Rational solutions of a (2 + 1)-dimensional Sharma-Tasso-Olver equation

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Abstract

We extend the (1 + 1)-dimensional Sharma-Tasso-Olver (STO) equation to a (2 + 1)-dimensional one by adding one additional term $u_{yy}$. A tri-linear form of the (2 + 1)-dimensional STO equation is obtained by the Painlevé analysis. A family of rational solutions for the (2 + 1)-dimensional STO equation is constructed by using the resulting tri-linear form. Associated 3-dimensional plot and density plot with particular choices of the involved parameters are given to show the characteristics of the rational solutions.

1. Introduction

Exact solutions of nonlinear partial differential equations are very important to predict and understand possible behaviors of physics phenomena [1–3]. As a kind of special localized wave solutions, special lump solutions which are rationally decaying in all directions in the space, were obtained for the Kadomtsev-Petviashvili by Manakov et. al. [4]. Subsequently, a class of rational solutions was investigated with many methods [5–10]. These wave solutions have potential applications in the open ocean and coastal areas [11]. Therefore, direct methods for construction of these kinds of solutions have attracted particular attention in mathematical physics [8–26]. Several traditional methods are developed to construct these kinds of waves, such as the inverse scattering transformation [12], the Bäcklund transformation [13], the Darboux transformation [14] and the Hirota bilinear method [15–31].

In this paper, we shall focus on rational solutions of a new (2 + 1)-dimensional STO equation by solving a tri-linear STO form that we are going to present. The paper is organized as follows. In Section 2, a (2 + 1)-dimensional STO equation is constructed by adding one additional term $u_{yy}$. A dipole-mode soliton solution and lump-like solutions of the (2 + 1)-dimensional STO equation can be obtained by solving the tri-linear form. The last section is devoted to summary and discussions.

2. Rational solutions of a (2 + 1)-dimensional STO equation

The (1 + 1)-dimensional STO equation reads

\[ u_{yy} + u_{tt} = 4u^2 u_{t} - u^3. \]
\[ u_t + \alpha u_x^2 + \frac{3}{2} \alpha u_{xx}^2 + \alpha u_{xxx} = 0, \tag{1} \]

where \( \alpha \) is an arbitrary constant. The \((1+1)\)-dimensional STO equation can describe the propagation of nonlinear dispersive waves in inhomogeneous media. This equation has gained much attentions due to its appearance in scientific applications. Many integrability properties for the STO Eq. (1) have been obtained [32–35]. The fission and fusion of the solitary wave were found by means of the Hirota direct method and the Bäcklund transformation [32]. The symmetry reduction procedure was used to obtain infinitely many symmetries and exact solutions [33,34].

The \((2+1)\)-dimensional KP equation can be taken a generalized two-spatial dimensional version of the \((1+1)\)-dimensional Korteweg-de Vries (KdV) equation. According to the form of the KP equation, we take a \((2+1)\)-dimensional STO equation as the following form as a generalization of the \((1+1)\)-dimensional STO equation

\[
\left( u_t + \alpha u_x^3 + \frac{3}{2} \alpha u_{xx}^2 + \alpha u_{xxx} \right)_x + \beta u_{yy} = 0, \tag{2} \]

where \( \beta \) is an arbitrary constant. By the truncated Painlevé analysis [36], the solution for the STO Eq. (2) reads

\[
u = \frac{u_0}{\phi} + u_1, \tag{3} \]

where \( u \) and \( u_1 \) are the solution of the STO Eq. (2). By substituting of the expansion (3) into (2) and vanishing the coefficient of \( \phi^{-5} \), we get

\[ u_0 = \phi. \tag{4} \]

By substituting (3) and (4) into (2), a tri-linear form for the \((2+1)\)-dimensional STO equation is obtained

\[
\phi^2 \phi_{xx} + 2\phi^2 \phi_{x1} - 2\phi \phi_{x1} \phi_1 + \alpha \phi^2 \phi_{xxx} - 2\alpha \phi \phi_{xx} \phi_1 - \alpha \phi \phi_{x1} \phi_1 + 2\alpha \phi \phi_{xx} \phi_{x1} + \beta \phi^2 \phi_{yy} - 2\beta \phi \phi_{xy} \phi_1 + 2\beta \phi \phi_{y} \phi_1 - \beta \phi \phi_{y} \phi_{y1} = 0. \tag{5} \]

We want to use a tri-linear Eq. (5) to calculate rational solutions. The quadratic function solutions to a \((2+1)\)-dimensional tri-linear Eq. (5) assume

\[
\phi = \xi^2 + \xi_1^2 + a_9 \xi_1 = a_9 x + a_2 y + a_3 t + a_4 \xi_2 = a_3 x + a_4 y + a_7 t + a_8, \tag{6} \]

where \( a_i, 1 \leq i \leq 9 \) are constant parameters to be determined. Substituting the expression (6) into (5) and vanishing the coefficients of different powers of \( x, y \) and \( t \), we can get the relationship among parameters for the Eq. (5). Two sets for the parameters \( a_i \) are expressed as the following forms.

Case I.

\[
a_2 = \frac{a_1 a_6}{a_5}, \quad a_3 = -\beta \frac{a_1 a_6^2}{a_5^2}, \quad a_7 = -\frac{\beta a_6^2}{a_5}. \tag{7} \]

The corresponding quadratic function solution of (5) yields

\[
\phi = \left( a_8 x + a_8 y + \beta \frac{a_1 a_6^2}{a_5^2} t + a_4 \right)^2 + \left( a_3 x + a_3 y - \frac{\beta a_6^2}{a_5} t + a_8 \right)^2 + a_9. \tag{8} \]

It is easily seen that the coefficients of \( x \) and \( y \) are direct ratio for the square terms in (8). It can not be given lump solution in case I. By substituting (4), (6) and (7) into (3), the solution of (2) writes

\[
u = \frac{2a_9 \xi_1 + 2a_9 \xi_2}{\phi}, \tag{9} \]

where \( \xi_1 = a_9 x + \frac{a_9}{a_5} y - \beta \frac{a_1 a_6^2}{a_5^2} t + a_4, \quad \xi_2 = a_3 x + a_3 y - \frac{\beta a_6^2}{a_5} t + a_8 \) and \( \phi \) satifies (8). By selecting the parameters \( \beta = 1, a_1 = -3, a_2 = 2, a_3 = 2, a_4 = 2, a_5 = 1, a_6 = 3, a_7 = 3 \), we get the solution

\[
u = \frac{2(13x + 13y - 13t - 4)}{(13 t^2 - 26x t + 26y t + 13x^2 + 26y^2 + 13y^2 + 8t - 8x - 8y + 8)^2}. \tag{10} \]

The 3-dimensional plot and density plot for a dipole-mode soliton (10) are depicted in Fig. 1. The dipole-mode soliton solution can be viewed a slit laser beam propagates in Kerr-type nonlinear, nonlocal media with exponential response function [37].

Case II.

\[
ad = -\beta \frac{a_1 a_6^2}{a_5^2} - a_2 a_5 a_6, \quad a_7 = -\beta \frac{a_1 a_6^2}{a_5^2} - a_2 a_5 a_6, \quad a_9 = 0. \tag{11} \]

which need to satisfy the condition \( a_1 a_6 \neq 0 \) to guarantee the well-posedness. It will yield the lump-like solution due to \( a_9 = 0 \). The corresponding quadratic function solution of (5) yields
\[ \phi = \left( a_2 x + a_3 y - \beta \frac{a_2 a_3^2 - a_1 a_3^2 + 2a_3 a_4 a_5}{a_1^2 + a_3^2} t + a_4 \right)^2 + \left( a_3 x + a_4 y - \beta \frac{a_3 a_5^2 - a_1 a_5^2 + 2a_3 a_5 a_4}{a_1^2 + a_5^2} t + a_5 \right)^2. \]  
\tag{12} 

The solution for (2) gives 
\[ u = \frac{2a_1 \xi_1 + 2a_2 \xi_2}{\phi}, \]  
\tag{13} 
where \( \xi_1 = a_2 x + a_3 y - \beta \frac{a_2 a_3^2 - a_1 a_3^2 + 2a_3 a_4 a_5}{a_1^2 + a_3^2} t + a_4 \), \( \xi_2 = a_3 x + a_4 y - \beta \frac{a_3 a_5^2 - a_1 a_5^2 + 2a_3 a_5 a_4}{a_1^2 + a_5^2} t + a_5 \) and \( \phi \) satisfies (12). The critical point of the lump-like wave (13) is calculated by taking the partial derivatives \( \phi_x \) and \( \phi_y \) be zero. The moving path of the lump-like wave is read \[ x = \frac{\beta (a_2^2 + a_3^2)}{a_1^2 + a_3^2} t + \frac{a_3 a_5 - a_1 a_6}{a_1^2 + a_5^2}, \]  
\[ y = \frac{\beta (a_2^2 + a_3^2)}{a_1^2 + a_5^2} t + \frac{a_2 a_5 - a_1 a_6}{a_1^2 + a_5^2}. \]  
\tag{14} 

The lump-like wave move along the route line 
\[ y = -\frac{2(a_1 a_2 + a_3 a_5)}{a_1^2 + a_5^2} x + \frac{a_5 (a_1 a_2^2 - a_1 a_5^2 + 2a_3 a_5 a_6) - a_4 (a_2 a_5^2 - a_3^2 a_5 + 2a_1 a_2 a_6)}{(a_2^2 + a_3^2)(a_1 a_6 - a_2 a_5)}, \]  
\tag{15} 
with the velocities 
\[ V_x = \frac{\beta (a_2^2 + a_3^2)}{a_1^2 + a_3^2}, \quad V_y = \frac{\beta (a_2^2 + a_5^2)}{a_1^2 + a_5^2}. \]  
\tag{16} 

To describe this kind of lump-like solution (13), we take the parameters \( \beta = 1, a_1 = -1, a_2 = -5, a_4 = 2, a_5 = -2, a_6 = 2, a_8 = 1 \). The lump-like solution reads 
\[ u = \frac{2(25x + 5y + 143t - 20)}{(841t^2 + 286xt - 58yt + 25x^2 + 10xy + 145y^2 - 200t - 40x - 80y + 25)^2}. \]  
\tag{17} 

The 3-dimensional plot and density plot for this lump-like solution (17) are presented in Fig. 2.
3. Summary and discussions

In summary, a (2 + 1)-dimensional STO equation is derived by adding one term $u_{xy}$. The tri-linear form of the (2 + 1)-dimensional STO equation is obtained by the Painlevé analysis. By solving the tri-linear form, we get two cases solutions of (8) and (12) by introducing the quadratic function solution. A dipole-mode soliton and the lump-like solution of the (2 + 1)-dimensional STO equation are generated by using the tri-linear form. The phenomena of the dipole-mode soliton and the lump-like solution are given by figures. Like the Hirota bilinear form, the lump-like solution can be also obtained with the tri-linear form. In this paper, we construct a new (2 + 1)-dimensional STO equation. We can study other integrability properties of the new (2 + 1)-dimensional STO equation in the future, such as lump-kink interaction solutions [39,40], lump-soliton interaction solutions [41–43], diversity of wave solutions [44] and interaction solutions among solitons and other complicated waves [45–49].

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References


