] = \begin{bmatrix} 0 & -\lambda(c_{m,x} - rb_m) & 0 \\ \lambda(c_{m,x} - rb_m) & 0 & -\lambda(b_{m,x} + rc_m) \\ 0 & \lambda(b_{m,x} + rc_m) & 0 \end{bmatrix}, \quad m \geqslant 0,$$

where for a matrix $A = (a_{ij})_{k \times l}$, we defined $(\lambda^m A)_+ = ((\lambda^m a_{ij})_+)_{k \times l}$, in which f_+ denotes the polynomial part of a particular series f:

$$f_+ = \sum_{i=0}^{n} f_i \lambda^i, \; \text{when } f = \sum_{i=-\infty}^{n} f_i \lambda^i, \quad n \geqslant 0. \tag{2.10}$$

The above matrix is not the same type as the Gateaux derivative matrix U', and so, we need to introduce modification terms in Lax operators to get soliton equations. In what follows, we shall present two kinds of such modification terms.

Noting

$$[U, e_1] = \lambda q e_2 - \lambda p e_3,$$

we take a sequence of Lax operators with the first kind of modification terms:

$$V_1^{[m]} = \lambda(\lambda^{2m+1}W)_+ + \Delta_{1,m}, \quad \Delta_{1,m} = \beta a_{m+1} e_1 \in \widetilde{\mathrm{so}}(3,\mathbb{R}), \quad m \geqslant 0, \tag{2.11}$$

where β is an arbitrarily given constant. Then, we arrive at

$$(V_1^{[m]})_x - [U, V_1^{[m]}] = \begin{bmatrix} 0 & -\lambda(c_{m,x} - rb_m + \beta p a_{m+1}) & -\beta a_{m+1,x} \\ \lambda(c_{m,x} - rb_m + \beta p a_{m+1}) & 0 & -\lambda(b_{m,x} + rc_m - \beta q a_{m+1}) \\ \beta a_{m+1,x} & \lambda(b_{m,x} + rc_m - \beta q a_{m+1}) & 0 \end{bmatrix}, \quad m \geqslant 0,$$

and further, the corresponding zero curvature equations

$$U_{t_m} - (V_1^{[m]})_x + [U, V_1^{[m]}] = 0, \quad m \geqslant 0, \tag{2.12}$$

equivalently yield a hierarchy of soliton equations:

$$u_{t_m} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}_{t_m} = K_{1,m} = \begin{bmatrix} b_{m,x} + rc_m - \beta q a_{m+1} \\ c_{m,x} - rb_m + \beta p a_{m+1} \\ \beta a_{m+1,x} \end{bmatrix}, \quad m \geqslant 0. \tag{2.13}$$

Every system in this counterpart soliton hierarchy is local, since the functions $a_i, b_i, c_i,\ i \geqslant 1$, are all local. The first two systems in the counterpart hierarchy (2.13) read

$$\begin{cases} p_{t_0} = p_x + qr + \frac{1}{2}\beta(p^2+q^2)q, \\ q_{t_0} = q_x - pr - \frac{1}{2}\beta p(p^2+q^2), \\ r_{t_0} = -\beta(pp_x + qq_x), \end{cases} \tag{2.14}$$

and

$$\begin{cases} p_{t_1} = q_{xx} - \frac{3}{2}p^2 p_x - \frac{1}{2}p_x q^2 - pq_x q - 2p_x r - pr_x - qr^2 - \frac{1}{2}(p^2+q^2)qr - \frac{3}{8}\beta(p^2+q^2)^2 q - \beta(p_x q - pq_x)q - \beta(p^2+q^2)qr, \\ q_{t_1} = -p_{xx} - 2q_x r - qr_x - \frac{1}{2}p^2 q_x - p_x pq - \frac{3}{2}q_x q^2 + \frac{1}{2}p(p^2+q^2)r + pr^2 + \beta p(p_x q - pq_x) + \beta p(p^2+q^2)r + \frac{3}{8}\beta p(p^2+q^2)^2, \\ r_{t_1} = \beta[p_{xx}q - pq_{xx} + \frac{3}{2}(p_x p + q_x q)(p^2+q^2) + 2(p_x p + q_x q)r + (p^2+q^2)r_x]. \end{cases} \tag{2.15}$$

Let us now take a sequence of Lax operators with the second kind of modification terms:

$$V_2^{[m]} = \lambda(\lambda^{2m+1}W)_+ + \Delta_{2,m}, \quad \Delta_{2,m} = \beta a_m e_1 \in \widetilde{\mathrm{so}}(3,\mathbb{R}), \quad m \geqslant 0, \tag{2.16}$$

where β is an arbitrarily given constant. Similarly, we obtain

$$(V_2^{[m]})_x - [U, V_2^{[m]}] = \begin{bmatrix} 0 & -\lambda(c_{m,x} - rb_m + \beta p a_m) & -\beta a_{m,x} \\ \lambda(c_{m,x} - rb_m + \beta p a_m) & 0 & -\lambda(b_{m,x} + rc_m - \beta q a_m) \\ \beta a_{m,x} & \lambda(b_{m,x} + rc_m - \beta q a_m) & 0 \end{bmatrix}, \quad m \geqslant 0,$$

and thus, the corresponding zero curvature equations

$$U_{t_m} - (V_2^{[m]})_x + [U, V_2^{[m]}] = 0,\ m \geqslant 0, \tag{2.17}$$

present a second hierarchy of soliton equations:

$$u_{t_m} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}_{t_m} = K_{2,m} = \begin{bmatrix} b_{m,x} + rc_m - \beta q a_m \\ c_{m,x} - rb_m + \beta p a_m \\ \beta a_{m,x} \end{bmatrix}, \quad m \geqslant 0, \tag{2.18}$$

where every member is local. The first two systems in the counterpart hierarchy (2.18) are

$$\begin{cases} p_{t_0} = p_x + qr - \beta p, \\ q_{t_0} = q_x - pr + \beta p, \\ r_{t_0} = 0, \end{cases} \tag{2.19}$$

and

$$
\begin{cases} p_{t_1} = q_{xx} - \frac{1}{2}p_x(p^2+q^2) - p(pp_x+qq_x) - 2p_xr - pr_x - \frac{1}{2}(p^2+q^2)qr - qr^2 - \frac{1}{2}\beta q(p^2+q^2), \\ q_{t_1} = -p_{xx} - 2q_xr - qr_x - (pp_x+qq_x)q - \frac{1}{2}(p^2+q^2)q_x + \frac{1}{2}p(p^2+q^2)r + pr^2 + \frac{1}{2}\beta p(p^2+q^2), \\ r_{t_1} = \beta(p_xp+qq_x). \end{cases} \tag{2.20}
$$

3. Hamiltonian and quasi-Hamiltonian structures

We shall show that all systems in the counterpart soliton hierarchies (2.13) and (2.18) are Liouville integrable. Let us first establish Hamiltonian structures for the first counterpart hierarchy (2.13) and quasi-Hamiltonian structures for the second counterpart hierarchy (2.18).

We shall use the trace identity [6,8]:

$$
\frac{\delta}{\delta u}\int \mathrm{tr}\left(\frac{\partial U}{\partial \lambda}W\right)dx = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\,\mathrm{tr}\left(\frac{\partial U}{\partial u}W\right), \quad \gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\mathrm{tr}(W^2)|, \tag{3.1}
$$

or generally, the variational identity (see [7,21]). It is direct to see that

$$
\frac{\partial U}{\partial\lambda} = \begin{bmatrix} 0 & -q & -2\lambda \\ q & 0 & -p \\ 2\lambda & p & 0 \end{bmatrix}, \quad \frac{\partial U}{\partial p} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\lambda \\ 0 & \lambda & 0 \end{bmatrix}, \quad \frac{\partial U}{\partial q} = \begin{bmatrix} 0 & -\lambda & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \frac{\partial U}{\partial r} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},
$$

and so, we have

$$
\mathrm{tr}\left(W\frac{\partial U}{\partial\lambda}\right) = -4\lambda a - 2qc - 2pb, \quad \mathrm{tr}\left(W\frac{\partial U}{\partial p}\right) = -2\lambda b, \quad \mathrm{tr}\left(W\frac{\partial U}{\partial q}\right) = -2\lambda c, \quad \mathrm{tr}\left(W\frac{\partial U}{\partial r}\right) = -2a.
$$

Then, the trace identity (3.1) gives rise to

$$
\frac{\delta}{\delta u}\int (2\lambda a + qc + pb)\,dx = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\begin{bmatrix} \lambda b \\ \lambda c \\ a \end{bmatrix}.
$$

A balance of coefficients of λ^{-2m-1} for each $m \geqslant 0$ in the equality leads to

$$
\frac{\delta}{\delta u}\int (2a_{m+1} + qc_m + pb_m)\,dx = (\gamma - 2m)\begin{bmatrix} b_m \\ c_m \\ a_m \end{bmatrix}, \quad m \geqslant 0.
$$

The identity with $m = 1$ tells $\gamma = 0$, and thus, we can obtain

$$
\frac{\delta}{\delta u}\mathcal{H}_m = \begin{bmatrix} b_m \\ c_m \\ a_m \end{bmatrix}, \quad m \geqslant 0, \tag{3.2}
$$

with the Hamiltonian functionals being defined by

$$
\mathcal{H}_0 = \int \left[\frac{1}{2}(p^2+q^2) + r\right]dx, \ \mathcal{H}_m = \int \left(-\frac{2a_{m+1} + qc_m + pb_m}{2m}\right)dx, \quad m \geqslant 1. \tag{3.3}
$$

Obviously, it follows from (2.8) that

$$
K_{1,m} = \begin{bmatrix} -c_{m+1} + qa_{m+1} - \beta qa_{m+1} \\ b_{m+1} - pa_{m+1} + \beta pa_{m+1} \\ \beta a_{m+1,x} \end{bmatrix} = \begin{bmatrix} -c_{m+1} + (1-\beta)qa_{m+1} \\ b_{m+1} - (1-\beta)pa_{m+1} \\ \beta a_{m+1,x} \end{bmatrix}, \quad m \geqslant 0,
$$

and so, we obtain

$$
K_{1,m} = J_1\begin{bmatrix} b_{m+1} \\ c_{m+1} \\ a_{m+1} \end{bmatrix}, \quad J_1 = \begin{bmatrix} 0 & -1 & (1-\beta)q \\ 1 & 0 & -(1-\beta)p \\ -(1-\beta)q & (1-\beta)p & (2\beta-1)\partial \end{bmatrix}, \quad m \geqslant 0. \tag{3.4}
$$

A direct and easy computation by Maple shows that J_1 is a Hamiltonian operator. It follows now that the first counterpart soliton hierarchy (2.13) has the Hamiltonian structures:

$$
u_{t_m} = K_{1,m} = J_1\frac{\delta\mathcal{H}_{m+1}}{\delta u}, \quad m \geqslant 0, \tag{3.5}
$$

where the Hamiltonian functionals, $\{\mathcal{H}_m\}_{m=0}^{\infty}$, are given by (3.3) and the Hamiltonian operator J_1 is defined as in (3.4).

Similarly, we can have

$$K_{2,m} = J_2 \begin{bmatrix} b_m \\ c_m \\ a_m \end{bmatrix}, \quad J_2 = \begin{bmatrix} \partial & r & -\beta q \\ -r & \partial & \beta p \\ \beta q & -\beta p & 2\beta\partial \end{bmatrix}, \quad m \geqslant 0. \tag{3.6}$$

This operator is skew-symmetric but does not satisfy the Jacobi identity, which can be shown by using computer algebra systems. From this, we see that the second counterpart soliton hierarchy (2.18) has the quasi-Hamiltonian structures:

$$u_{t_m} = K_{2,m} = J_2 \frac{\delta \mathcal{H}_m}{\delta u}, \quad m \geqslant 0, \tag{3.7}$$

where $\mathcal{H}_m$'s are given by (3.3) and J_2 is defined as in (3.6).

The resulting functionals correspond to common conservation laws for each soliton system in the counterpart soliton hierarchy (2.13) or (2.18). We point out that such differential polynomial conservation laws can also be generated either directly by computer algebra codes (see, e.g., [22]) or from some Riccati equation obtained from the underlying matrix spectral problems (see, e.g., [23–25]).

Based on the Hamiltonian structures in (3.5) and the quasi-Hamiltonian structures in (3.7), we can now state that the counterpart soliton hierarchies (2.13) and (2.18) are Liouville integrable, i.e., they possesses infinitely many commuting conserved functionals and symmetries:

$$\{\mathcal{H}_k, \mathcal{H}_l\}_{J_i} = \int \left(\frac{\delta \mathcal{H}_k}{\delta u}\right)^T J_i \frac{\delta \mathcal{H}_l}{\delta u}\, dx = 0, \quad i = 1,2, \quad k,l \geqslant 0, \tag{3.8}$$

and

$$[K_{i,k}, K_{i,l}] = K'_{i,k}(u)[K_{i,l}] - K'_{i,l}(u)[K_{i,k}] = 0, \quad i = 1,2, \quad k,l \geqslant 0. \tag{3.9}$$

These commuting relations are also consequences of the Virasoro algebra of Lax operators. See [26–28] for a detailed and systematical study on algebraic structures of Lax operators and zero curvature equations.

4. Concluding remarks

Starting with the matrix loop algebra $\widetilde{so}(3,\mathbb{R})$, we introduced a counterpart matrix spectral problem of the D-Kaup–Newell spectral problem by using the same linear combination of basis matrices, and generated two integrable counterparts of the D-Kaup–Newell soliton hierarchy. All members in the resulting counterpart soliton hierarchies are Hamiltonian or quasi-Hamiltonian, and so, Liouville integrable, i.e., possess infinitely many commuting symmetries and conserved densities.

It is noted that the soliton hierarchy (2.13) is different from a soliton hierarchy presented in [29]. Although the two spectral problems of the two hierarchies look very similar and the two first nonlinear systems are almost the same, it is direct to see that the two hierarchies are very different from each other, if one checks the derivative terms in the two second nonlinear systems. Moreover, we did not find a second class of Hamiltonian structures for our soliton hierarchy (2.13) which needs to be compatible with the presented one (3.5). This is pretty different from the bi-Hamiltonian situation, particularly in the Drinfel'd–Sokolov hierarchies corresponding to the affine twisted Lie algebras [30], and also shows a big difference between our soliton hierarchy (2.13) and the soliton hierarchy by Xia et al. in [29], which was proved to possess a bi-Hamiltonian structure. Quasi-Hamiltonian structures can imply integrability as showed in the recent literature (see, e.g., [7]), but of course, they do not behave like bi-Hamiltonian structures, either.

We point out that among typical discussed spectral matrices associated with $\widetilde{so}(3,\mathbb{R})$ are the following three:

$$\begin{aligned} U(u,\lambda) &= \lambda e_1 + p e_2 + q e_3, \\ U(u,\lambda) &= \lambda^2 e_1 + \lambda p e_2 + \lambda q e_3, \\ U(u,\lambda) &= \lambda e_1 + \lambda p e_2 + \lambda q e_3, \end{aligned}$$

where $u = (p,q)^T$ includes two dependent variables. Our examples are two new soliton hierarchies with three dependent variables, fitting into the zero curvature formulation. We hope more examples of such soliton hierarchies with three dependent variables or even more dependent variables, such as four or five dependent variables, can be presented. It is recognized that given a starting matrix loop algebra, it normally needs only a considerable amount of time and computational dexterity to compute hierarchies of integrable equations. Higher-order matrix spectral problems can lead to soliton hierarchies but normally need a more considerable investment of time (see, e.g., [31–36]). Integrable couplings (see, e.g., [37–41]) associated with enlarged matrix loop algebras [42] provide more specific examples of soliton hierarchies generated from higher-order matrix spectral problems. They possess triangular forms [37,21] and their conserved densities can be generated by applying the variational identity [7,21].

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