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Lump Solutions for Two Mixed Calogero-Bogoyavlenskii-Schiff and Bogoyavlensky-Konopelchenko Equations*

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Abstract *Based on the Hirota bilinear operators and their generalized bilinear derivatives, we formulate two new (2+1)-dimensional nonlinear partial differential equations, which possess lumps. One of the new nonlinear differential equations includes the generalized Calogero-Bogoyavlenskii-Schiff equation and the generalized Bogoyavlensky-Konopelchenko equation as particular examples, and the other has the same bilinear form with different D_p-operators. A class explicit lump solutions of the new nonlinear differential equation is constructed by using the Hirota bilinear approaches. A specific case of the presented lump solution is plotted to shed light on the charateristics of the lump.*

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1 Introduction

The investigation of exact solutions to nonlinear partial differential equations is one of the most important problems. Many kinds of soliton solutions are studied by a variety of methods including the inverse scattering transformation,[1] the Darboux transformation,[2−3] the Hirota bilinear method,[4] and symmetry reductions,[5] etc.[6−10] Recently, lump solutions which are rational, analytical and localized in all directions in the space,[11−20] have attracted much attention. As another kind of exact solutions, it exsits potential applications in physics, partically in atmospheric and oceanic sciences.[21]

The Hirota bilinear method in soliton theory provides a powerful approach to finding exact solutions.[4] A kind of lump solutions can be also obtained by means of the Hirota bilinear formuation. Recently, the generalized bilinear operators are proposed by exploring the linear superposition principle.[22] Many new nonlinear systems are constructed by using the generalized Hirota bilinear operators.[23−26] The lump solutions and integrable propertites for those new nonlinear systems are interesting topic in nonlinear science.

The paper is organized as follows. In Sec. 2, a new nonlinear differential equation is constructed by means of the bilinear formulation. The new nonlinear equation includes a Calogero-Bogoyavlenskii-Schiff equation and a Bogoyavlensky-Konopelchenko (gCBS-BK) equation. A class of gCBS-BK-like equations can be obatined by using the generalized bilinear method. In Sec. 3, a lump solution to the newly presented gCBS-BK systems is obtained bsaed on the *Maple* symbolic computations. Two figures are given theoretically and graphically. The last section is devoted to summary and discussions.

2 A Generalized gCBS-BK Equation

We consider a (2+1)-dimensioanl nonlinear partial differential equation

$$\begin{aligned}&u_t+u_{xxy}+3u_xu_y+\delta_1u_y+\delta_2w_{yy}+\delta_3u_x\\&+\delta_4(3u_x^2+u_{xxx})+\delta_5(3w_{yy}^2+w_{yyyy})\\&+\delta_6(3u_yw_{yy}+u_{yyy})=0\,,\quad u_x=w\,,\end{aligned}\tag{1}$$

where $\delta_i, i=1,2,\ldots,6$ are arbitrary constants. While the constants satisfy $\delta_3=\delta_4=\delta_5=\delta_6=0$ and $\delta_5=\delta_6=0$, (1) becomes a generalized Calogero-Bogoyavlenskii-Schiff (CBS) equation[18] and a generalized[18,27] Bogoyavlensky-Konopelchenko (BK) equation,[19] respectively. The CBS equation was constructed by the modified Lax formalism and the self-dual Yang-Mills equation respectively.[28−29]

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The BK equation is described as the interaction of a Riemann wave propagating along y-axis and a long wave propagating along x-axis.[30] These two equations have been widely studied in different ways.[31–32] The (2+1)-dimensional nonlinear differential equation (1) is thus called gCBS-BK equation. The Hirota bilinear form of gCBS-BK equation (1) has

$$\begin{aligned}&D_tD_x + D_x^3D_y + \delta_1 D_xD_y + \delta_2 D_y^2 + \delta_3 D_x^2 + \delta_4 D_x^4 + \delta_5 D_y^4 + \delta_6 D_xD_y^3\\ &= 2(f_{xt}f - f_tf_x + f_{xxxy}f - f_{xxx}f_y - 3f_{xxy}f_x + 3f_{xx}f_{xy} + \delta_1(f_{xy}f - f_xf_y)\\ &+ \delta_2(f_{yy}f - f_y^2) + \delta_3(f_{xx}f - f_x^2) + \delta_4(f_{xxxx}f - 4f_xf_{xxx} - 3f_{xx}^2)\\ &+ \delta_5(ff_{yyyy} - 4f_yf_{yyy} + 3f_{yy}^2) + \delta_6(ff_{xyyy} - f_xf_{yyy} - 3f_yf_{xyy} + 3f_{xy}f_{yy})) = 0,\end{aligned} \tag{2}$$

by the relationship between u, w, and f

$$w = 2(\ln f)_{xx} = \frac{2(f_{xx}f - f_x^2)}{f^2}, \qquad u = 2(\ln f)_x = \frac{2f_x}{f}. \tag{3}$$

Based on the generalized bilinear thoery,[22] the generalized bilinear operators read

$$\begin{aligned}&(D_{p,x}^m D_{p,t}^n) f(x,t)\cdot f(x',t') = (\partial_x + \alpha_p\partial_{x'})^m(\partial_t + \alpha_p\partial_{t'})^n f(x,t)f(x',t')|_{x'=x,t'=t}\\ &= \sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j}\alpha_p^i\alpha_p^j\frac{\partial^{m-i}}{\partial x^{m-i}}\frac{\partial^i}{\partial x'^{(i)}}\frac{\partial^{n-j}}{\partial t^{n-j}}\frac{\partial^j}{\partial t'^{(j)}}f(x,t)f(x',t')|_{x'=x,t'=t}\\ &= \sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j}\alpha_p^i\alpha_p^j\frac{\partial^{m+n-i-j}f(x,t)}{\partial x^{m-i}\partial t^{n-j}}\frac{\partial^{i+j}f(x,t)}{\partial x^i\partial t^j},\end{aligned} \tag{4}$$

where $m, n \ge 0$ and $\alpha_p^s = (-1)^{r_p(s)}$ if $s = r_p(s) \bmod p$. Here α_p is a symbol. For a prime number $p > 2$, we can not write the relationship

$$\alpha_p^i\alpha_p^j = \alpha_p^{i+j}, \qquad i, j \ge 0. \tag{5}$$

Taking the prime number $p = 3$, we have

$$\alpha_3 = -1, \quad \alpha_3^2 = 1, \quad \alpha_3^3 = 1, \quad \alpha_3^4 = -1, \quad \alpha_3^5 = 1, \quad \alpha_3^6 = 1, \quad \ldots, \tag{6}$$

and then, we have the concrete operators

$$\begin{aligned}&D_{3,t}D_{3,x}f\cdot f = 2f_{xt}f - 2f_xf_t\,, \quad D_{3,x}^3D_{3,y} = 6f_{xx}f_{xy}\,, \quad D_{3,x}D_{3,y} = 2f_{xy}f - 2f_xf_y\,,\\ &D_{3,y}^2 = 2f_{yy}f - 2f_y^2\,, \quad D_{3,x}^2 = 2f_{xx}f - 2f_x^2\,, \quad D_{3,x}^4 = 6f_{xx}^2\,, \quad D_{3,y}^4 = 6f_{yy}^2\,, \quad D_{3,x}D_{3,y}^3 = 6f_{yy}f_{xy}\,.\end{aligned} \tag{7}$$

By the above analysis, the corresponding bilinear form of the gCBS-BK equation (1) in $p = 3$ reads

$$\begin{aligned}&D_{3,t}D_{3,x} + D_{3,x}^3D_{3,y} + \delta_1 D_{3,x}D_{3,y} + \delta_2 D_{3,y}^2 + \delta_3 D_{3,x}^2 + \delta_4 D_{3,x}^4 + \delta_5 D_{3,y}^4 + \delta_6 D_{3,y}D_{3,y}^3\\ &= 2(f_{xt}f - f_tf_x + 3f_{xx}f_{xy} + \delta_1(f_{xy}f - f_xf_y) + \delta_2(f_{yy}f - f_y^2) + \delta_3(f_{xx}f - f_x^2)\\ &+ 3\delta_4 f_{xx}^2 + 3\delta_5 f_{yy}^2 + 3\delta_6 f_{yy}f_{xy}) = 0|\,.\end{aligned} \tag{8}$$

Bell polynomial theories suggest a dependent variable transfomation

$$u = 2(\ln f)_x, \tag{9}$$

to transfrom bilinear equations to nonlinear equations. By selecting the variable transformation (9), a gCBS-BK-like equation is obtained from the generalized bilinear form (8)

$$\begin{aligned}&u_t + \frac{3}{4}u^2u_y + \frac{3}{2}u_xu_y + \frac{3}{4}uu_xw_y + \frac{3}{8}u^3w_y\\ &+ \delta_1u_y + \delta_2w_{yy} + \delta_3u_x + \frac{3}{8}\delta_4(u^2 + 2u_x)^2\\ &+ \frac{3}{2}\delta_5(w_y^2 + w_{yy})^2 + \frac{3}{8}\delta_6(w_y^2 + w_{yy})(uw_y + 2u_y) = 0\,,\\ &u_x = w\,.\end{aligned} \tag{10}$$

By selecting the prime number $p = 3$, we get a new gCBS-BK-like equtaion (10). We can aslo select $p = 5, 7, 9, \ldots$ to get new nonlinear partial differential equations. This provides a useful method to get new nonlinear systems that possess bilinear forms. In this paper, we shall focus on the gCBS-BK equation (1) and the gCBS-BK-like equation (10) for the prime number $p = 3$.

3 A Search for Lump Solution

Based on the bilinear form, a quadratic function solution to the (2+1)-dimensional bilinear gCBS-BK equation (2) and bilinear gCBS-BK-like equation (8), is defined by

$$\begin{aligned}&f = \xi_1^2 + \xi_2^2 + a_9\,,\\ &\xi_1 = a_1x + a_2y + a_3t + a_4\,,\\ &\xi_2 = a_5x + a_6y + a_7t + a_8\,,\end{aligned} \tag{11}$$

where a_i, $1 \le i \le 9$ are constant parameters to be determined. Substituting the expression (11) into Eqs. (2) and (8) and vanishing the coefficients of different powers of x, y, and t, we can get the same relationship among parameters for Eqs. (2) and (8). The following set of solutions for the parameters a_3, a_7, and a_9

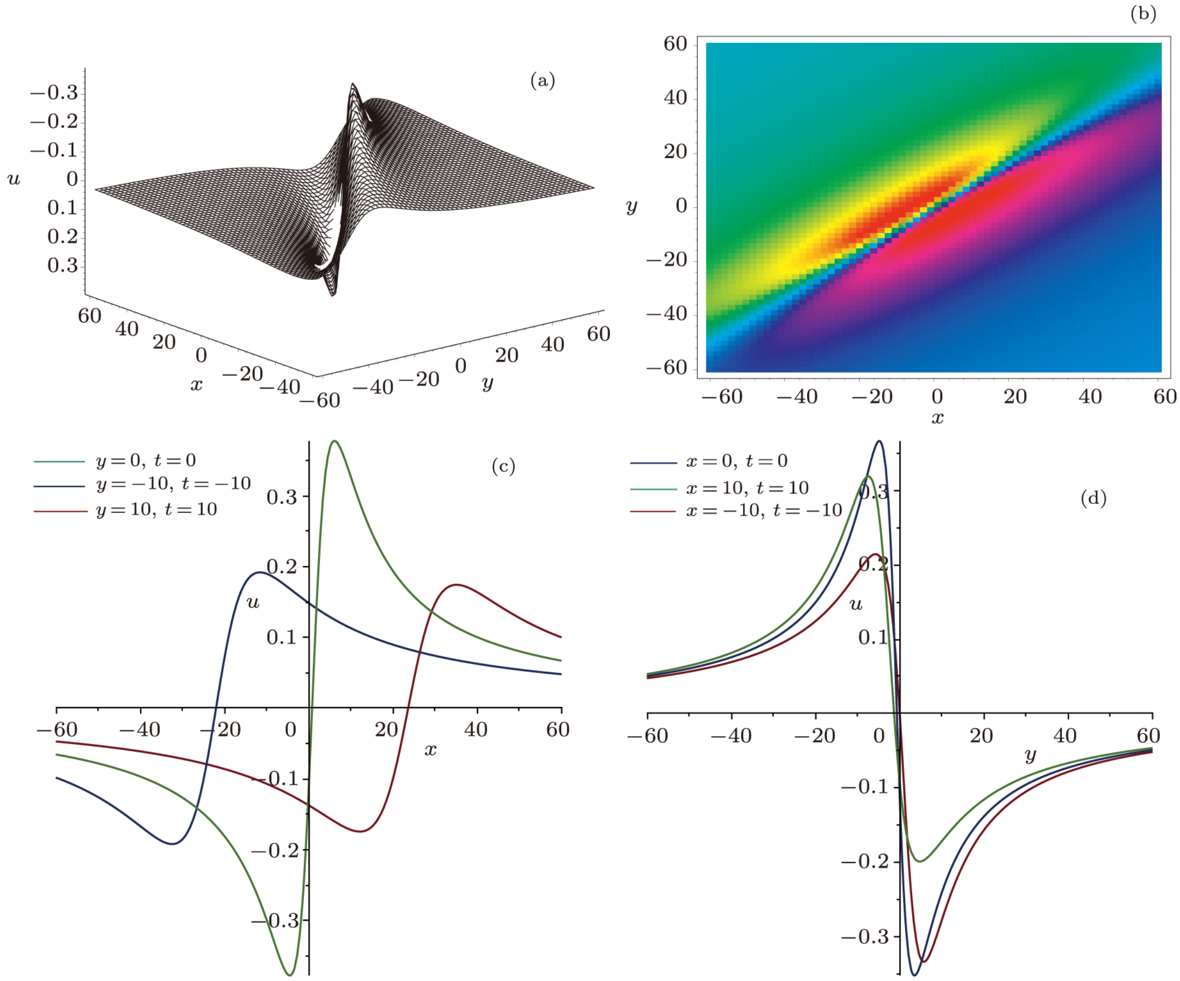

Fig. 1 (Color online) Profiles of the lump solution (13). (a) 3D lump plot with the time $t = 0$, (b) the corresponding density plot, (c) the curve by selecting different parameters y and t, (d) the curve by selecting different parameters x and t.

$$a_3 = -\delta_1 a_2 - \delta_3 a_1 - \frac{\delta_2(a_1a_2^2 - a_1a_6^2 + 2a_2a_5a_6)}{a_1^2+a_5^2},$$

$$a_7 = -\delta_1\Big(\frac{a_1a_2}{a_5} + a_6\Big) - \delta_2\frac{(a_1a_2+a_5a_6)^2 - (a_1a_6 - a_2a_5)^2}{a_5(a_1^2+a_5^2)} - \delta_3\frac{a_1^2+a_5^2}{a_5} - \frac{a_1a_3}{a_5},$$

$$a_9 = -\frac{3(a_1^2+a_5^2)^2}{\delta_2(a_1a_6-a_2a_5)^2}\Big(a_1a_2 + a_5a_6 + \delta_4(a_1^2+a_5^2) + \delta_5\frac{(a_2^2+a_6^2)^2}{a_1^2+a_5^2} + \delta_6\frac{(a_2^2+a_6^2)(a_1a_2+a_5a_6)}{a_1^2+a_5^2}\Big), \tag{12}$$

which need to satisfy the following conditions

(i) $a_5 \neq 0$, to guarantee the well-posedness for f;

(ii) $\delta_2\Big(a_1a_2 + a_5a_6 + \delta_4(a_1^2+a_5^2) + \delta_5\frac{(a_2^2+a_6^2)^2}{a_1^2+a_5^2} + \delta_6\frac{(a_2^2+a_6^2)(a_1a_2+a_5a_6)}{a_1^2+a_5^2}\Big) < 0$, to have the positivity of f;

(iii) $a_1a_6 - a_2a_5 \neq 0$, to ensure the localization of u, w in all directions in the space.

The parameters take $a_1 = 1, a_2 = -2, a_4 = -2, a_5 = -2, a_6 = 2, a_8 = 1, \delta_1 = 1, \delta_2 = 1, \delta_3 = 1, \delta_4 = 1, \delta_5 = 1, \delta_6 = 2$. By substituting Eq. (11) into Eq. (9) and combining the relationship (12), we get the lump solution

$$u = -\frac{16(27t - 25x + 30y + 20)}{100x^2 + 160y^2 - 240xy + 240y - 160x + 240t + 304yt - 216tx + 148t^2 + 2875}\,. \tag{13}$$

The 3D plot, density plot, and curve plot for this lump solution are depicted in Fig. 1. The parameters take $a_1 = 1$, $a_2 = 1$, $a_4 = 1$, $a_5 = -2$, $a_6 = 3$, $a_8 = 1$, $\delta_1 = 1$, $\delta_2 = 1$, $\delta_3 = 1$, $\delta_4 = 1$, $\delta_5 = -2$, $\delta_6 = 2$. The lump solution has the following form

$$u = \frac{4(5x - 5y - 1)}{5x^2 + 10y^2 - 10xy + 8y - 2x + 6t + 10yt + 5t^2 + 182}\,. \tag{14}$$

The 3D plot, density plot and curve plot for the lump solution are shown in Fig. 2.

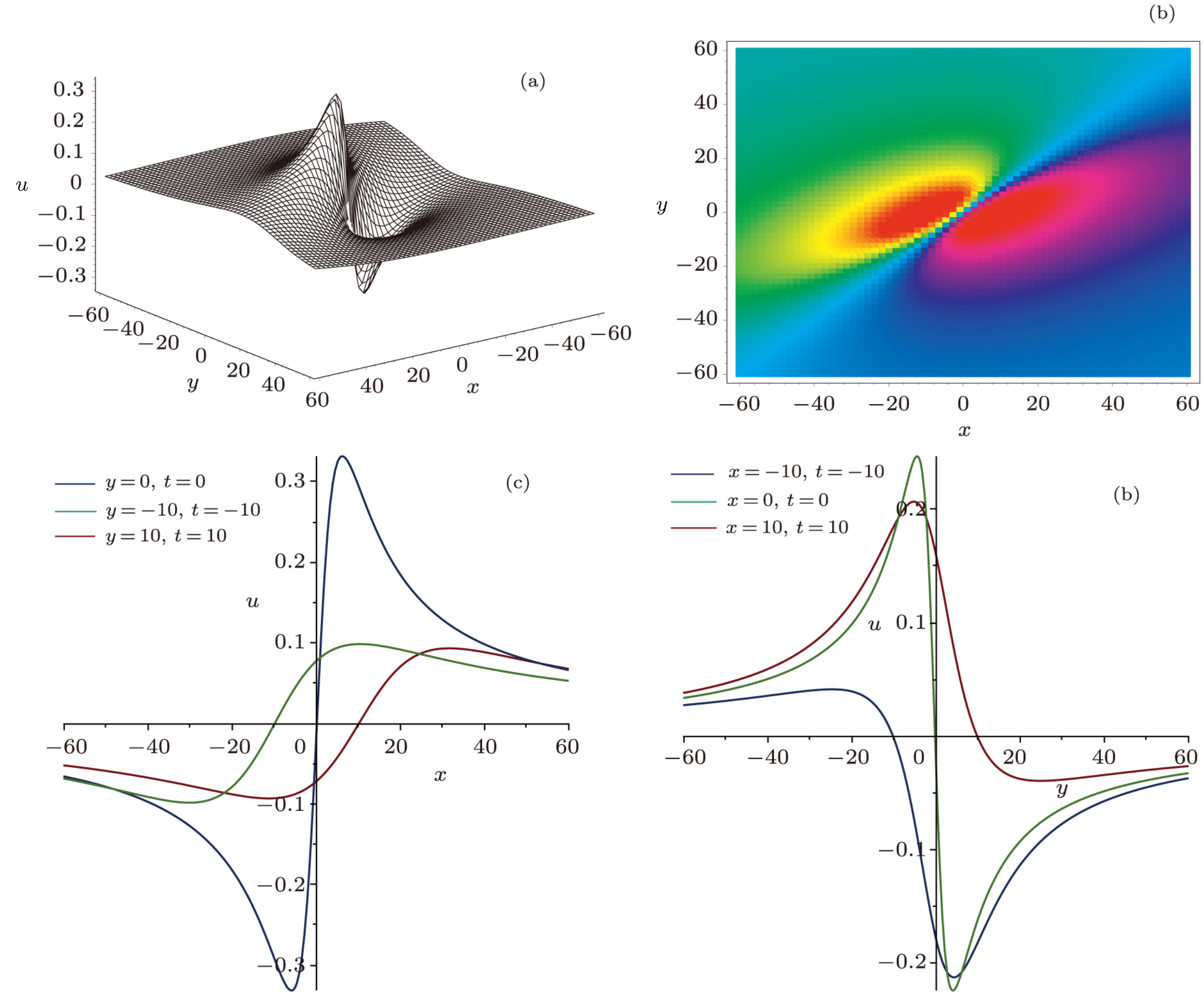

Fig. 2 (Color online) Profiles of the lump solution (14). (a) 3D lump plot with the time $t = 0$, (b) the corresponding density plot, (c) the curve by selecting different parameters y and t, (d) the curve by selecting different parameters x and t.

4 Summary and Discussions

In summary, the gCBS-BK equation was derived in terms of Hirota bilinear forms. By selecting the prime number $p = 3$, a gCBS-BK-like equation was formulated by the generalized Hirota operators. The lump solution of the gCBS-BK equation and the gCBS-BK-like equation was generated by their Hirota bilinear forms. The phenomena of lump solutions were presented by figures. The results provide a new example of (2+1)-dimensional nonlinear partial differential equations, which possess lump solutions. Other new nonlinear equations can be also obtained by seleting the prime numbers $p = 5, 7, \ldots$ It is demonstrated that the generalized Hirota operators are very useful in constructing new nonlinear differential equations, which possess nice math properties. In the meanwhile, lump-kink interaction solutions,[34−35] lump-soliton interaction solutions,[36] lump type solutions for the (3+1)-dimensional nonlinear differential equations[36−38] and solitons-cnoidal wave interaction solutions[39−41] are important and will be explored in the future.

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