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# Characteristics and interactions of solitary and lump waves of a (2 + 1)-dimensional coupled nonlinear partial differential equation

**Bo Ren · Wen-Xiu Ma · Jun Yu**

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**Abstract** A (2 + 1)-dimensional coupled nonlinear partial equation which possesses a Hirota bilinear form is introduced. Based on the Hirota bilinear form, two solitary waves are constructed. In the meanwhile, lump waves are derived by using a positive quadratic function. By combining an exponential function with a quadratic function, interaction solutions between a lump and a one-kink soliton, and between a bi-lump and a one-soliton solution are generated. Some special concrete interaction solutions are depicted in both analytical and graphical ways.

**Keywords** Nonlinear partial differential equation · Hirota bilinear form · Solitary wave · Lump wave · Interaction solution

B. Ren (✉) · J. Yu
Institute of Nonlinear Science, Shaoxing University, Shaoxing 312000, China
e-mail: renbosemail@163.com

W.-X. Ma
Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620-5700, USA

W.-X. Ma
Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China

W.-X. Ma
Department of Mathematics, King Abdulaziz University, Jeddah, Saudi Arabia

W.-X. Ma
College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, Shandong, China

W.-X. Ma
International Institute for Symmetry Analysis and Mathematical Modelling, Department of Mathematical Sciences, North-West University, Mafikeng Campus, Private Bag X2046, Mmabatho 2735, South Africa

## 1 Introduction

Nonlinear partial differential equations are applied to solving some complex problems in a variety of science and engineering [1–9]. Finding exact solutions plays an important role in nonlinear science. Among these exact solutions, solitary waves and lump solutions can be used to study natural phenomena appeared in fluids, engineering and nonlinear optics [10–14]. Lump waves which have attracted much attention are localized in all directions of spaces [12]. The study of this field is mainly by means of the Darboux transformation [15–18] and the Hirota bilinear method [19–28]. To describe complex physical phenomena, hybrid interaction solutions are widely investigated by combining different variable functions [29–39]. Interaction solutions among multi-soliton and other complicated waves are discussed by the localization procedure related to the nonlocal symmetry and the consistent tanh expansion method [29–32]. Interaction solutions between lump waves and multi-soliton are studied by using the Hirota bilinear method [33–40].

In this paper, we consider a $(2+1)$-dimensional coupled nonlinear partial differential equation (cNPDE)

$$\begin{aligned}
&u_{xt}+\frac{3}{2}u_x u_{xx}+\frac{1}{4}u_{xxxx}+\delta_1 w_x+\delta_2 u_{xy}\\
&\quad+\delta_3 u_{xx}+\frac{\delta_4}{4}(u_{xxxy}+3u_x u_{xy}\\
&\quad+3u_{xx}u_y)+\frac{\delta_5}{4}(w_{xxy}+3u_{xy}w+3u_y w_x)\\
&\quad+\delta_6\left(3ww_x+\frac{1}{2}w_{xyy}\right)=0,\\
&u_{yy}-w_x=0,
\end{aligned} \tag{1}$$

where $\delta_i$ $(i=1,2,\ldots,6)$ are arbitrary constants. Equation (1) reduces to a $(2+1)$-dimensional potential Kadomtsev–Petviashvili (pKP) equation by choosing $\delta_2=\delta_3=\delta_4=\delta_5=\delta_6=0$, which describes the dynamics of a wave with a small amplitude. The periodic kink wave and the group-invariant solutions of the pKP equation have been derived [41,42]. The nonlocal symmetry and interaction solutions of the pKP equation have been given by the localization procedure related to nonlocal symmetries [43].

This paper is organized as follows: in Sect. 2, we construct the Hirota bilinear form of Eq. (1) by using the Painlevé–Bäcklund transformation. In Sect. 3, we obtain two solitary waves by introducing a perturbation expansion. Lump waves are presented by solving the corresponding Hirota bilinear form in Sect. 4. In Sect. 5, interaction solutions between a lump and a one-kink soliton, and between a bi-lump and a one-soliton solution are derived by adding an exponential function to a quadratic function. The last section is a simple summary and discussion.

## 2 A bilinear form of a coupled nonlinear partial differential equation

Based on the Painlevé analysis [44], a Painlevé–Bäcklund transformation of Eq. (1) reads

$$u=\frac{u_0}{\phi}+u_1,\quad w=\frac{w_0}{\phi^2}+\frac{w_1}{\phi}+w_2, \tag{2}$$

where $\phi$ is an auxiliary function of the variables $x$, $y$ and $t$. The functions of $u_1$ and $w_2$ are arbitrary seed solution of Eq. (1). Substituting (2) into (1) and balancing the coefficients $\phi^{-5}$ and $\phi^{-3}$, we get

$$u_0=2\phi_x,\quad w_0=-2\phi_y^2. \tag{3}$$

Balancing the coefficient $\phi^{-4}$ gives

$$w_1=2\phi_{yy}. \tag{4}$$

Substituting (3), (4) and the seed solution $u_1=0$, $w_2=0$ into (2), we get

$$u=\frac{2\phi_x}{\phi},\quad w=-\frac{2\phi_y^2}{\phi^2}+\frac{2\phi_{yy}}{\phi}. \tag{5}$$

A bilinear form of (1) is yielded

$$\begin{aligned}
&2\phi\phi_{xt}-2\phi_t\phi_x+\frac{1}{2}\phi\phi_{xxxx}-2\phi_x\phi_{xxx}\\
&\quad+\frac{3}{2}\phi_{xx}^2+2\delta_1(\phi\phi_{yy}-\phi_y^2)\\
&\quad+2\delta_2(\phi\phi_{xy}-\phi_x\phi_y)+2\delta_3(\phi\phi_{xx}-\phi_x^2)\\
&\quad+\delta_4(\phi\phi_{xxxy}-\phi_{xxx}\phi_y-3\phi_x\phi_{xxy}+3\phi_{xx}\phi_{xy})\\
&\quad+\delta_5(\phi\phi_{xyyy}-\phi_x\phi_{yyy}-3\phi_y\phi_{xyy}+3\phi_{xy}\phi_{yy})\\
&\quad+\delta_6(\phi\phi_{yyyy}-4\phi_y\phi_{yyy}+3\phi_{yy}^2)=0.
\end{aligned} \tag{6}$$

The bilinear equation (6) has the following equivalent formula:

$$\begin{aligned}
&D_tD_x+\frac{1}{4}D_x^4+\delta_1D_y^2+\delta_2D_xD_y+\delta_3D_x^2\\
&\quad+\frac{\delta_4}{2}D_x^3D_y+\frac{\delta_5}{2}D_xD_y^3+\frac{\delta_6}{2}D_y^4=0,
\end{aligned} \tag{7}$$

with the $D$-operators defined by

$$\begin{aligned}
&D_x^lD_y^nD_t^m f(x,y,t)\cdot g(x',y',t')\\
&\quad=\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^l\left(\frac{\partial}{\partial y}-\frac{\partial}{\partial y'}\right)^n\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^m\\
&\qquad f(x,y,t)\cdot g(x',y',t')|_{x=x',y=y',t=t'}.
\end{aligned} \tag{8}$$

## 3 Solitary waves of a coupled nonlinear partial differential equation

The Hirota bilinear method has been widely used to solve a class of nonlinear evolution equations [45]. Based on the Hirota bilinear method, we assume that a two-front wave for $\phi$ has a perturbation expansion

$$\phi=1+\exp(\theta_1)+\exp(\theta_2), \tag{9}$$

where $\theta_1=a_1x+b_1y+c_1t$, $\theta_2=a_2x+b_2y+c_2t$, and $a_1$, $b_1$, $c_1$, $a_2$, $b_2$ and $c_2$ are arbitrary constants. Inserting (9) into (6) and solving the coefficients of different powers of the exponent functions, a relation among the arbitrary constants reads

$$c_1 = -\frac{a_1^3}{4} - \delta_1 \frac{b_1^2}{a_1} - \delta_2 b_1 - \delta_3 a_1 - \delta_4 \frac{a_1^2 b_1}{4} - \delta_5 \frac{b_1^3}{4} - \delta_6 \frac{b_1^4}{2a_1},$$
$$c_2 = -\frac{a_2^3}{4} - \delta_1 \frac{b_2^2}{a_2} - \delta_2 b_2 - \delta_3 a_2 - \delta_4 \frac{a_2^2 b_2}{4} - \delta_5 \frac{b_2^3}{4} - \delta_6 \frac{b_2^4}{2a_2}, \quad (10)$$

where $\delta_6$ satisfies

$$\delta_6 = \Big(\frac{3}{2}a_1^2 a_2^2 (a_1 - a_2)^2 - 2\delta_1 (a_1 b_2 - a_2 b_1)^2 + \frac{1}{2}\delta_4 a_1 a_2 (a_1 - a_2)(a_1^2 b_2 + 2a_1 a_2 (b_1 - b_2) - a_2^2 b_1) + \frac{3}{2}\delta_5 a_1 a_2 b_1 b_2 (a_1 - a_2)(b_1 - b_2)\Big)\Big/\big(a_1^2 b_2^4 - 2a_1 a_2 b_1 b_2 (2b_1^2 - 3b_1 b_2 + 2b_2^2) + a_2^2 b_1^4\big). \quad (11)$$

Substituting (9) into (5) yields a two-front wave

$$u = \frac{2(a_1 \exp(\theta_1) + a_2 \exp(\theta_2))}{1 + \exp(\theta_1) + \exp(\theta_2)},$$
$$w = -\frac{2(b_1 \exp(\theta_1) + b_2 \exp(\theta_2))^2}{(1 + \exp(\theta_1) + \exp(\theta_2))^2} + \frac{2(b_1^2 \exp(\theta_1) + b_2^2 \exp(\theta_2))}{1 + \exp(\theta_1) + \exp(\theta_2)}, \quad (12)$$

where $c_1, c_2$ and $\delta_6$ satisfy (10) and (11). We show a two-front wave for $u$ and $w$ with specific parameters $a_1 = -\frac{1}{2}, a_2 = \frac{1}{2}, b_1 = \frac{1}{2}, b_2 = \frac{1}{2}, \delta_1 = -1, \delta_2 = 1, \delta_3 = 2, \delta_4 = 1, \delta_5 = 2$ in Fig. 1, and another kind of a two-front wave with specific parameters $a_1 = -\frac{1}{3}, a_2 = -\frac{1}{2}, b_1 = 1, b_2 = \frac{1}{2}, \delta_1 = -1, \delta_2 = 1, \delta_3 = 2, \delta_4 = 1, \delta_5 = 2$ in Fig. 2. The solution of $w$ is shown as "U"-shaped and "Y"-shaped in Figs. 1b and 2b, respectively. Characteristics of two

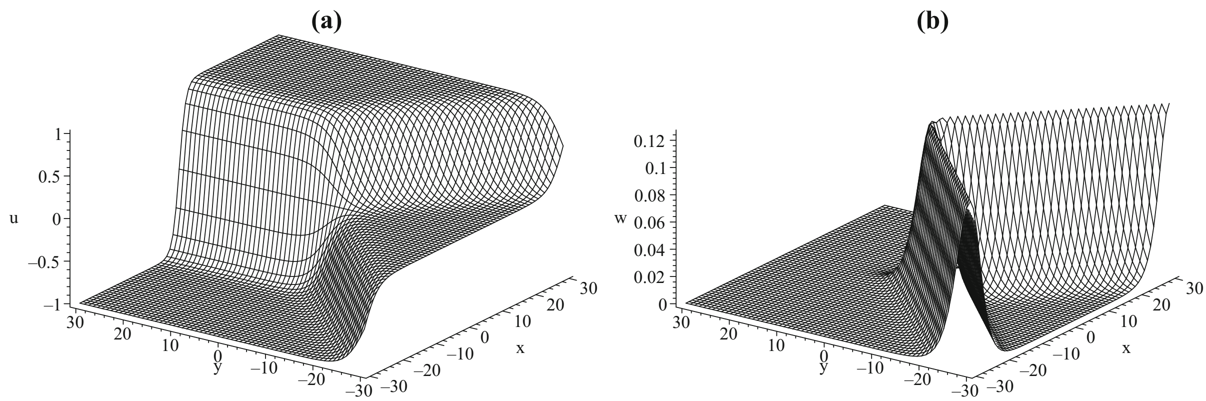


**Fig. 1** Profile of a two-front wave (12): **a** 3-dimensional plot of $u$ with $t = 0$, **b** 3-dimensional plot of $w$ with $t = 0$

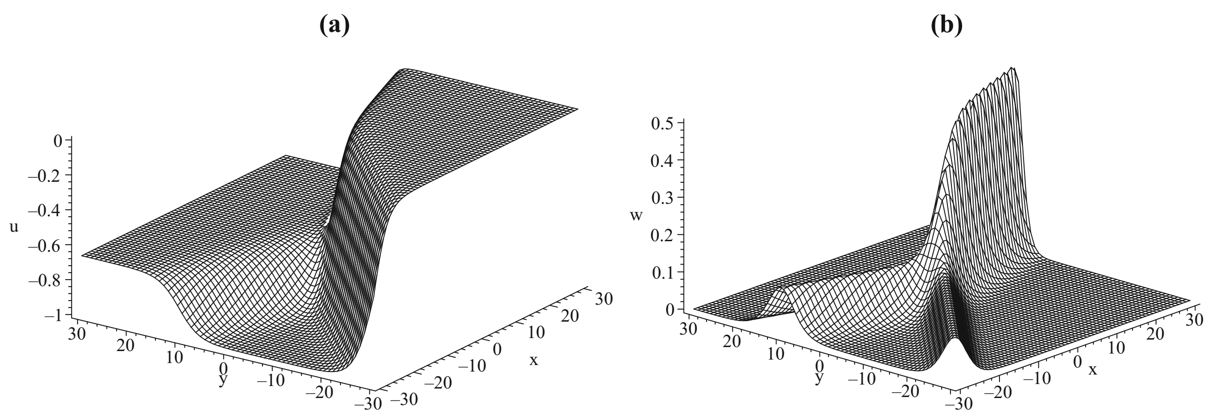


**Fig. 2** Profile of a two-front wave (12): **a** 3-dimensional plot of $u$ with $t = 0$, **b** 3-dimensional plot of $w$ with $t = 0$

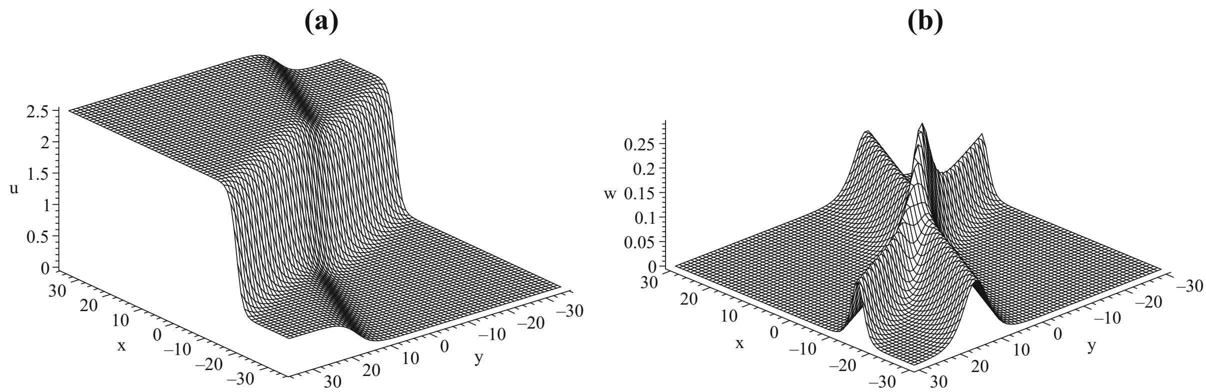


**Fig. 3** Profile of two-soliton solution (16): **a** 3-dimensional plot of $u$ with $t = 0$, **b** 3-dimensional plot of $w$ with $t = 0$

front waves are thus different by selecting different parameters.

For a two-soliton solution, we assume

$$\phi = 1 + \exp(\theta_1) + \exp(\theta_2) + a_{12}\exp(\theta_1 + \theta_2), \quad (13)$$

where $a_1$, $b_1$, $c_1$, $a_2$, $b_2$, $c_2$ and $a_{12}$ are arbitrary parameters to be determined. Substituting (13) into (6) and solving the coefficients of different powers of the exponent functions, a relation among the arbitrary constants is

$$\begin{aligned} c_1 &= -\frac{a_1^3}{4} - \delta_1\frac{b_1^2}{a_1} - \delta_2 b_1 - \delta_3 a_1 - \delta_4\frac{a_1^2 b_1}{4} \\ &\quad - \delta_5\frac{b_1^3}{4} - \delta_6\frac{b_1^4}{2a_1}, \\ c_2 &= -\frac{a_2^3}{4} - \delta_1\frac{b_2^2}{a_2} - \delta_2 b_2 - \delta_3 a_2 \\ &\quad - \delta_4\frac{a_2^2 b_2}{4} - \delta_5\frac{b_2^3}{4} - \delta_6\frac{b_2^4}{2a_2}, \end{aligned} \quad (14)$$

where $\delta_5$ and $\delta_6$ satisfy

$$\begin{aligned} \delta_5 &= \frac{1}{A}\Big(\frac{2a_1^2a_2^2}{b_1b_2}\big(b_1a_2(b_1^2+3b_2^2) - a_1b_2(3b_1^2+b_2^2)\big) \\ &\quad + \delta_4\frac{a_1a_2}{b_1b_2}\big(a_2^2b_1(b_1^2+2b_2^2) \\ &\quad + 2a_1a_2b_1b_2(b_2^2-b_1^2) - a_1^2b_2^2(2b_1^2+b_2^2)\big) \\ &\quad + 8\delta_1b_1b_2(a_1b_2-a_2b_1)\Big), \\ \delta_6 &= \frac{1}{A}\Big(\frac{3}{2}a_1^2a_2^2(a_1^2-a_2^2) - 2\delta_1(a_1^2b_2^2-a_2^2b_1^2) \\ &\quad + \frac{\delta_4}{2}a_1a_2(a_1^2-a_2^2)(a_1b_2+a_2b_1)\Big), \\ A &= a_1^2b_2^4 + 2a_1a_2b_1b_2(b_1^2-b_2^2) - a_2^2b_1^4. \end{aligned} \quad (15)$$

Substituting (13) into (5) yields a two-soliton solution

$$\begin{aligned} u &= \frac{2(a_1\exp(\theta_1)+a_2\exp(\theta_2)+a_{12}(a_1+a_2)\exp(\theta_1+\theta_2))}{1+\exp(\theta_1)+\exp(\theta_2)+a_{12}\exp(\theta_1+\theta_2)}, \\ w &= -\frac{2(b_1\exp(\theta_1)+b_2\exp(\theta_2)+a_{12}(b_1+b_2)\exp(\theta_1+\theta_2))^2}{(1+\exp(\theta_1)+\exp(\theta_2)+a_{12}(a_1+a_2)\exp(\theta_1+\theta_2))^2} \\ &\quad + \frac{2(b_1^2\exp(\theta_1)+b_2^2\exp(\theta_2)+a_{12}(b_1+b_2)^2\exp(\theta_1+\theta_2))}{1+\exp(\theta_1)+\exp(\theta_2)+a_{12}(a_1+a_2)\exp(\theta_1+\theta_2)}. \end{aligned} \quad (16)$$

To illustrate this two-soliton solution (16), we select the parameters $a_1 = 1, a_2 = \frac{1}{4}, b_1 = \frac{1}{2}, b_2 = \frac{1}{2}, a_{12} = 2, \delta_1 = -1, \delta_2 = 2, \delta_3 = 1, \delta_4 = 3, \delta_5 = 2$. The interactions between two kink solitons and two solitons are shown in Fig. 3a, b, respectively.

## 4 Lump waves of a coupled nonlinear partial differential equation

To get lump waves of Eq. (1), we take a quadratic function $\phi$ as

$$\begin{aligned} \phi &= g^2 + h^2 + a_9, \\ g &= a_1x + a_2y + a_3t + a_4, \\ h &= a_5x + a_6y + a_7t + a_8. \end{aligned} \quad (17)$$

where $a_i$ $(i = 1, 2, \ldots, 9)$ are arbitrary parameters. By substituting (17) into (7) and balancing different powers of $x$, $y$ and $t$, we get the solutions of $a_i$'s

$$\begin{aligned} a_3 &= -\frac{\delta_1(a_1a_2^2 + 2a_2a_5a_6 - a_1a_6^2)}{a_1^2+a_5^2} - \delta_2a_2 - \delta_3a_1, \\ a_9 &= -\frac{3\delta_4(a_1^2+a_5^2)^2(a_1a_2+a_5a_6)}{4\delta_1(a_1a_6-a_2a_5)^2} \end{aligned}$$

$$- \frac{3\delta_5(a_1^2+a_5^2)(a_2^2+a_6^2)(a_1a_2+a_5a_6)}{4\delta_1(a_1a_6-a_2a_5)^2} - \frac{3\delta_6(a_1^2+a_5^2)(a_2^2+a_6^2)^2}{4\delta_1(a_1a_6-a_2a_5)^2} - \frac{3(a_1^2+a_5^2)^3}{4\delta_1(a_1a_6-a_2a_5)^2},$$

$$a_7 = -\frac{\delta_1(2a_1a_2a_6 - a_2^2a_5 + a_5a_6^2)}{a_1^2+a_5^2} - \delta_2a_6 - \delta_3a_5, \tag{18}$$

which should satisfy the following constraint conditions:

$$\delta_1a_5 \neq 0, \quad a_1a_6 - a_2a_5 \neq 0,$$
$$\delta_1\Big[(a_1^2+a_5^2)(a_1^2+a_5^2+\delta_4(a_1a_2+a_5a_6)) + (a_2^2+a_6^2)(2\delta_6(a_2^2+a_6^2) + \delta_5(a_1a_2+a_5a_6))\Big] < 0, \tag{19}$$

so that the localization of $u$ and $w$ in all directions of the $(x, y)$-plane is guaranteed. A class of lump waves of Eq. (1) is thus generated

$$u = \frac{4a_1g + 4a_5h}{\phi},$$
$$w = -\frac{8(a_2g+a_6h)^2}{\phi^2} + \frac{4a_2^2+4a_6^2}{\phi}, \tag{20}$$

where

$$\phi = g^2+h^2 - \frac{3\delta_4(a_1^2+a_5^2)^2(a_1a_2+a_5a_6)}{4\delta_1(a_1a_6-a_2a_5)^2} - \frac{3\delta_5(a_1^2+a_5^2)(a_2^2+a_6^2)(a_1a_2+a_5a_6)}{4\delta_1(a_1a_6-a_2a_5)^2} - \frac{3\delta_6(a_1^2+a_5^2)(a_2^2+a_6^2)^2}{4\delta_1(a_1a_6-a_2a_5)^2} - \frac{3(a_1^2+a_5^2)^3}{4\delta_1(a_1a_6-a_2a_5)^2},$$
$$g = a_1x+a_2y - \left(\frac{\delta_1(a_1a_2^2+2a_2a_5a_6-a_1a_6^2)}{a_1^2+a_5^2} + \delta_2a_2+\delta_3a_1\right)t+a_4,$$
$$h = a_5x+a_6y - \left(\frac{\delta_1(2a_1a_2a_6+a_2^2a_5+a_5a_6^2)}{a_1^2+a_5^2} + \delta_2a_6-\delta_3a_5\right)t+a_8. \tag{21}$$

To catch the moving path of the lump waves in (20), the critical point of the lump waves is calculated by solving $\phi_x = \phi_y = 0$. The exact moving path of the lump waves is written as

$$x = x(t) = \frac{(a_2a_7-a_3a_6)t - (a_2a_8-a_4a_6)}{a_1a_6-a_2a_5},$$
$$y = y(t) = \frac{(a_1a_7-a_3a_5)t - (a_1a_8-a_4a_5)}{a_1a_6-a_2a_5}, \tag{22}$$

which can describe the traveling path of the lump waves along a straight line

$$y = \frac{a_3a_5-a_1a_7}{a_2a_7-a_3a_6}x + \frac{a_3a_8-a_4a_7}{a_2a_7-a_3a_6}, \tag{23}$$

with $a_3$, $a_7$ and $a_9$ satisfying (18). The parameters are selected as $a_1 = -1, a_2 = 2, a_4 = -3, a_5 = 1, a_6 = 3, a_8 = 2, \delta_1 = -1, \delta_2 = 2, \delta_3 = 3, \delta_4 = 2, \delta_5 = 1, \delta_6 = 1$ in Figs. 4 and 5. A lump wave of $u$ is plotted in Fig. 4. The spatial structure of a lump wave is described in Fig. 4a. From Fig. 4a, we can easily know that the lump wave has a localized characteristic at $t = 0$. A bi-lump wave of $w$ is plotted in Fig. 5. The spatial structure of a bi-lump wave is described in Fig. 5a at $t = 0$. Figures 4b and 5b represent the corresponding density plots of the lump wave. Figure 4c displays the contour plot of the lump wave at $t = -35, t = 0, t = 36$. Figure 5c is the contour plot of the lump wave at $t = -20, t = 0, t = 20$. The blue line of Figs. 4c and 5c is the relevant moving progress (23), i.e., $y = \frac{2}{19}x + \frac{9}{19}$. The wave along $x$-axis of the lump wave is depicted in Figs. 4d and 5d.

## 5 Interaction solution between a lump and a one-soliton solution

Interaction solutions between lumps and other type solutions can be constructed by mixing a quadratic function with other type functions. In order to find interaction solution between lump waves and a one-soliton solution, we assume that an interaction solution is determined by a sum of a quadratic function and an exponential function

$$\phi = g^2 + h^2 + a_9 + k_1\exp(k_2x + k_3y + k_4t + k_5),$$
$$g = a_1x + a_2y + a_3t + a_4,$$
$$h = a_5x + a_6y + a_7t + a_8, \tag{24}$$

with $k_i$ $(i = 1, 2, \ldots, 5)$ being five undetermined real parameters. By substituting (24) into (6) and vanishing the different powers of $x$, $y$ and $t$, we obtain the following two cases of constraining relations for the parameters:

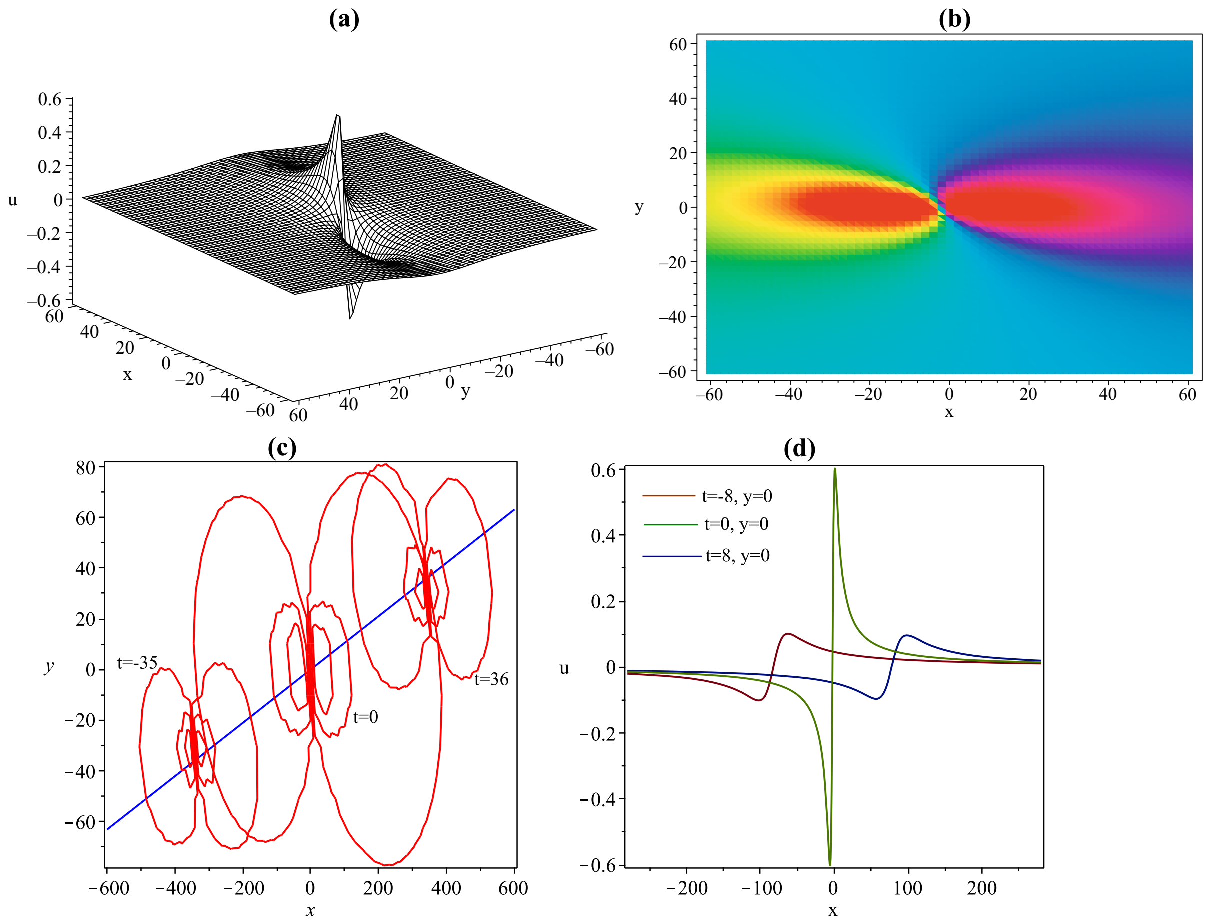


**Fig. 4** Profile of a lump wave (20): **a** 3-dimensional plot with $t = 0$, **b** the corresponding density plot, **c** the red line is the contour plot of the lump wave at $t = -35$, $t = 0$, $t = 36$, and the blue line is the relevant moving progress (23), i.e., $y = \frac{2}{19}x + \frac{9}{19}$, **d** the wave propagation pattern of the wave along $x$-axis by selecting $y = 0$ and different time. (Color figure online)

### Case I

$$a_3 = -\frac{\delta_1(a_1a_2^2 + 2a_2a_5a_6 - a_1a_6^2)}{a_1^2 + a_5^2} - \delta_2 a_2 - \delta_3 a_1, \quad (25)$$

$$a_7 = -\frac{\delta_1(a_5a_6^2 - a_5a_2^2 + 2a_1a_2a_6)}{a_1^2 + a_5^2} - \delta_2 a_6 - \delta_3 a_5,$$

$$k_4 = -\frac{k_2^3}{4} - \delta_1\frac{k_3^2}{k_2} - \delta_2 k_3 - \delta_3 k_2 - \delta_4\frac{k_3k_2^2}{4} - \delta_5\frac{k_3^4}{4} - \delta_6\frac{k_3^4}{2k_2},$$

$$a_9 = \Big[3\delta_1 k_2^2 A(k_2^2B - k_3^2A)\big((k_3^2A + k_2^2B)^2 + k_2^2k_3^2(D^2 - 3C^2)\big) + 8k_3^5A^2C(k_3C - 2k_2B) + 16k_2^3k_3^3BD^2(2k_3C - k_2B) + 12k_2^2k_3^4AB(D^2 - C^2) + 4k_2^3B(k_2^3B^3 - 8k_2k_3^2BC^2 + 12k_3^3C^3)\Big] \Big/ \big(4k_2^2k_3^4\delta_1 D^2 E\big) - \frac{3k_2AC}{k_3^3D^2}(k_2^2B - k_3^2A),$$

$$\delta_4 = -\frac{2k_2}{k_3} + \frac{8\delta_1(k_2B - k_3C)(k_3^2A - k_2^2B)}{3k_2^2k_3AE},$$

$$\delta_5 = \frac{2k_2^3}{k_3^3} + \frac{2C}{3k_3^2A} + \frac{k_3^4A + k_2^2B^2 - 2k_2^2k_3^2D^2}{3k_2k_3^3AE},$$

$$\delta_6 = -\frac{k_2^4}{2k_3^4} + \frac{2\delta_1(k_2^2B - k_3^2A)(3k_3^2A - k_2^2B - 2k_2k_3C)}{3k_3^4AE},$$

$$A = a_1^2 + a_5^2,$$

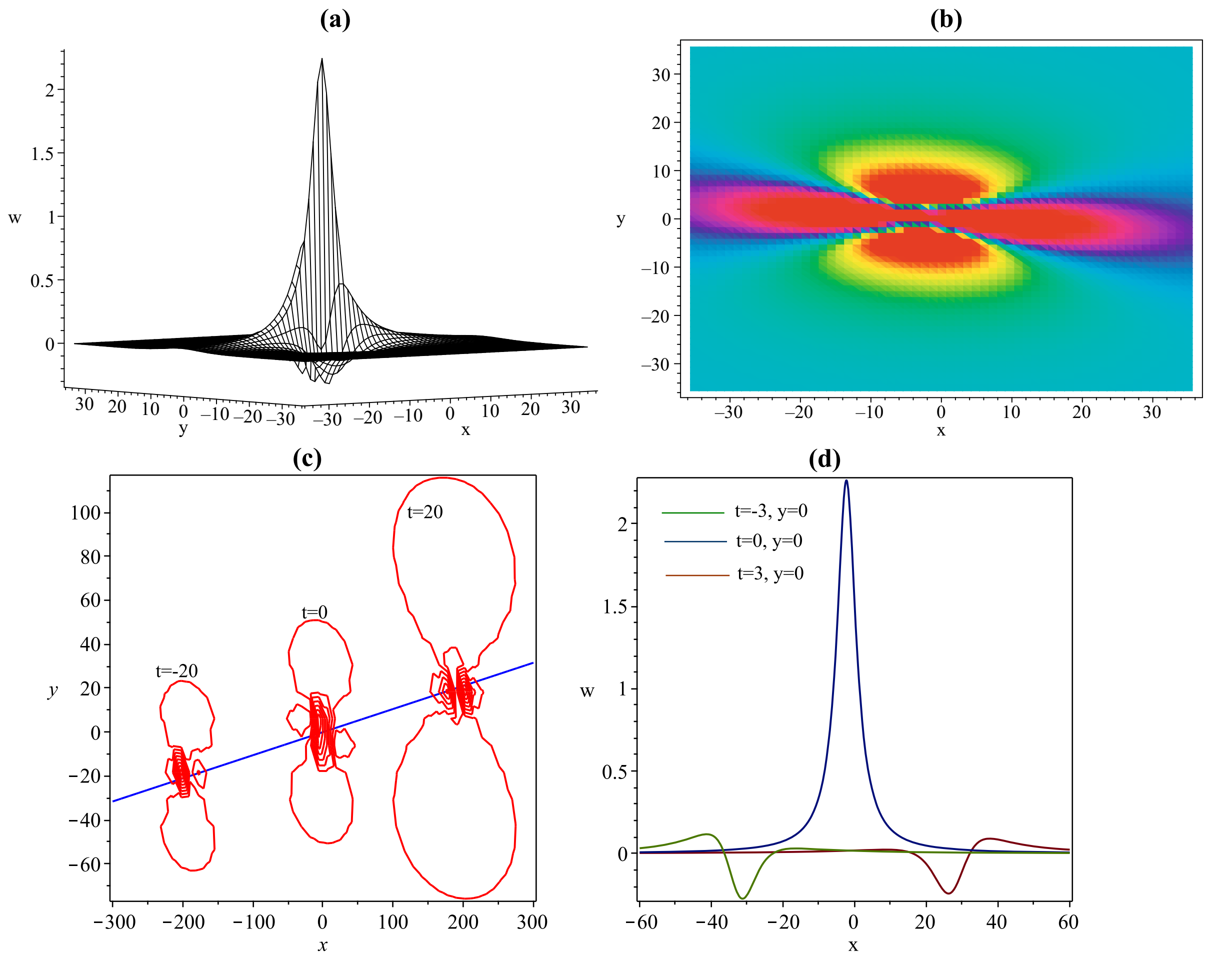


**Fig. 5** Profile of a bi-lump wave (20): **a** 3-dimensional plot of $w$ with the time $t=0$, **b** the corresponding density plot, **c** the red line is the contour plot of the lump wave at $t=-20, t=0, t=20$, and the blue line is the relevant moving progress (23), i.e., $y=\frac{2}{19}x+\frac{9}{19}$, **d** the wave propagation pattern of the wave along $x$-axis by selecting $y=0$ and different time. (Color figure online)

$$B=a_2^2+a_6^2,\quad C=a_1a_2+a_5a_6,$$
$$D=a_1a_6-a_2a_5,\quad E=k_3^2A+k_2^2B-2k_2k_3C,$$

which should satisfy the following constraint conditions:

$$\delta_1a_5k_2k_3\neq 0,\quad a_1a_6-a_2a_5\neq 0,\quad a_9>0, \tag{26}$$

so that the localization of $u$ and $w$ in all directions of the $(x, y)$-plane is guaranteed. By substituting (24) into (5) and combining the parameters relations (25), we get the following interaction solution of Eq. (1):

$$u=\frac{4a_1g+4a_5h+2k_1k_2\exp(f)}{\phi},$$
$$w=-\frac{2(2a_2g+2a_6h+k_1k_3\exp(f))^2}{\phi^2}+\frac{2(2a_2^2+2a_6^2+k_1k_3^2\exp(f))}{\phi}, \tag{27}$$

where

$$\phi=g^2+h^2+a_9+k_1\exp(f),$$
$$g=a_1x+a_2y-\left(\frac{\delta_1(a_1a_2^2+2a_2a_5a_6-a_1a_6^2)}{a_1^2+a_5^2}+\delta_2a_2+\delta_3a_1\right)t+a_4,$$

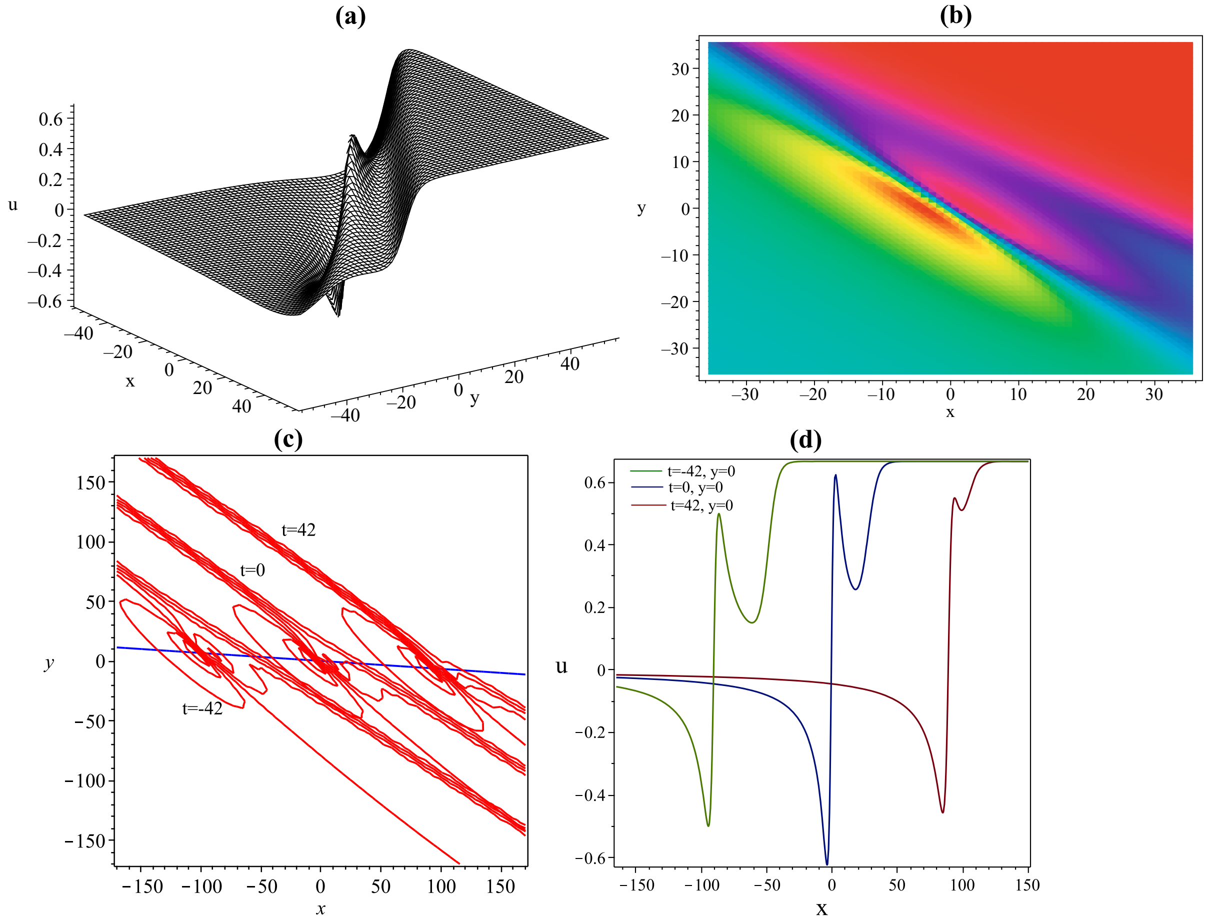


**Fig. 6** Profile of an interaction solution between a lump and a one-kink soliton solution (27): **a** 3-dimensional plot with $t = 0$, **b** the corresponding density plot, **c** the red line is contour plot at $t = -42, t = 0, t = 42$ and the blue line is the relevant moving progress (23), i.e., $y = -\frac{1}{15}x - \frac{17}{75}$, **d** the wave propagation pattern of the wave along $x$-axis by selecting $y = 0$ and different time $t$. (Color figure online)

$$h = a_5 x + a_6 y - \left( \frac{\delta_1 (a_5 a_6^2 - a_5 a_2^2 + 2a_1 a_2 a_6)}{a_1^2 + a_5^2} + \delta_2 a_6 + \delta_3 a_5 \right) t + a_8,$$

$$f = k_2 x + k_3 y - \left( \frac{k_2^3}{4} + \delta_1 \frac{k_3^2}{k_2} + \delta_2 k_3 + \delta_3 k_2 + \delta_4 \frac{k_3 k_2^2}{4} + \delta_5 \frac{k_3^4}{4} + \delta_6 \frac{k_3^4}{2k_2} \right) t + k_5. \quad (28)$$

The parameters are selected as $a_1 = 1, a_2 = 3, a_4 = 1, a_5 = 5, a_6 = 5, a_8 = 3, k_1 = 1, k_2 = \frac{1}{3}, k_3 = \frac{1}{2}, k_5 = 1, \delta_1 = -1, \delta_2 = 2, \delta_3 = 1$ in Figs. 6 and 7. The interaction solution between a lump and a one-kink soliton of $u$ is presented in Fig. 6a at $t = 0$. Figure 5b displays the corresponding density plot of the lump–kink wave. Figure 6c represents the homologous contour plot at time $t = -42, t = 0, t = 42$. The interaction solution between a bi-lump and a one-soliton solution of $w$ is presented in Fig. 7a at $t = 0$. The corresponding density is plotted in Fig. 7b. Figure 7c is the homologous contour plot at time $t = -52, t = 0, t = 46$. The blue line shown in Figs. 6c and 7c is the relevant moving progress of the lump wave (23), i.e., $y = -\frac{1}{15}x - \frac{17}{75}$. The wave along $x$-axis of the corresponding interaction solution is shown in Figs. 6d and 7d.

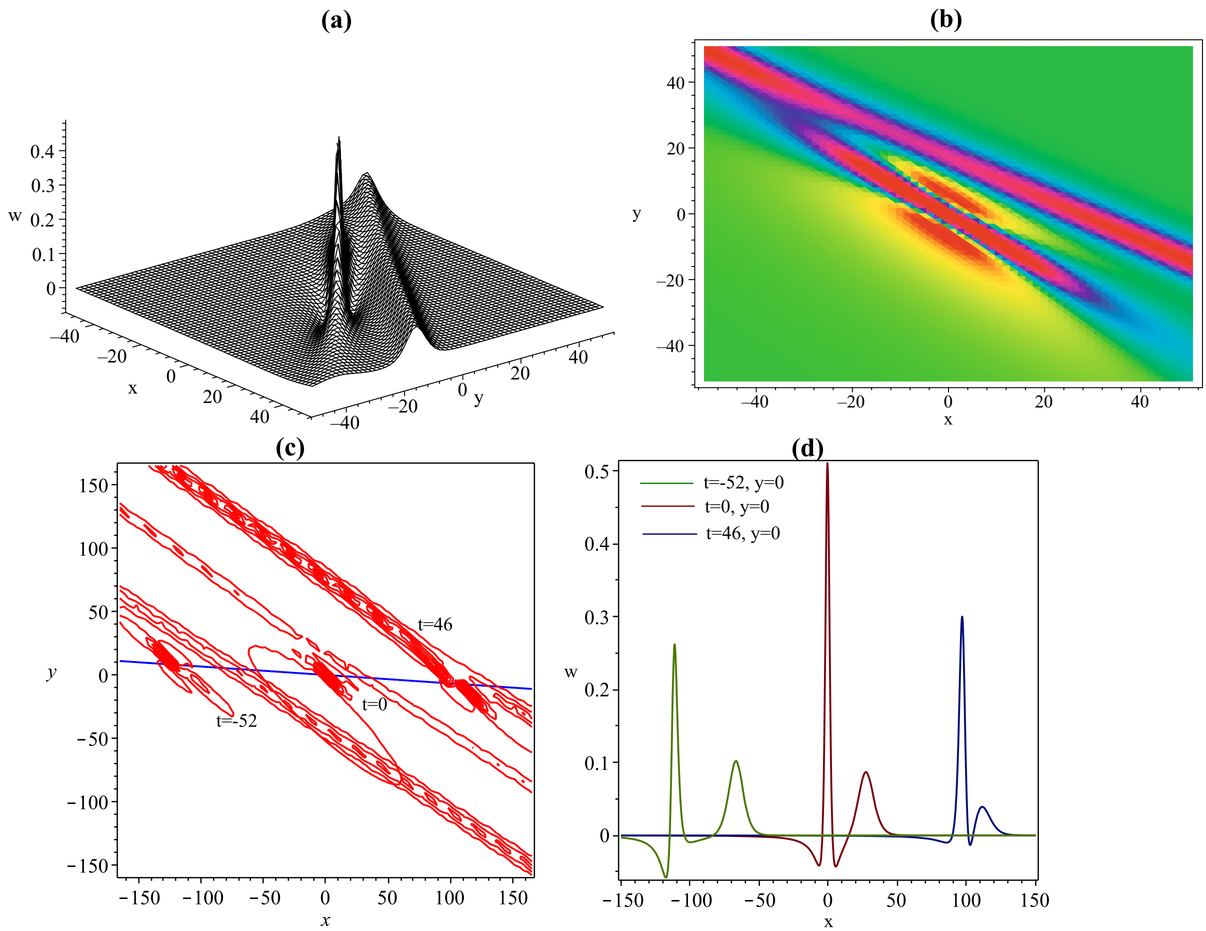


**Fig. 7** Profile of an interaction solution between a bi-lump and a one-soliton solution (27): **a** 3-dimensional plot with $t=0$, **b** the corresponding density plot, **c** the red line is contour plot at $t=-52, t=0, t=46$ and the blue line is the relevant moving progress (23), i.e., $y=-\frac{1}{15}x-\frac{17}{75}$, **d** the wave propagation pattern of the wave along $x$-axis by selecting $y=0$ and different time $t$

**Case II**

$$a_1=\frac{2a_2\delta\sqrt{\delta_1}}{k_2},$$
$$a_3=-\frac{\sqrt{\delta_1}a_2k_2(2a_5^2k_2^2+4\delta\delta_1a_2^2-\delta k_2^2a_5^2)}{2(a_5^2k_2^2+4\delta_1a_2^2)}-\frac{2\delta\sqrt{\delta_1}\delta_3a_2}{k_2}-\delta_2a_2,$$
$$a_6=\frac{a_5k_2}{2\delta\sqrt{\delta_1}},\quad a_7=-\frac{a_5k_2^2(a_5^2k_2^2-4\delta_1a_2^2+8\delta\delta_1a_2^2)}{4(a_5^2k_2^2+4\delta_1a_2^2)}-\frac{\delta_2k_2a_5}{2\sqrt{\delta_1}}-\delta_3a_5,$$
$$k_4=-\frac{k_2^3}{4}-\delta_1\frac{k_3^2}{k_2}-\delta_2k_3-\delta_3k_2-\delta_4\frac{k_3k_2^2}{4}-\delta_5\frac{k_3^4}{4}-\delta_6\frac{k_3^4}{2k_2},$$
$$\delta_4=-\frac{4\delta_1}{\delta\sqrt{\delta_1}k_2},\quad \delta_5=-\frac{8\delta_1^2-\delta_6k_2^4}{\delta\sqrt{\delta_1}k_2^3},\tag{29}$$

which $\delta^2=1$ and should satisfy the following constraint conditions:

$$\delta_1a_5k_2\neq0,\quad a_9>0,\tag{30}$$

so that localization of $u$ and $w$ in all directions of the $(x,y)$-plane is guaranteed. By substituting (24) into (5) and combining the parameters relations (29), we get the following interaction solution of Eq. (1):

$$u = \frac{4a_1 g + 4a_5 h + 2k_1 k_2 \exp(f)}{\phi},$$
$$w = -\frac{2(2a_2 g + 2a_6 h + k_1 k_3 \exp(f))^2}{\phi^2} + \frac{2(2a_2^2 + 2a_6^2 + k_1 k_3^2 \exp(f))}{\phi}, \quad (31)$$

where

$$\phi = g^2 + h^2 + a_9 + k_1 \exp(f),$$
$$g = a_1 x + a_2 y - \left(\frac{\sqrt{\delta_1} a_2 k_2 (2a_5^2 k_2^2 + 4\delta\delta_1 a_2^2 - \delta k_2^2 a_5^2)}{2(a_5^2 k_2^2 + 4\delta_1 a_2^2)} + \frac{2\delta\sqrt{\delta_1}\delta_3 a_2}{k_2} + \delta_2 a_2\right) t + a_4,$$
$$h = a_5 x + a_6 y - \left(\frac{a_5 k_2^2 (a_5^2 k_2^2 - 4\delta_1 a_2^2 + 8\delta\delta_1 a_2^2)}{4(a_5^2 k_2^2 + 4\delta_1 a_2^2)} + \frac{\delta_2 k_2 a_5}{2\sqrt{\delta_1}} + \delta_3 a_5\right) t + a_8,$$
$$f = k_2 x + k_3 y - \left(\frac{k_2^3}{4} + \delta_1 \frac{k_3^2}{k_2} + \delta_2 k_3 + \delta_3 k_2 + \delta_4 \frac{k_3 k_2^2}{4} + \delta_5 \frac{k_3^4}{4} + \delta_6 \frac{k_3^4}{2k_2}\right) t + k_5. \quad (32)$$

Similarly to the Case I, we can get interaction solutions between a lump and a one-kink soliton, and between a bi-lump and a one-soliton solution by using (31).

## 6 Conclusion

In this work, the Hirota bilinear form of Eq. (1) was derived by the truncated Painlevé analysis. Based on the obtained bilinear form, solitary waves were firstly constructed via a perturbative expansion (shown in Figs. 1, 2, 3). Then, some lump waves were found by using a positive quadratic function. Finally, the interaction solutions, between a lump wave and a one-kink soliton, and between a bi-lump wave and a one-soliton solution, were proposed by adding an additional exponential function to a positive quadratic function (shown in Figs. 4, 5, 6, 7).

In addition, we could also construct some new integrable systems by using the generalized bilinear operators [46], which are given by

$$D_{p,t} D_{p,x} + \frac{1}{4} D_{p,x}^4 + \delta_1 D_{p,y}^2 + \delta_2 D_{p,x} D_{p,y} + \delta_3 D_{p,x}^2 + \frac{\delta_4}{2} D_{p,x}^3 D_{p,y} + \frac{\delta_5}{2} D_{p,x} D_{p,y}^3 + \frac{\delta_6}{2} D_{p,y}^4 = 0, \quad (33)$$

with the prime numbers $p = 3, 5, 7, \cdots$. We are going to study hybrid solutions and integrable properties of Eq. (33).

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**Compliance with ethical standards**

**Conflict of interest** The authors declare that they have no conflict of interest.

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