Modern Physics Letters B
Vol. 33, No. 20 (2019) 1950229 (10 pages)

DOI: 10.1142/S0217984919502294

Families of analytic solutions for (2 + 1) model in unbounded domain via optimal Lie vectors with integrating factors

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Received 30 January 2019
Accepted 30 April 2019
Published 4 July 2019

Through the commutator table and the adjoint table between the infinitesimals, we apply two stages of Lie symmetry reduction to reduce the (2 + 1)-dimensional Boiti–Leon–Manna–Pempinelli (BLMP) equation to ordinary differential equations (ODE's). Some of these ODE's had no quadrature. We derive several new solutions for these non-solvable ODE's using Integrating Factors property.

Keywords: Integrating factors; Lie transformation; (2 + 1)-dimensional Boiti–Leon–Manna–Pempinelli (BLMP) equation; analytical solutions.

1. Introduction

Gilson *et al.*[1] derived (2 + 1)-dimensional Boiti–Leon–Manna–Pempinelli (BLMP) equation during their researches on water propagation

$$w_{yt} + w_{xxxy} - 3w_{xy}w_x - 3w_y w_{xx} = 0\,. \tag{1}$$

Mabrouk *et al.* in Ref. 2 derive the Lax pair of BLMP solitary wave solutions using the group transformation and present some explicit solutions for BLMP. In Ref. 3,

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the Wronskian determinant solutions of the $(2+1)$-dimensional BLMP equation were investigated and exact solutions for the (BLMP) equation were obtained. In Ref. 4, BLMP equation bilinear form is obtained based on the binary Bell polynomials and a general Riemann theta function in terms of the Hirota bilinear form and new exact solutions are presented. In Ref. 5, exact and explicit solutions of the BLMP equation are derived using the homogeneous balance method and the auto-Backlund transformation of the (BLMP) equation.

In this paper, we use Lie vectors to reduce the (BLMP) equation to ODE's. Some of the obtained ODE's had no quadrature. This is where our work begins; using integrating factors we evaluate exact solutions.

The paper is constituted of six sections.

In Sec. 2. We derive Lie symmetry vectors, using Maple.

In Sec. 3. We construct the optimal system using the adjoint Lie vectors.[6,7]

In Sec. 4. We reduce the (BLMP) to ordinary differential equation (ODE) in three steps. For each Lie (BLMP) vector, the following steps apply.

- BLMP partial differential equation variables $(x, y; t)$ are reduced to a PDE in two variables (r, s) whose Lie symmetries are evaluated.
- These symmetries are used for a further reduction of independent variables from $(r; s)$ to one variable (η).
- Non-solvable ODE equations are reduced to solvable ones, through their corresponding integrating factors.

In Sec. 5. We analyze all Lie vectors obtained. Each case is listed in Table 4.

In Sec. 6. Conclusions.

2. Mathematical Model

An investigation of its Lie vectors results in eight Lie vectors;

$$\begin{aligned} &X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = y\frac{\partial}{\partial y}, \quad X_4 = \frac{\partial}{\partial x}, \quad X_5 = \frac{\partial}{\partial w}, \\ &X_6 = t\frac{\partial}{\partial w}, \quad X_7 = t\frac{\partial}{\partial x} - \frac{x}{3}\frac{\partial}{\partial w}, \quad X_8 = \frac{x}{3}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{1}{3}w\frac{\partial}{\partial w}. \end{aligned} \tag{2}$$

3. Derivation of Lie Optimal Vectors

The commutative product of $X_1 - X_8$ vectors is first evaluated and listed in Table 1. Adjoint matrices of the vectors are then constructed using the relation;

$$Ad(e^{\varepsilon V})w_0 = w_0 - \varepsilon[V, w_0] + \frac{\varepsilon^2}{2}[V, [V, w_0]] \dots . \tag{3}$$

The adjoint matrices are constructed, following Refs. 13 and 14 work steps. Finally, through a backward procedure, epsilons in the adjoint matrix product listed in

Table 1. Commutator table.

$[V_1, V_2]$	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_1	0	0	0	0	0	X_5	X_4	X_1
X_2	0	0	X_2	0	0	0	0	0
X_3	0	$-X_2$	0	0	0	0	0	0
X_4	0	0	0	0	0	$\frac{-1}{3}X_5$	$-\frac{1}{3}X_5$	$\frac{1}{3}X_4$
X_5	0	0	0	0	0	0	0	$-\frac{1}{3}X_5$
X_6	$-X_5$	0	0	$\frac{1}{3}X_5$	0	0	0	$\frac{-4}{3}X_6$
X_7	$-X_4$	0	0	$\frac{1}{3}X_5$	0	0	0	$-\frac{2}{3}X_7$
X_8	$-X_1$	0	0	$\frac{-1}{3}X_4$	$\frac{1}{3}X_5$	$\frac{4}{3}X_6$	$\frac{2}{3}X_7$	0

Table 2. Adjoint table.

$Ad(V_1, V_2)$	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_1	X_1	X_2	X_3	X_4	X_5	$X_6 - \varepsilon_1 X_5$	$X_7 - \varepsilon_1 X_4$	$X_8 - \varepsilon_1 X_1$
X_2	X_1	X_2	$X_3 - \varepsilon_2 X_2$	X_4	X_5	X_6	X_7	X_8
X_3	X_1	$X_2 e^{\varepsilon_3}$	X_3	X_4	X_5	X_6	X_7	X_8
X_4	X_1	X_2	X_3	X_4	X_5	X_6	$X_7 + \frac{\varepsilon_4}{3}X_5$	$X_8 - \frac{\varepsilon_4}{3}X_4$
X_5	X_1	X_2	X_3	X_4	X_5	X_6	X_7	$X_8 + \frac{\varepsilon_5}{3}X_5$
X_6	$X_1 + \varepsilon_6 X_5$	X_2	X_3	X_4	X_5	X_6	X_7	$X_8 + \frac{4\varepsilon_6}{3}X_6$
X_7	$X_1 + \varepsilon_7 X_4 - \frac{\varepsilon_7^2}{6}X_5$	X_2	X_3	$X_4 - \frac{\varepsilon_7}{3}X_5$	X_5	X_6	X_7	$X_8 + \frac{2\varepsilon_7}{3}X_7$
X_8	$X_1 e^{\varepsilon_8}$	X_2	X_3	$X_4 e^{\varepsilon_8/3}$	$X_5 e^{-\varepsilon_8/3}$	$X_6 e^{-4\varepsilon_8/3}$	$X_7 e^{-2\varepsilon_8/3}$	X_8

Table 2 are evaluated and the optimal Lie vectors derived;

$$X_8, \quad X_7 + X_8, \quad X_5 + X_7 \,.$$

These vectors will be used to reduce Boiti equation to an ordinary differential equation.

4. Reduction of (BLMP) Equation Order

The original partial differential equation (1) function of three independent variables; $(x; y; t)$ is first reduced to a partial differential equation in two independent variables, $(r; s)$, using the optimal Lie vectors (2). These reductions are listed in Table 3.

Table 3. Reduction of variables (x, y, t) to (r, s).

Case	Symmetry variables	Reduction of Eq. (1)
X_8	$r = y,\ s = \dfrac{t}{x^3}$, $F(r,s) = xw(x,y,t)$	$(-F_{rs}+6F_r+114F_{rs}s+135F_{rss}s^2 + 27s^3(F_{rsss})+27s^2F_s(F_{rs})+63sF_sF_r + 9sFF_{rs}+9FF_r+27s^2F_rF_{ss})s = 0$
X_7+X_8	$r = -0.5\dfrac{-2x+3t}{(t)^{1/3}},\ s = y$, $F(r,s) = -\left(-w(x,y,t) + \dfrac{3}{8}t - 0.5x\right)(t)^{1/3}$	$F_s + F_{rs}r + 9F_{rs}F_r + 9F_{rr}F_s - 3F_{rrrs} = 0$
X_5+X_7	$r = y,\ s = -x+t$, $F(r,s) = w(x,y,t) - \dfrac{2}{3}s - 1$	$3F_{ss}F_r + F_{rs} + 3F_{rs}F_s + F_{sssr} = 0$
$X_1+X_4+X_8$	$r = y,\ s = \dfrac{t+1}{27+27x+9x^2+x^3}$, $F(r,s) = (x+3)w(xyt)$	$(-F_{rs}+6F_r+114F_{rs}s+135F_{rss}s^2 + 27s^3(F_{rsss})+27s^2F_s(F_{rs})+63sF_sF_r + 9sFF_{rs}+9FF_r+27s^2F_rF_{ss})s^2 = 0$
$X_1+X_6+X_7$	$r = y,\ s = -2x+t^2$, $F(r,s) = w(x,y,t) - \left(-\dfrac{1}{3}t+1\right) - \dfrac{1}{9}t^3$	$6F_{rs}-12F_{ss}F_r-12F_{rs}F_s-8F_{sssr} = 0$

4.1. *Reduction of* (1) *using* X_8 *Lie vector*

Equation (1) is transformed through the optimal vector $X_8 = \frac{x}{3}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{1}{3}w\frac{\partial}{\partial w}$ to;

$$-F_{rs} + 6F_r + 114F_{rs}s + 135F_{rss}s^2 + 27s^3(F_{rsss}) + 27s^2F_s(F_{rs}) + 63sF_sF_r + 9sFF_{rs} + 9FF_r + 27s^2(F_rF_{ss})s = 0\,. \tag{4}$$

This equation has no closed form solution but possesses six Lie vectors. we will choose here to work only with V_5 Lie vector;

$$V_5 = \frac{\partial}{\partial r} + s^{4/3}\frac{\partial}{\partial s} + \frac{-9sF+1}{27s^{2/3}}\frac{\partial}{\partial F}\,, \tag{5}$$

it leads to an ODE with no analytic solution, while the rest of the Lie vectors lead to solvable or non-solvable ODE's as described in Table 4. Using V_5 transform (4) to a nonlinear fourth degree ordinary differential equation of the form

$$\theta_{\eta\eta\eta\eta} - 2\theta_{\eta\eta}\theta_\eta^2 = 0, \quad \text{where } \eta = \frac{3+rs^{1/3}}{s^{1/3}}, \quad \theta(\eta) = \left(F(r,s) + \frac{1}{18s}\right)s^{1/3}\,. \tag{6}$$

This equation is not solvable. We here will use integrating factors to solve this equation.

Table 4. Analysis of each Lie vector.

Case	Lie vectors of reduced equation	Analysis of each vector	Integrating factors wave form
X_8	V_1, V_2, V_3, V_4, V_5, V_6	Solution of the ODE with no need to Lie reductions gives a 4th-order ODE with no analytical solution. Solution using integrating factors (two Lie reductions). Get a solution using fourth level of Lie reductions.	$w(x,y,t) = -\frac{x^2}{18t} - \frac{6}{x\left(3+y\left(\frac{t}{x^3}\right)^{\frac{1}{3}} + c_1\left(\frac{t}{x^3}\right)^{\frac{1}{3}}\right)} + \frac{c_2}{(t)^{1/3}}$
$X_7 + X_8$	V_1, V_2, V_3, V_4, V_5, V_6	Solution of the ODE with no need to Lie reductions. Solution using integrating factors. Get a solution using fourth levels of Lie reductions gives a 4th-order ODE with no analytical solution.	$w(x,y,t) = \frac{3}{8}t - 0.5x + \left(-\frac{1}{18}\left(-\frac{1}{2}\frac{-2x+3t}{(t)^{\frac{1}{3}}}\right)^2 + \frac{2}{\frac{-2x+3t}{2(t)^{\frac{1}{3}}} + y + c_1} + c_2\right)(t)^{-\frac{1}{3}}$
$X_5 + X_7$	V_1, V_2, V_3, V_4, V_5, V_6	Solution using integrating factors. Solution of the ODE with no need to Lie reductions gives a 4th-order ODE with no analytical solution.	$w(x,y,t) = -\tan(0.5(-y-x+t) + 0.5c_1) + 0.5c_2 + \frac{2}{3}(-x+t) + 1$ $w(x,y,t) = -\tan(-0.5(\ln(y)-x+t) + 0.5c_1) + 0.5c_2 + \frac{2}{3}(-x+t) + 1\,.$

4.1.1. *Using Lie integrating factor to get an explicit solution*

We first deduce Eq. (6) integrating factors using maple.

$$\mu_1 = \theta_\eta, \quad \mu_2 = 1\,. \tag{7}$$

The integrating factors reduces Eq. (7) to;

$$3{\theta_{\eta\eta}}^2 - 2\theta_\eta^3 = 0\,. \tag{8}$$

Equation (8) has a closed form solution of the form;

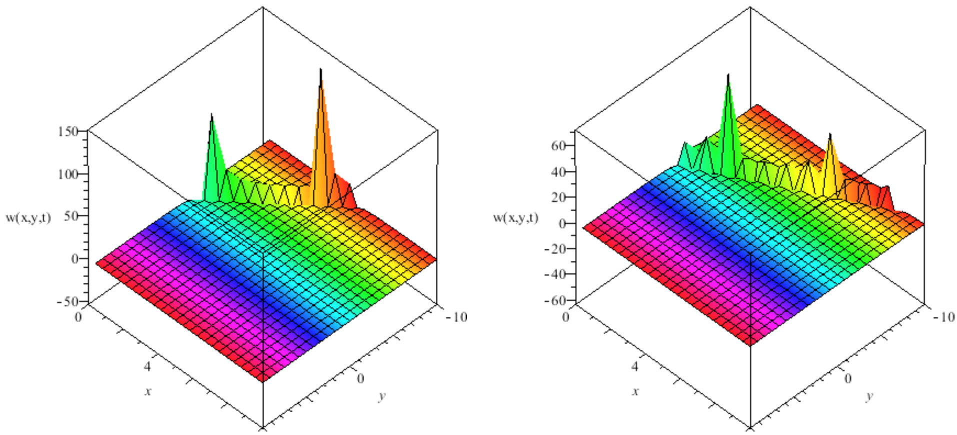
$$\theta(\eta) = -\frac{6}{\eta + c_1} + c_2\,, \tag{9}$$

where c_1, c_2 are integration constants.

Then back substituting to (x, y, t) where $r = y, s = \frac{t}{x^3}, F(r, s) = xw(x, y, t)$ we obtain

$$w(x,y,t) = -\frac{x^2}{18t} - \frac{6}{x\left(3 + y\left(\frac{t}{x^3}\right)^{\frac{1}{3}} + c_1\left(\frac{t}{x^3}\right)^{\frac{1}{3}}\right)} + \frac{c_2}{(t)^{1/3}}\,. \tag{10}$$

This result is plotted in Fig. 1 at various values of time.



(a) (b)

Fig. 1. (Color online) (a) $w(x, y, t)$ at $c_1 = 1$, $c_2 = 1$, $t = 5$ s and (b) $w(x, y, t)$ at $c_1 = 1$, $c_2 = 1$, $t = 20$ s.

By increasing the time value, the wave peaks move towards to right and the amplitude decrease.

4.2. *Reduction of* (1) *using* $X_7 + X_8$ *Lie vector*

This Lie vector reduces Eq. (1) to

$$F_s + F_{rs}r + 9F_{rs}F_r + 9F_{rr}F_s - 3F_{rrrs} = 0\,. \tag{11}$$

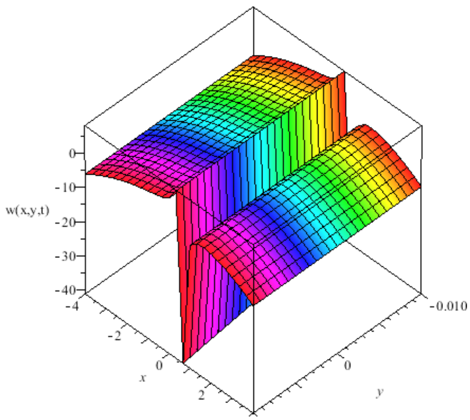
This equation has no closed form solution and possesses six Lie vectors;

$$\begin{aligned}
&V_1 = \frac{\partial}{\partial s} + \frac{\partial}{\partial F}, \quad V_2 = s\frac{\partial}{\partial s} + \frac{\partial}{\partial F}, \quad V_3 = \frac{\partial}{\partial r} + \frac{\partial}{\partial s} - \frac{1}{9}r\frac{\partial}{\partial F}\,,\\
&V_4 = \frac{\partial}{\partial r} + s\frac{\partial}{\partial s} - \frac{1}{9}r\frac{\partial}{\partial F}, \quad V_5 = r\frac{\partial}{\partial r} + \frac{\partial}{\partial s} - \left(F + \frac{1}{6}r^2\right)\frac{\partial}{\partial F}\,,\\
&V_6 = r\frac{\partial}{\partial r} + s\frac{\partial}{\partial s} - \left(F + \frac{1}{6}r^2\right)\frac{\partial}{\partial F}\,.
\end{aligned} \tag{12}$$

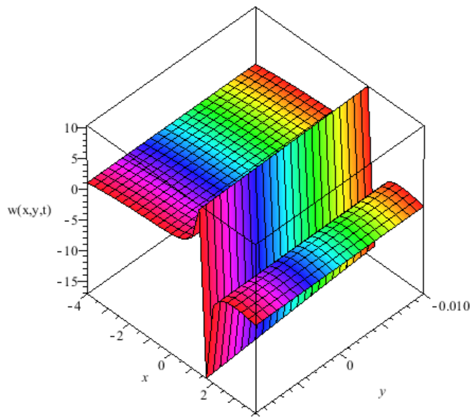
We did test all above Lie vector and found that they lead to solvable ODE's as described in Table 4. In this table we notice that V_3 transform (11) to a non-solvable nonlinear fourth degree ordinary differential equation of the form

$$\theta_{\eta\eta\eta\eta} + 6\theta_{\eta\eta}\theta_\eta^2 = 0\,, \tag{13}$$

where $\theta(\eta) = F(r, s) + \frac{1}{18}r^2$ and $\eta = -r + s$. We here use the integrating factors to solve this equation.



(a)



(b)

Fig. 2. (Color online) (a) $w(x,y,t)$ at $c_1 = 1$, $c_2 = 1$, $t = 0.1$ s and (b) $w(x,y,t)$ at $c_1 = 1$, $c_2 = 1$, $t = 0.5$ s.

4.2.1. *Using integrating factor technique to get an explicit solution*

We first deduce the integrating factors of Eq. (13) using maple.

$$\mu_1 = \theta_\eta, \quad \mu_2 = 1. \tag{14}$$

The first integrating factor reduces Eq. (13) to

$$\theta_{\eta\eta}^2 - 2\theta_\eta^3 = 0. \tag{15}$$

This equation has closed form solution of the form

$$\theta(\eta) = \frac{-2}{\eta + c_1} + c_2, \tag{16}$$

where c_1, c_2 are integration constants. Back substituting in (16) with $r = -0.5\frac{-2x+3t}{(t)^{\frac{1}{3}}}$, $s = y$ and $F(r,s) = -(-w(x,y,t) + \frac{3}{8}t - 0.5x)(t)^{1/3}$ we obtain

$$w(x,y,t) = \frac{3}{8}t - 0.5x + \left(-\frac{1}{18}\left(-\frac{1}{2}\frac{-2x+3t}{(t)^{\frac{1}{3}}}\right)^2 + \frac{2}{\frac{-2x+3t}{2(t)^{\frac{1}{3}}} + y + c_1} + c_2\right)(t)^{-\frac{1}{3}}. \tag{17}$$

This result is plotted in Fig. 2.

Rapid decreasing of the amplitude during small increasing of the time.

4.3. *Reduction of (1) using $X_5 + X_7$ Lie vector*

This Lie vector reduces (1) to

$$3F_{ss}F_r + F_{rs} + 3F_{rs}F_s + F_{sssr} = 0. \tag{18}$$

This equation has no closed form solution and possesses six Lie vectors obtained by using maple software

$$\begin{aligned} &V_1 = \frac{\partial}{\partial r} + \frac{\partial}{\partial s}, \quad V_2 = r\frac{\partial}{\partial r} + \frac{\partial}{\partial s}, \quad V_3 = \frac{\partial}{\partial r} + \frac{\partial}{\partial F}, \quad V_4 = r\frac{\partial}{\partial r} + \frac{\partial}{\partial F}\,, \\ &V_5 = \frac{\partial}{\partial r} + s\frac{\partial}{\partial s} - \left(F + \frac{2}{3}s\right)\frac{\partial}{\partial F}, \quad V_6 = r\frac{\partial}{\partial r} + s\frac{\partial}{\partial s} - \left(F + \frac{2}{3}s\right)\frac{\partial}{\partial F}\,, \end{aligned} \tag{19}$$

where V_1 and V_2 transformation of (18) are giving ODE's with no quadrature while the rest of the vectors V_3 give solvable ODE's or equations with no Lie vectors. We will concentrate on V_1 and V_2 transformations in the foregoing steps.

- V_1 transform Eq. (18) to a nonlinear fourth degree ordinary differential equation of the form

$$\theta_{\eta\eta\eta\eta} + 6\theta_{\eta\eta}\theta_\eta^2 + \theta_{\eta\eta} = 0\,. \tag{20}$$

  This equation is not solvable. We here will use the integrating factors to solve this equation.

4.3.1. *Using the Lie integrating factor to get an explicit solution*

We first deduce the Integrating Factors of (20) using maple.

$$\mu_1 = \theta_\eta, \quad \mu_2 = 1\,. \tag{21}$$

The integrating factor reduces Eq. (20) to

$$\theta_{\eta\eta}^2 + \theta_\eta^2 + 2\theta_\eta^3 = 0\,. \tag{22}$$

This equation has a closed form solution

$$\theta(\eta) = -0.5\eta + c_1\,, \tag{23}$$

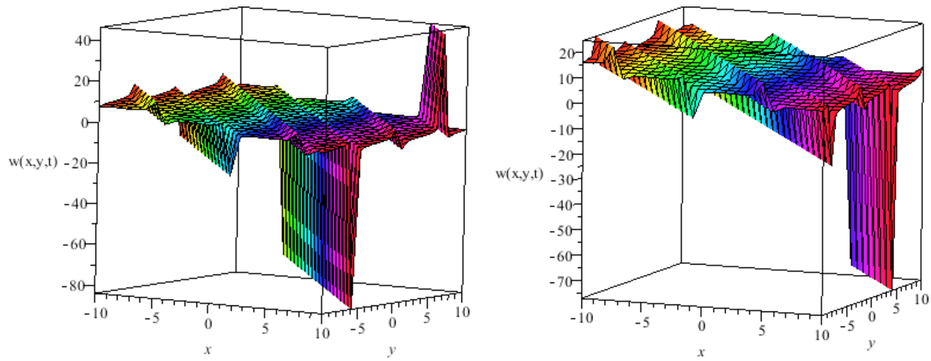
where $\eta = -r + s$ and $r = y$, $s = -x + t$,

$$\theta(\eta) = -\tan(0.5\eta + 0.5c_1) + 0.5c_2\,, \tag{24}$$

where c_1, c_2 are integration constants $\theta(\eta) = F(r,s)$. Back substituting, we obtain

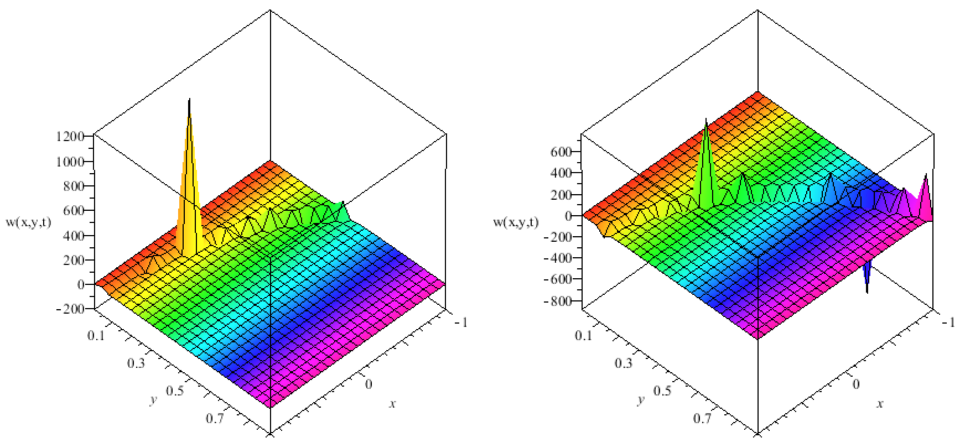
$$\begin{aligned} F(r,s) &= w(x,y,t) - \frac{2}{3}s - 1\,, \\ w(x,y,t) &= -\tan(0.5(-y - x + t) + 0.5c_1) + 0.5c_2 + c_1 + \frac{2}{3}(-x + t) + 1\,. \end{aligned} \tag{25}$$

This result is plotted in Fig. 3.



(a) (b)

Fig. 3. (Color online) (a) $w(x,y,t)$ at $c_1 = 1$, $c_2 = 1$, $t = 0$ s and (b) $w(x,y,t)$ at $c_1 = 1$, $c_2 = 1$, at $t = 10$ s.



(a) (b)

Fig. 4. (Color online) (a) $w(x,y,t)$ at $c_1 = 1$, $c_2 = 1$, $t = 0.2$ s and (b) $w(x,y,t)$ at $c_1 = 1$, $c_2 = 1$, $t = 1$ s.

Travelling wave solution obtained, increasing the time move the peaks towards right and decreasing the amplitude.

- **Using $\mathbf{V_2}$**, it will transform (18) to same Eq. (22). But in back substitution process, we find that $\eta = -\ln(r) + s$, so Eq. (24) return to (x, y, t) in the form

$$w(x,y,t) = -\tan(-0.5(\ln(y) - x + t) + 0.5c_1) + 0.5c_2 + \frac{2}{3}(-x + t) + 1\,. \quad (26)$$

This result is plotted in Fig. 4 for various values of times.

Our result duplicated in Fig. 4 as a novel solution for Eq. (1).

5. Analysis of the Reduced Equations for Each Lie Vector

In this section the reduced equations solutions, considered in the four previous paragraphs are summarized in Table 4.

6. Conclusions

New explicit solutions of $(2+1)$ BLMP equation are obtained using two Lie reduction stages with the Integrating Factors method property for ODEs having no quadrature. We did replace successive Lie reductions process by integrating factors. As a summary the advantages of using Integrating Factors are:

- Get new and different solutions if we use Lie symmetry reduction from A to z.
- Reduce the reduction stages by using Lie symmetry reduction.
- The Integrating Factors method leads to a solution in less steps than the Lie reduction.
- The Integrating Factors method overcome the problems as depicted in Ref. 8 of Lie symmetry reduction method (back substitution problems).

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