A non-autonomous Gardner equation and its integrability: Solitons, positons and breathers

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\textbf{A B S T R A C T}

The work studies some integrable properties and soliton type solutions of a non-autonomous Gardner equation with damping and forcing terms. A bilinear form, a bilinear Bäcklund transformation and a Lax pair are derived for the considered Gardner equation explicitly. \(K\)-Soliton solution with proper existence condition, smooth positons, breathers and their interaction solutions are presented via the bilinear form. Moreover, the amplitude as well as velocity of the soliton solutions are derived, and a first-order breather solution and a second-order smooth positon are generated from the two-soliton solution. The interaction between a single-breather solution and the single-soliton solution and the interaction of a second-order smooth positon and the single-soliton solution are studied analytically, based on the three-soliton solution. Profiles of various types of the obtained solutions and their interactions are illustrated graphically.

1. Introduction

The study of the dispersion of solitary waves in diverse nonlinear systems has recently attracted a lot of attention. Soliton approaches are widely applicable in many physics and engineering disciplines. In a number of physical systems, the Korteweg–de Vries (KdV) equation, or some of its relatives, has come to be recognized as a classic model for the characterization of long waves with weak nonlinearity and weak dispersion \cite{1,2}. For example, internal waves of gravity in canals with altering section widths \cite{3,4}, ion-acoustic waves in plasmas \cite{5,6}, Bose–Einstein condensates in weakly interaction molecular gases, and shallow water flows in canals and seas have all been studied using the KdV equation and its variants with the quadratic nonlinearity \cite{7,8}. Again, the modified KdV-type equations along with cubic nonlinearity have emerged in areas like interfacial waves in a different-layer liquid with changing depths \cite{9} and Alfvén waves in different plasma environment \cite{10,11}. One particular type of extended KdV equation, also known as the Gardner equation, was created with the KdV quadratic nonlinearity as well as the modified KdV cubic nonlinearity. It can refer to characterize the dust-acoustic waves in a dusty plasma \cite{12}, the internal waves in organized shear flows in the sea or atmosphere \cite{13}, and the propagation of wave in a plasma environment consisting with negative ions \cite{14}. The Gardner model shows the rivalry between cubic and quadratic nonlinearities, and dispersion. Based on the asymptotic derivation, the Gardnar equation, defined as follows,

\begin{equation}
\dot{x} + A x x + B x^2 + C x x x = 0
\end{equation}

can describe various events of fluid dynamics in different environments well \cite{1,2}. Here, the study of the dynamics of the basic localized travelling waves of the Gardner model is the primary objective of the current research. A lot of studies show how the extended KdV equation has lately gained popularity as a framework for the explanation of internal solitary waves in shallow waters \cite{15–17}.

Again, it is generally known that particle interactions produce a damping impact to increase in any physical environment. There are numerous more events that can result in dissipation in a dynamical system, such as the resonant energy transfer between molecules and an electrostatic wave in a plasma atmosphere. Investigations conducted on space plasma revealed a considerable impact of various types of outwardly induced damping on wave transmission in plasma environments \cite{18–26}. Additionally, external forces may manifest themselves

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in certain circumstances, such as when flowing water crosses a bottom topography or waves are produced by going ships [27,28]. In accordance with the aforementioned factors, in this essay, we focus on the subsequent non-autonomous Gardner having with external forcing and damping which is presented as,

\[ x_t + A x x_x + B x^2 x_x + C x_{xx} + L x = A(t). \]  

(2)

Here, the function \( f(x,t) \) incorporates the space and time variables \( x \) and \( t \). The coefficients, \( A,B \) and \( C \) represent the coefficients of quadratic nonlinearity, cubic nonlinearity, and dispersion respectively, whereas the damping and forcing coefficients are, respectively represented by \( L \) and \( A(t) \).

The most efficient method for locating various soliton solutions using the dependent variable conversion and the traditional parameter expansion is Hirota’s bilinear method [29,30]. On the grounds of Bell polynomials [31], Lambert showed a compact and constructive approach [32,33] for generating Lax pairs and bilinear Bäcklund transformations (B'Ts) of some integrable systems. As a result of bilinear B'Ts, it is possible to formulate new NLEE solutions based on the existing ones [34]. Many times, the integrability of a nonlinear partial differential equation can be determined in consequence of the Painlevé property [35,36], Lax pair [37], and symmetries [38]. By virtue of a Lax pair for a nonlinear system, a chain of integrable properties viz. Hamiltonian structures [39], infinitely many conserved quantities [40,41], bi-Hamiltonian structures [42], and Darboux transformations [43], can be shown. Hence, the AKNS scheme [44] is exercised to fabricate the Lax pair of Eq. (2), and the integrability of the said system is claimed under some constraints.

Recently, there has been increasing interest in observing more complex nonlinear coherent structures, such as multi-solitons, multi-shocks, breathers, lumps, and rogue waves, etc. [45–52] to nonlinear evolution equations. For the nonlinear Maccari system, Ma et al. [53] investigated the soliton resonances, soliton molecules, especially the V-type and Y-type soliton molecules. In the Caudrey-Dodd-Gibbon equation, Li and Ma [54] also show breathers, soliton molecules, soliton fusions and fissions, and lump waves under constrained conditions. Furthermore, they examined some parametric constraints of a \((3 + 1)\)-dimensional Geng equation [55] in consideration of hybrid soliton and breather waves, solution molecules, and breather molecules. There is no doubt that breathers as a localized periodic wave are of great significance in water wave dynamics, ion-acoustic wave theory in plasma environment, optics, and biophysics, etc. In general, a breather is an unsteady wave that travels in one direction [56,57]. It is additionally established that the modified KdV and Gardner equations possess breather solutions that correspond with breathing wave packets [58–62] in the event of a positive cubic nonlinearity. Solitons and breathers work together to define the asymptotics of the wave field. The breather solutions have the same polarity as this family of solitons and have densities that range from zero to the previously stated algebraic soliton. While the interactions of two solitons with the same polarization were substantially comparable to the instance for the KdV equation, Slyn'yaev [61] got the two-soliton solution for this case under some conditions, using the Darboux transformation, and demonstrated that the interaction of two solitons with opposing directions generated a few virtually distinct characteristics. The main objective of this paper is to find explicit two-soliton solution for this case under some conditions, using the above-mentioned factors. Here, we use Hirota’s method to determine exact solutions to the KdV equation [67], the Hirota-Satsuma coupled KdV system [68], etc. It is interesting to note that, during a soliton-positon collision, the soliton keeps its same shape, while a positon’s carrier wave and its envelope both exhibit finite phase shifts [65,69]. It has the property of being super reflectionless that positons slowly decay and are oscillating solutions which arise in many completely integrable nonlinear partial differential equations [70]. Spectral problems associated with positon solutions have positive eigenvalues, embedded in continuous spectrums. Positon solutions many times remain singular ones for various models, viz. the defocusing mKdV equation [71,72], the SG equation [73], and the Toda lattice [74]. Again, relations among solitons, positons, and breathers are illustrated and studied in [75]. A number of examples of \( n \)-pole solutions contain a smooth positon solution of \( nth \)-order [76,77]. The finding of breather positons, which are essentially a transition phase from higher-order breather waves to rogue waves, is a highly important development [78].

To the very best of our understanding, there is no information in the currently available research about multi-breathers, interacting between breathers and solitons, or positon solutions in the occurrence of damped and forced terms. The issue at hand is how to easily and swiftly extract higher-level soften positons and breather positons through the general \( N \)-soliton solution. Specifically, we are attempting to obtain explicit expressions for solitons and breathers’ interaction in this paper. We address an extremely inventive limit method that gives second-order smooth positons, higher-order smooth positons and breather positons for Eq. (2). The article is arranged as follows:

In Section 2, a bilinear form, a bilinear Bäcklund transformation, a Lax pair are constructed to check the integrability of Eq. (2) under some constraint. Section 3 presents \( K \)-solitons which are derived directly from the bilinear form of the said equation. The interactions between breathers and solitons are derived from the \( K \)-soliton solution in Section 4. In Section 5, using the \( K \)-soliton solution, smooth positons, breathers positons, and soliton positons are achieved using an inventive limit method. In Section 6, the interacting natures of breathers and solitons, and the propagating properties of smooth positons, are illustrated numerically, and significant effects of damping and forcing terms are illustrated numerically with sincere care. Finally, the article is concluded in Section 7.

2. Bilinear form, bilinear BT, and Lax pair

It is crucial to look for many soliton solutions to the non-autonomous Gardner equation in order to comprehend many nonlinear elements in various scientific domains. There are various methods for locating numerous soliton solutions to nonlinear evolution equation problems. Hirota’s approach is particularly appealing, because it is both elegant and straightforward. It can also be used to obtain phase changes. Here, we use Hirota’s method to determine exact solutions to the non-autonomous Gardner equation [29].

2.1. Bilinear form

Using the transformation

\[ \chi = R(t) \left( \ln \frac{H}{G} \right) + \chi_0 + M(t). \]  

(3)

we get the following bilinear form, resulting from an application of the transformation to Eq. (2).

\[ D_t^2 H \cdot G = 0, \]  

(4a)

\[ [D_t + C D_t^2 + P_t] D_t \]  

\[ H \cdot G = 0, \]  

(4b)

which satisfies the conditions

\[ R(t) = -6C, \quad A = -2B[R(t) + M(t)], \]  

(5)

where \( R(t) \), \( M(t) \) and \( P(t) \) are given by

\[ R(t) = 3e^{-L t}, \quad M(t) = e^{-L t} \int e^{L t} A(t) dt, \]  

\[ P(t) = A[M(t) + R(t)] + B[R(t) + M(t)]^2. \]  

(6)
Here $s_0$ is chosen as an integrating constant. Further, if $L = 0$ and $\Delta(t) = 0$, it is clear that $x = x_0(t)$ is a seed solution of Eq. (1), in which $x_0(t)$ is a free real disturbance parameters. Again, if $L \neq 0$ and $\Delta(t) \neq 0$, the seed solution of Eq. (2) can be presented as $x = s_0 e^{-Lt} + e^{-Lt} \int e^{L(t)} \text{d}t$. Now, a different form can be used to describe the bilinear equations (4a)-(4b) of Eq. (2) as below,

$$\mathcal{K}_1(D_x, D_t, D^{1}_{x}, \ldots) H \cdot \mathcal{G} = 0, \quad \mathcal{K}_2(D_x, D_t, D^{1}_{x}, \ldots) H \cdot \mathcal{G} = 0,$$

where $\mathcal{K}_1$ in a function of $D_x, D_t, D^{1}_{x}$ and $\mathcal{K}_2$ in a function of $D^{1}_{x}$ without constant term. The remaining works deals with conclusions about 1, 2, 3, and $K$-soliton solution conditions that were formed utilizing these bilinear equations (7a)-(7b).

### 2.2. Bilinear BT

Now let us present a bilinear BT of the non-autonomous Gardner equation. Assuming that $(H', G')$ and $(H, G)$ are two different solutions of Eq. (2), we will consider the following:

$$Q_1 = [D^2(H' G') H G] - [D^2(H, G) H'] G' = 0, \quad Q_2 = [(D_x + P(t) D_x + CD^2_x H') G] - [(D_x + P(t) D_x + CD^2_x H) G'] = 0.$$

By considering

$$D_x H' = \lambda_1 H'^{G'}, \quad D_x H \cdot G = \lambda_2 H'^{G'},$$

where $\lambda_1, \lambda_2$ are arbitrary constants, we notice that they satisfy the first equation $Q_1$ and from the second equation $Q_2$, we find

$$(D_x + (P(t) + 3 \lambda_1 \lambda_2) D_x + D^3_x) H' \cdot H = 0, \quad (D_x + (P(t) + 3 \lambda_1 \lambda_2) D_x + D^3_x) G' \cdot G = 0.$$

Therefore, a bilinear BT for Eq. (2) becomes

$$D_x H' \cdot G = \lambda_1 H'^{G'}, \quad D_x H \cdot G = \lambda_2 H'^{G'},$$

where $\lambda_1, \lambda_2$ are arbitrary constants, we notice that they satisfy the first equation $Q_1$ and from the second equation $Q_2$, we find

$$(D_x + (P(t) + 3 \lambda_1 \lambda_2) D_x + D^3_x) H' \cdot H = 0, \quad (D_x + (P(t) + 3 \lambda_1 \lambda_2) D_x + D^3_x) G' \cdot G = 0.$$
The dispersion relation is satisfied automatically by using the dispersion
where $\chi$ and $\xi$ are satisfied automatically by using the dispersion
relation (23) is all that the
Figs. 1(a) and 1(b) show that the magnitudes of the forcing ($g_0$) and
damping ($L$) terms have a significant impact on the velocities of
the solitons. As expected, faster wave velocities are produced by
higher values of $g_0$, whereas slower wave velocities are produced by
higher levels of the damping component (see, Fig. 1(c)). The damped and
forced soliton amplitudes (Fig. 1(d)–1(f)) play a similar role to the
soliton velocity.

3.2. 2-soliton solution

We trim the formulas (20) to $H_2$ and $G_2$ and the set given below to
create the 2-soliton solution for Eq. (2),

\[ H_2 = e^{\psi_1} + e^{\psi_2}, \quad G_2 = e^{\psi_1 + \psi_2}, \]
\[ g_1 = r_1 e^{\psi_1} + r_2 e^{\psi_2}, \quad G_2 = e^{\psi_1 + \psi_2}, \]
\[ w_1 = A_1 x + w_1 t - i \frac{\pi}{2} + \psi_0. \]

By making this decision and taking into account that $H_2 = 0$, $G_2 = 0$,
the coefficient of $e^2$ is satisfied automatically by using the dispersion
relation (23). We derive the first-order solution to Eq. (2) by setting
$\epsilon = 1$, without losing generality as,

\[
\chi(x,t) = \sqrt{g_0} e^{-L_2} \left[ \frac{d}{dx} \ln \left( 1 + e^{\psi_1} \right) + e^{-L_2} \int e^{L_2} A(t) dt \right],
\]
where $\psi_1 = \psi_1(x,t) - \left[ P(t)A_1 + C A_1^2 \right] t - i \frac{\pi}{2} + \psi_0$ and $P(t) = A[M(t) + \chi_2 R(t)] + B[\chi_2 R(t) + M(t)]^2$. If the parameters $A_1, \psi_0$ are taken as
real constants, the first-order solution is called a one-soliton solution.
The dispersion relation $\kappa_2(A_1) = 0$ i.e., relation (23) is all that the
one-soliton requirement requires as well.

3.1.1. Amplitude and velocity of solitons

Now, we define the amplitude of the soliton solution as

\[ A_{\text{mp}} = \sqrt{g_0} A_1 e^{-L_2} + M(t). \]

The amplitude is almost unchanged during its propagation and the
typical face of the solitary waves is depicted as

\[ A_1 x = \left[ P(t)A_1 + C A_1^2 \right] t + \psi_0. \]

In the spatial direction, the wave velocity is given as follows:

\[ V_x = \left[ P(t) + C A_1^2 \right] + i P'(t). \]

A particular case

In consideration of $A(t) = g_0 \cos(\Omega t)$, the amplitude of the one-
soliton solution is derived as

\[ A_{\text{mp}} = \sqrt{g_0} A_1 e^{-L_2} + g_0 \frac{L \cos(\Omega t) + \Omega \sin(\Omega t)}{L^2 + \Omega^2}, \]
and the velocity reads

\[ V_x = -g_0 \frac{L \cos(\Omega t) + \Omega \sin(\Omega t)}{L^2 + \Omega^2} \times \frac{\sqrt{g_0} e^{-L_2} \left( \frac{\partial}{\partial t} \ln H(x,t) \right)}{g_0} + e^{-L_2(t)} A(t) dt. \]
3.3. 3-soliton solution

Here, our objective is to obtain a 3-soliton for Eq. (2) by truncating
the expression (20) to $H_3$ and $G_3$. The third-order auxiliary function $H$
and $G$ yields the following when $\varepsilon = 1$ is taken into account,

$$
H = 1 + e^{\phi_1} + e^{\phi_2} + e^{\phi_3} + c_{12}e^{\phi_1+\phi_2} + c_{12}e^{\phi_1+\phi_3}
+ c_{12}e^{\phi_2+\phi_3},
$$

$$
G = 1 - (e^{\phi_1} + e^{\phi_2} + e^{\phi_3} + c_{12}e^{\phi_1+\phi_2} + c_{12}e^{\phi_1+\phi_3}
+ c_{12}e^{\phi_2+\phi_3},
$$

$$
\psi_i = A_i x - \left[ P(t) A_i + CA_i^2 \right] t - \frac{\sigma_i \phi_i}{2} + \phi_i^0,
$$

$$
c_{ij} = (A_i - A_j)^2, \quad i < j, \quad i, j = 1, 2, 3.
$$

(37a)

(37b)

(37c)

Generally, for the case of (37) with the required conditions (5) and (6),
we have discovered several sorts of interaction structures between three
solitons. In order to achieve a third-order soliton solution, equations from (37)
can be substituted into $\chi(x, t) = \frac{\sigma_0 e^{-\Delta t}}{\pi \ln (H(x, t)}} + e^{-\Delta t} \int e^{\Delta t} A(t) dt$. If all the parameters $A_1, A_2, A_3$ and $\phi_1^0, \phi_2^0, \phi_3^0$ are taken
as real constants, the corresponding third-order solution becomes a
3-soliton in addition to the three soliton:

$$
\sum_{\sigma = 1}^{N} K_1(\sigma_1, w_1 + \sigma_2 w_2 + \sigma_3 w_3, A_1 + \sigma_2 A_2 + \sigma_3 A_3) K_2(\sigma_1 A_1 - \sigma_2 A_2) \times
K_3(\sigma_2 A_2 - \sigma_3 A_3) K_3(\sigma_2 A_2 - \sigma_3 A_3) = 0
$$

(38)

3.4. K-soliton solution

Similarly, in accordance with Hirota’s bilinear method using the
bilinear equations (4a)–(4b), the K-soliton solution of Eq. (2) is the
following:

$$
\chi(x, t) = \chi(x, t) = \sigma_0 e^{-\Delta t} \left[ \frac{\Delta t}{\ln (H(x, t)}} + e^{-\Delta t} \int e^{\Delta t} A(t) dt, \right.
$$

where

$$
H = \sum_{d=0,1} \exp \left( \sum_{r<s} d_r d_s M_{rs} + \sum_{r=s} d_r \right),
$$

$$
G = \sum_{d=0,1} (-1)^{\sum_{d<s} d_r} \exp \left( \sum_{r<s} d_r d_s M_{rs} + \sum_{r=s} d_r \right),
$$

$$
\psi_r = A_r x - \left[ P(t) A_r + CA_r^2 \right] t - \frac{\sigma_r \phi_r}{2} + \phi_r^0,
$$

$$
c_{rs} = e^{\Delta t} = \frac{(A_r - A_s)^2}{(A_r + A_s)^2}, \quad 1 \leq r < s \leq K,
$$

(40)

(41a)

(41b)

(41c)

(41d)

with

$$
P(t) = A[M(t) + x_0 R(t)] + B[x_0 R(t) + M(t)]^2.
$$

(42)

In this soliton, $r, s$ are assumed to have values of 1, 2, …, $K$, where $K$
denotes the soliton number and $\phi_r^0$ the phase constants. Moreover, $\sum_{d=0,1}$
and $\sum_{r<s}$ express the summation to the conceivable combinations of
$d = 0, 1 (r, s = 1, 2, \ldots, K)$. The real constants $\phi_r^0 (r = 1, 2, \ldots, K)$ are
arbitrarily taken.

We shall demonstrate that a K-soliton solution (41a)–(41b) exists for
the Hirota bilinear equation (7a)–(7b), if and only if,

$$
\sum_{\sigma = 1}^{N} \left[ \sum_{r=1}^{n} \sigma_r \psi_r \right] \prod_{r<s} K_1(\sigma_r A_r - \sigma_s A_s) = 0, \quad \text{for} \quad n = 1, 3, 5, \ldots \leq K
$$

(43)

and

$$
\sum_{\sigma = 1}^{N} \left[ \prod_{r=1}^{n} \sigma_r \psi_r \right] \sum_{r=1}^{n} \sigma_r A_r \prod_{r<s} K_1(\sigma_r A_r - \sigma_s A_s) = 0, \quad \text{for} \quad n = 2, 4, 6, \ldots \leq K
$$

(44)

where

$$
K_1(\sum_{r=1}^{n} \sigma_r \psi_r) = (\sum_{r=s}^{n} \sigma_r \psi_r) + C(\sigma_1 A_1 + \sigma_2 A_2 + \ldots + \sigma_n A_n)^3
$$

$$
P(t)(\sigma_1 A_1 + \sigma_2 A_2 + \ldots + \sigma_n A_n),
$$

(45)

(46)
where $\sum_{i=1}^{n}$ is the sum of all types of conceivable combinations of $\sigma_i$ (each $\sigma_i$ takes 1 or -1), and $\prod_{i=1}^{n}$ is the product of all types of probable combinations of the $n$ elements.

It is evident that for $n=1$, the identity (43) clearly holds and for $n=2$, the identity (44) holds also. We will now demonstrate these identities, (43) and (44). Let us consider the left-hand side of Eq. (43), and Eq. (44) as $H_1(A_1, A_2, \ldots, A_n)$ and $H_2(A_1, A_2, \ldots, A_n)$ respectively. The terms $H_1$ and $H_2$ are discovered to have the following characteristics:

(i) The polynomial $H_1$ is symmetric as well as homogeneous.

(ii) If $A_1 = A_2$ then $H_1(A_1, A_2, \ldots, A_n) = 2(2A_1)^2 \prod_{i=1}^{n}(A_i^2 - A_1^2)H_1(A_1, A_2, \ldots, A_n)$.

(iii) The polynomial $H_2$ is also symmetric and homogeneous.

For $n = 1$, the identity (43) is simply verified. Assume now that $n = 1$ is the limit of the identity. After that, it is shown that using the properties (i), (ii), $H_1$ can be factored by a homogeneous polynomial of degree $2n-1$ as,

$$\prod_{i=1}^{n}(A_i^2 - A_1^2)^2,$$

which, is symmetric too. However, Eq. (43) demonstrates that the degree of $H_1$ is $n(n-1) + 3$ (which is less than $2n(n-1)$ for $n > 1$).

The identity has been already established and $H_2$ must be zero for $n$.

Now, for $n = 2$, the identity (44) can be easily confirmed. Assume that the identity is valid for $n = 2$. Eventually, we discover that $H_2$ may be factored by a symmetric homogeneous polynomial

$$\prod_{i=1}^{n} A_i \prod_{i=1}^{n}(A_i^2 - A_1^2)^2,$$

of degree $n^2$ by utilizing the properties (iv), (v), and (vi). In contrast, Eq. (44) reveals that the degree of $H_1$ is $n(n-1) + 2$. Therefore, the identity has been established and $H_1$ must be zero for $n$. It follows from this that the Hirota bilinear equation (41a)-(41b) has the K-soliton solution, suggesting that the non-autonomous Gardner equation does as well.

4. Breather, breather-soliton interaction solution from K-soliton solution

From the K-soliton solution (40), we explore breathers and breather-soliton; finally, their complicated interacting behaviour is demonstrated through numerical graphs and figures.

4.1. 1-order breather solution from a two-soliton solution

To find a 1-order breather solution from the previous 2-soliton solution, we consider

$$A_1 = p_1 + iq_1, \quad A_2 = p_1 - iq_1, \quad A_1^0 = \psi_{11}^0 + i\psi_{11}^0, \quad A_2^0 = \psi_{12}^0 - i\psi_{12}^0.$$  

Now, the functions $H$ and $G$ in Eq. (36) are expressed as

$$H = 1 + 2e^{\psi_{11}^0 - \bar{\psi}_{11}^0} \cos(\psi_{12}),$$

$$G = 1 - 2e^{\psi_{11}^0 - \bar{\psi}_{11}^0} \cos(\psi_{12}),$$

$$\psi_{11} = p_1 x + (P(t)p_1 + C(3p_1^2 q_1 - q_1^2))t + \psi_{11}^{01},$$

$$\psi_{12} = q_1 x - (P(t)q_1 + C(3p_1^2 q_1 - q_1^2))t + \psi_{12}^{01},$$

with

$$\psi_{11}^{01}, \psi_{12}^{01}, \text{ being real constants.}$$

Then, we have the first-order breather solution:

$$X_{br} = s_0 e^{-2tL} \left[ \ln \frac{H}{G} \right] + e^{-2tL} \int e^{2t} A(t)dt,$$

where $H, G$ are determined by (51a), (51b), respectively.

The parameters are chosen in a realistic manner, i.e., $p_1 \neq 0$, and $exp(\psi_{12}) > 1$, and thus, the one-breather solution can be generated. Similarly, a general breather can also be obtained; in particular, the period of the one-breather in the $x$-direction is $2\pi$. On the other hand, the breather propagating in the $(x, t)$-plane is localized $t$-direction and periodically occurs in the $x$-direction. Accordingly, we assert the breather solution of Eq. (52) has a periodic oscillating localized wave profile moving with the speed

$$V_{br} = (P(t) + C(3p_1^2 q_1 - q_1^2))t + t^2.$$  

4.2. Interaction between 1-order breather and 1-soliton

The interaction solution between a 1-order breather and a 1-soliton to the non-autonomous Gardner equation can be derived from a 3-soliton solution by setting the parameters as follows:

$$A_1 = p_1 + iq_1, \quad A_2 = p_1 - iq_1, \quad A_3 = 1. (\text{constant})$$

Substituting these into Eq. (37), one can obtain the corresponding interaction solution to the non-autonomous Gardner equation:

$$X_{brs} = s_0 e^{-2tL} \left[ \ln \frac{H}{G} \right] + e^{-2tL} \int e^{2t} A(t)dt,$$

where

$$H = 1 + 2e^{\psi_{11}^0 - \bar{\psi}_{11}^0} \cos(\psi_{12}),$$

$$G = 1 - 2e^{\psi_{11}^0 - \bar{\psi}_{11}^0} \cos(\psi_{12}),$$

$$\psi_{11} = p_1 x + (P(t)p_1 + C(3p_1^2 q_1 - q_1^2))t + \psi_{11}^{01},$$

$$\psi_{12} = q_1 x - (P(t)q_1 + C(3p_1^2 q_1 - q_1^2))t + \psi_{12}^{01},$$

$$\psi_{13} = A_2 x - (P(t)A_2 + C(A_2^2 q_1 - q_1^2))t + \psi_{13}^{01},$$

with

$$\psi_{11}^{01}, \psi_{12}^{01}, \psi_{13}^{01}, \text{ being real constants.}$$

Then, we have the first-order breather solution:

$$X_{brs} = s_0 e^{-2tL} \left[ \ln \frac{H}{G} \right] + e^{-2tL} \int e^{2t} A(t)dt,$$

where $H, G$ are determined by (51a), (51b), respectively.

The parameters are chosen in a realistic manner, i.e., $p_1 \neq 0$, and $exp(\psi_{12}) > 1$, and thus, the one-breather solution can be generated. Similarly, a general breather can also be obtained; in particular, the period of the one-breather in the $x$-direction is $2\pi$. On the other hand, the breather propagating in the $(x, t)$-plane is localized $t$-direction and periodically occurs in the $x$-direction. Accordingly, we assert the breather solution of Eq. (52) has a periodic oscillating localized wave profile moving with the speed

$$V_{br} = (P(t) + C(3p_1^2 q_1 - q_1^2))t + t^2.$$  

In this section, from the solution (41) with Eq. (42) and $K = 4$, using various types of breather-solitons interacting with solitons, we are able
to generate several brand-new interaction structures. We examine how breather-solitons interact with different types of solitons, by applying a complex conjugate condition technique. We select the fourth-order auxiliary function $H$ and $G$ as

$$H = 1 + 2e^{\text{str}_2} + 2e^{\text{str}_3} + e^{\text{str}_4} + \sum_{i,j,k \leq 4} c_{ijkl} e^{\text{str}_i + \text{str}_j + \text{str}_k + \text{str}_l},$$

$$G = 1 - 2e^{\text{str}_2} - e^{\text{str}_3} + e^{\text{str}_4} + \sum_{i,j,k \leq 4} c_{ijkl} e^{\text{str}_i + \text{str}_j + \text{str}_k + \text{str}_l},$$

where $c_{ijkl}$ for $i,j,k \leq 4$.

### 4.3.1. 2-Order breather solution

The 2-order breather type wave can be directly constructed from a 4-soliton solution by (42) with $K = 4$. The fixed restrictive conditions can be fulfilled. Similarly to Eq. (50) by taking advantage of the substitution

$$A_1 = p_1 + i q_1, \ A_2 = p_1 - i q_1, \ A_3 = p_2 + i q_2, \ A_4 = p_2 - i q_2,$$

$$\zeta_1 = \zeta_1^{(0)}, \ \zeta_2 = \zeta_2^{(0)}, \ \zeta_3 = \zeta_3^{(0)}, \ \zeta_4 = \zeta_4^{(0)},$$

we obtain the corresponding 2-breather solution:

$$\chi_{2\text{br}} = s_0 e^{L_z} \left[ \frac{H}{G} \right] + e^{L_z} \int e^{L_z} A(t) dt,$$

with (see Box I) where

$$H = 1 + 2e^{\text{str}_1} \cos(\psi_{12}) + 2e^{\text{str}_2} \cos(\psi_{13}) - q_{112}^2 \frac{e^{\text{str}_1}}{p_1^2} - q_{113}^2 \frac{e^{\text{str}_1}}{p_2^2} + c_{1234} \left( \frac{(l + l_1)^2(l + l_2)^2}{(k_1)^4} + \frac{(l + l_3)^2(l + l_4)^2}{(k_1)^4} \right),$$

$$G = 1 - 2e^{\text{str}_1} \cos(\psi_{12}) - 2e^{\text{str}_2} \cos(\psi_{13}) + q_{112}^2 \frac{e^{\text{str}_1}}{p_1^2} + q_{113}^2 \frac{e^{\text{str}_1}}{p_2^2} + c_{1234} \left( \frac{(l + l_1)^2(l + l_2)^2}{(k_1)^4} + \frac{(l + l_3)^2(l + l_4)^2}{(k_1)^4} \right),$$

$$\psi_{11} = p_1 x - (P_1 p_1 + C(p_1^2 - 3p_1 q_1^2)) t + q_1^0,$$

$$\psi_{12} = q_1 x - (P_1 q_1 + C(-q_1^2 + 3p_1 q_1^2)) t + q_1^0,$$

$$\psi_{31} = p_2 x - (P_2 p_2 + C(p_2^2 - 3p_2 q_2^2)) t + q_2^0,$$

$$\psi_{42} = q_2 x - (P_2 q_2 + C(-q_2^2 + 3p_2 q_2^2)) t + q_2^0,$$

and

$$c_{12} = q_{112}^2, \ c_{23} = q_{113}^2, \ c_{13} = \frac{(l + l_1)^2}{k_1^2}, \ c_{14} = \frac{(l + l_2)^2}{k_1^2},$$

$$c_{24} = \frac{(l + l_3)^2}{k_1^2}, \ c_{34} = \frac{(l + l_4)^2}{k_1^2},$$

$$l = p_1^2 + p_2^2 + q_1^2 + q_2^2, \ l_1 = 2(q_2 p_2 + p_1 q_2), \ l_2 = 2(q_1 p_2 - p_1 q_2),$$

$$k = (p_1 + p_2)^2 + (q_1 + q_2)^2,$$

also, $\xi_{11}, \xi_{12}, \xi_{13}, \xi_{14}, \xi_{21}, \xi_{23}, \xi_{24}, \xi_{31}, \xi_{32}, \xi_{34}, \xi_{42}, \xi_{43}, \xi_{44}$ are arbitrary real constants.

### 4.3.2. Interaction between a 1-order breather and 2-soliton

An interaction solution between a 1-breather and 2-soliton is obtained by choosing $A_1 = p_1 + i q_1, A_2 = p_1 - i q_1$. Now, we determine the 1-order breather and 2-soliton solution by substituting $H, G$ in the equation given below,

$$\chi_{12} = s_0 e^{L_z} \left[ \frac{H}{G} \right] + e^{L_z} \int e^{L_z} A(t) dt,$$

where (see Box II) with

$$\psi_{11} = p_1 x - [P_1 p_1 + C(p_1^2 - 3p_1 q_1^2)] t + q_1^0,$$

$$\psi_{12} = q_1 x - [P_1 q_1 + C(-q_1^2 + 3p_1 q_1^2)] t + q_1^0,$$

$$\psi_{31} = p_2 x - [P_2 p_2 + C(p_2^2 - 3p_2 q_2^2)] t + q_2^0,$$

$$\psi_{42} = q_2 x - [P_2 q_2 + C(-q_2^2 + 3p_2 q_2^2)] t + q_2^0.$$
If we set some of the parameters of Eq. (37) as \( s_{1}^{0} = \eta_{1}^{0} + \ln \left( \frac{2}{\pi} \right), \) \( s_{2}^{0} = \eta_{2}^{0} + \ln \left( \frac{2}{\pi} \right), \) \( s_{3}^{0} = \eta_{1}^{0} + \ln \left( \frac{2}{\pi} \right), \) \( A_{1} = A_{1} + \delta, \) \( A_{2} = A_{2} + \delta, \) \( \text{then taking limit as } \delta \to 0 \text{ yields a 2nd-order smooth positon solution to the non-autonomous Gardner equation (2):} \)

\[
X_{\text{3rd}} = s_{0} e^{-L_{1}} \left[ \ln \frac{H}{G} \right] + e^{-L_{1}} \int e^{L_{1}} A(t) dt, \tag{71a}
\]
\[
H = 1 + a \partial_{A_{1}} e^{\theta_{1}^{(1)} - i \varphi} - a^{2} \frac{2}{(2A_{1})^{2}} e^{2 \theta_{1}^{(1)} - i \varphi}, \tag{71b}
\]
\[
G = 1 - a \partial_{A_{1}} e^{\theta_{1}^{(1)} - i \varphi} - a^{2} \frac{2}{(2A_{1})^{2}} e^{2 \theta_{1}^{(1)} - i \varphi}, \tag{71c}
\]

with \( \theta_{1} = A_{1} x - (P(t)A_{1} + C A_{1}) y + \eta_{1}^{0}. \)

Proof. Utilizing Eq. (70), \( G \) in Eq. (36) is presented as

\[
G = 1 - a \partial_{A_{1}} e^{\theta_{1}^{(1)} - i \varphi} - a^{2} \frac{2}{(2A_{1})^{2}} e^{2 \theta_{1}^{(1)} - i \varphi}.
\]

where \( \theta_{1} = A_{1} x - (P(t)A_{1} + C A_{1}) y + \eta_{1}^{0}, \) \( i = 1, 2, \) which represents the semi-rational expression, when \( \delta \to 0: \)

\[
G = 1 - a \partial_{A_{1}} e^{\theta_{1}^{(1)} - i \varphi} - a^{2} \frac{2}{(2A_{1})^{2}} e^{2 \theta_{1}^{(1)} - i \varphi}.
\]

In a similar manner, \( H \) in Eq. (36) is transformed to

\[
H = 1 + a \partial_{A_{1}} e^{\theta_{1}^{(1)} - i \varphi} - a^{2} \frac{2}{(2A_{1})^{2}} e^{2 \theta_{1}^{(1)} - i \varphi}.
\]

Hence, Eq. (71) is simply verified.

### 5.2. 3rd-order smooth positon from a three-soliton solution

To obtain a third-order smooth positon, we show the following proposition by a similar argument to Proposition 1.

**Proposition 2.** If we choose the parameters of (37) as \( s_{1}^{0} = \eta_{1}^{0} + \ln \left( \frac{2}{\pi} \right), \) \( s_{2}^{0} = \eta_{2}^{0} + \ln \left( \frac{2}{\pi} \right), \) \( \delta_{1}^{0} = \delta_{2}^{0} = 0, \) \( A_{1} = A_{1} + \delta, \) \( A_{2} = A_{2} + \delta, \) \( \text{then taking limit as } \delta \to 0 \text{ yields a smooth third-order positon solution:} \)

\[
X_{\text{3rd}} = s_{0} e^{-L_{1}} \left[ \ln \frac{H}{G} \right] + e^{-L_{1}} \int e^{L_{1}} A(t) dt, \tag{75}
\]

where

\[
H = 1 + a \left( \frac{\partial \theta_{1}}{\partial A_{1}} \right)^{2} + \frac{\partial^{2} \theta_{1}}{\partial A_{1}^{2}} e^{\theta_{1}^{(1)} - i \varphi} - a^{2} \frac{2}{(2A_{1})^{2}} e^{2 \theta_{1}^{(1)} - i \varphi},
\]

\[
G = 1 - a \left( \frac{\partial \theta_{1}}{\partial A_{1}} \right)^{2} + \frac{\partial^{2} \theta_{1}}{\partial A_{1}^{2}} e^{\theta_{1}^{(1)} - i \varphi} - a^{2} \frac{2}{(2A_{1})^{2}} e^{2 \theta_{1}^{(1)} - i \varphi}.
\]
5.3. Interaction between a smooth positon and a soliton

Proposition 3. By setting the parameters $\xi_0^2 = \eta_3^0 + \ln(\frac{\alpha}{\beta})$, $\xi_3^2 = \eta_3^0 + \ln(\frac{\alpha}{\beta})$, $A_2 = A_1 + \delta$ in Eq. (37) and using the limit technique method as $\delta \to 0$, we obtain the corresponding interaction solution between a smooth 2-positon and a 1-soliton:

$$x_{2p+1s}(t) = x_{2p}(t) + x_{1s}(t),$$

where

$$H = 1 + ae^{\frac{\xi_3}{2} + z} \frac{\partial \chi}{\partial A_1} + ae^{\frac{\xi_3}{2} - z} \frac{\partial \chi}{\partial A_1} - \frac{\alpha^2 - \beta^2}{4A_1^2} \xi_1e^{2\theta_1 - i\pi},$$

and the complex parameters $\xi_0^2$ and $\xi_3^2$ have already been defined in Proposition 1. Again, with the exception of the computation step, the argument for Proposition 3 is nearly identical to that for Proposition 1.

6. Results and discussions

We will now talk about a few intriguing characteristics of the presented 1-order breather, soliton-breather, and 2-order breather and positons of the non-autonomous Gardner equation, by employing Hirota's bilinear approach. The properties that are discovered above can be expressed physically in the following ways.

In Fig. 2(a), we see that $x_{2p}$ evolves across a line which makes a constant angle with the $x$ and $t$ axes in the absence of damping and forcing ($L = 0, d(t) = 0$). Fig. 2(b) and 2(c) clearly demonstrate the impact of damping and forcing terms (periodic forcing component $d(t) = \kappa_0 \cos(\Omega t)$). From the visual presentation in Fig. 2(b), it is clear that the breather has lost its structure for damping effect as time passes. On the other hand, a periodic nature is found on the background of the wave structure in Fig. 2(c) because of acting of external forces, which also leads to destruct the smooth form of the breather. Fig. 3(a)-3(c) exhibit the propagation properties of a breather under an action of hyperbolic forcing and constant damping. Externally applied hyperbolic forcing leads to formation of a kinky-breather type wave. In Fig. 3(b), we see the significant impact of damping in the formation of a breather on the hyperbolic background. To exhibit a clear visual presentation of acting of external excitation, contour plots are drawn in Fig. 3(c).

We will further separate our discussion into two examples, solitons of being depressed along with those of elevation, in order to provide an even greater overview of the phases of interactions that occur between a soliton and a breather. We select the first scenario (Fig. 4) as a representative illustration of a breather heading to the left slamming into a depression soliton moving to the right. The breather can be visualized immediately as two small hills on either side of a large valley, with polarity switching after half a cycle. The soliton will initially crash with the 'small hill' on the left, leaving the central valley nearly frozen. Despite each of the valleys properly merging, the soliton interactions in the middle valley occur via transaction identity after traversing the minor hill on the left. When the centre valley ultimately interacts in the middle valley occur via transaction identity after nearly frozen. Despite each of the valleys properly merging, the soliton initially crash with the 'small hill' on the left, leaving the central valley moving to the right. The breather can be expressed physically in the following ways.

$$\xi_{2p+1s}(t) = x_{2p}(t) + x_{1s}(t),$$

where

$$H = 1 + ae^{\frac{\xi_3}{2} + z} \frac{\partial \chi}{\partial A_1} + ae^{\frac{\xi_3}{2} - z} \frac{\partial \chi}{\partial A_1} - \frac{\alpha^2 - \beta^2}{4A_1^2} \xi_1e^{2\theta_1 - i\pi},$$

and take a limit as $\delta \to 0$, we can obtain a second-order breather-positon solution to the non-autonomous Gardner equation as follows:

$$x_{2p+1s}(t) = x_{2p}(t) + x_{1s}(t),$$

where

$$H = 1 + ae^{\frac{\xi_3}{2} + z} \frac{\partial \chi}{\partial A_1} + ae^{\frac{\xi_3}{2} - z} \frac{\partial \chi}{\partial A_1} - \frac{\alpha^2 - \beta^2}{4A_1^2} \xi_1e^{2\theta_1 - i\pi},$$

and

$$\xi_{2p+1s}(t) = x_{2p}(t) + x_{1s}(t),$$

where

$$H = 1 + ae^{\frac{\xi_3}{2} + z} \frac{\partial \chi}{\partial A_1} + ae^{\frac{\xi_3}{2} - z} \frac{\partial \chi}{\partial A_1} - \frac{\alpha^2 - \beta^2}{4A_1^2} \xi_1e^{2\theta_1 - i\pi},$$

and take a limit as $\delta \to 0$, we can obtain a second-order breather-positon solution to the non-autonomous Gardner equation as follows:

$$x_{2p+1s}(t) = x_{2p}(t) + x_{1s}(t),$$

where

$$H = 1 + ae^{\frac{\xi_3}{2} + z} \frac{\partial \chi}{\partial A_1} + ae^{\frac{\xi_3}{2} - z} \frac{\partial \chi}{\partial A_1} - \frac{\alpha^2 - \beta^2}{4A_1^2} \xi_1e^{2\theta_1 - i\pi},$$

and take a limit as $\delta \to 0$, we can obtain a second-order breather-positon solution to the non-autonomous Gardner equation as follows:

$$x_{2p+1s}(t) = x_{2p}(t) + x_{1s}(t),$$

where

$$H = 1 + ae^{\frac{\xi_3}{2} + z} \frac{\partial \chi}{\partial A_1} + ae^{\frac{\xi_3}{2} - z} \frac{\partial \chi}{\partial A_1} - \frac{\alpha^2 - \beta^2}{4A_1^2} \xi_1e^{2\theta_1 - i\pi},$$

and take a limit as $\delta \to 0$, we can obtain a second-order breather-positon solution to the non-autonomous Gardner equation as follows:

$$x_{2p+1s}(t) = x_{2p}(t) + x_{1s}(t),$$

where

$$H = 1 + ae^{\frac{\xi_3}{2} + z} \frac{\partial \chi}{\partial A_1} + ae^{\frac{\xi_3}{2} - z} \frac{\partial \chi}{\partial A_1} - \frac{\alpha^2 - \beta^2}{4A_1^2} \xi_1e^{2\theta_1 - i\pi},$$

and take a limit as $\delta \to 0$, we can obtain a second-order breather-positon solution to the non-autonomous Gardner equation as follows:

$$x_{2p+1s}(t) = x_{2p}(t) + x_{1s}(t),$$

where

$$H = 1 + ae^{\frac{\xi_3}{2} + z} \frac{\partial \chi}{\partial A_1} + ae^{\frac{\xi_3}{2} - z} \frac{\partial \chi}{\partial A_1} - \frac{\alpha^2 - \beta^2}{4A_1^2} \xi_1e^{2\theta_1 - i\pi},$$

and take a limit as $\delta \to 0$, we can obtain a second-order breather-positon solution to the non-autonomous Gardner equation as follows:

$$x_{2p+1s}(t) = x_{2p}(t) + x_{1s}(t),$$

where

$$H = 1 + ae^{\frac{\xi_3}{2} + z} \frac{\partial \chi}{\partial A_1} + ae^{\frac{\xi_3}{2} - z} \frac{\partial \chi}{\partial A_1} - \frac{\alpha^2 - \beta^2}{4A_1^2} \xi_1e^{2\theta_1 - i\pi},$$

and take a limit as $\delta \to 0
diminishing tendency in the evolution of the breather structure because of damping.

In Fig. 5(a)–5(c), a special type of periodic breather is studied. Purely, an X-type breather is formed when a smooth space (in the absence of damping and forcing terms) is considered (Fig. 5(a)). As before, the breather loses its periodic structure in Fig. 5(b) and 5(c) when an external excitation is performed along with a damping.

In Fig. 6(a)–6(c), we see the interaction of a breather and a rarefactive X-shaped soliton in the smooth as well as non smooth spaces. Fig. 6(a)–6(c) show how the propagating properties vary when external excitations in the form of trigonometric and hyperbolic forces act. Under the action of trigonometric forces, the constant background changes to be a periodic base whereas a kink type background appears when hyperbolic forcing is proceed. Additionally, there is a significant bend in the characteristic line of the soliton as well as the breather by reason of the external excitations and dampings (Fig. 6(b) and 6(c)).

Fig. 7(a)–7(c) exhibit the 3D Profiles of 2nd-order smooth positon solution of Eq. (71), under consideration of an external periodic force.
Fig. 5. The three dimensional graphs of 2-breather solution, by considering : $A_1 = 1.6 + 1.2i$, $A_2 = 1.6 - 1.2i$, $A_3 = 1.4 + 1.5i$, $A_4 = 1.3 - 1.5i$, $B = 25$, $\phi_1^0 = 0$, $\phi_2^0 = 0$, $\phi_3^0 = 0$, $\phi_4^0 = 0$, $\gamma_0 = 0.1$, and (a) when $\gamma_0 = 0.0$, $L = 0$, (b) when $A(t) = g_0 \cos(\Omega t)$ and $g_0 = 0.15$, $L = 0.05$, $\Omega = 1.5$, (c) when $A(t) = g_0 \frac{\partial}{\partial r}$ and $g_0 = 0.3$, $L = 0.05$, $\Omega = 1.5$.

Fig. 6. The three dimensional graphs of 1-breather-2-soliton solution, by considering : $A_1 = 1 + i$, $A_2 = 1 - i$, $A_3 = 1$, $A_4 = 1.5$, $B = 25$, $\phi_1^0 = 0$, $\phi_2^0 = 0$, $\phi_3^0 = 0$, $\phi_4^0 = 0$, $\gamma_0 = 0.1$, and (a) when $\gamma_0 = 0.0$, $L = 0$, (b) when $A(t) = g_0 \cos(\Omega t)$ and $g_0 = 0.3$, $L = 0.05$, $\Omega = 1.5$, (c) when $A(t) = g_0 \frac{\partial}{\partial r}$ and $g_0 = 0.3$, $L = 0.05$, $\Omega = 1.5$.

Fig. 7. 3D Profiles of two order smooth positon solution given by (71), by considering : $A(t) = g_0 \cos(\Omega t)$, $A_1 = 1.5$, $B = 25$, $\phi_0^1 = 0$, $\phi_0^2 = 0$, $\phi_0^3 = 0$, $\phi_0^4 = 0$, $\gamma_0 = 0.1$, and (a) when $g_0 = 0.0$, $L = 0$, (b) when $g_0 = 0$, $L = 0.05$, $\Omega = 1$, (c) when $g_0 = 0.3$, $L = 0.05$, $\Omega = 1$.

$(A(t) = g_0 \cos(\Omega t))$ with the adjacency of a damping. Fig. 7(a) presents a smooth positon in a smooth background when damping and forcing are all absent. In Fig. 7(b), the smoothness of the positon is affected due to an act of damping and asymptotically the positon dies out. Additionally, the direction of the positon remarkably changed because of a damping. On the other hand, under the influence of periodic forces, a periodic type of wave backdrop appears in Fig. 7(c). In an approximate ‘bound state’, a set of upward and downward waves move to the right (Fig. 8). In the left far field, the upward wave follows the depressed wave, but in the opposite right afar field, the order is inverted. Physically, it is comparable to a breather with almost zero frequency. A dual pole/two-positon solution and a 2-soliton are fundamentally different from one another since the former’s peaks are spaced apart like the logarithm of time $t$, making the separation distance almost constant.
The peaks of a 2-soliton arrangement will differ as the product of the velocity difference and time. A similar pattern of obsession also appears for the 3rd-order smooth position solution in Fig. 8(a)–8(c). It is significant to note that a damping plays a significant part in both of the aforementioned examples in bending the direction of positon. The interactive wave profiles of a smooth 2-positons and a 1-soliton solution are shown in Fig. 9(a)–9(c). The dampening effect in Fig. 9(b) dulled the positon's structural details. It is noteworthy to observe that a dampening also aids in positioning the positon's orientation. Fig. 9(c) depicts the propagation of the positon on a periodic backdrop due to the application of periodic forcing.

7. Conclusion

This article uses the traditional Hirota’s bilinear approach to present a number of new analytic multi-solitonic, breather, and positon solutions for the non-autonomous Eq. (2). Moreover, the integrability of the non-autonomous Gardner equation is judged by means of its Bäcklund transformation and Lax pair. The final outcomes can be briefly summarized below,

• The integrability of the non-autonomous Gardner equation is checked via the existence of Bäcklund transformation and Lax pair under some constraint; and new integrability conditions are derived.
• The $K$-soliton conditions are examined and a set of new analytic $K$-soliton solution for the presented bilinear Gardner system is derived.
• The corresponding breathers are straightforwardly constructed. For a visual outlook of breathers and positons in the presented bilinear system, some 3D graphs are presented.
• The numerical graphs demonstrate that a damping in the non-autonomous Gardner framework leads to diminish the amplitude as well as speed of the soliton, and in the asymptotic states the soliton may dies out finally.
• Various external forces affect the foundation of a wave background; for instance, a trigonometric force can generate periodic background, and the periodicity increases as the external force strength increases. However, applying the hyperbolic forcing term results in kink-type grounding, and as the magnitude of forcing continues to increase, the height of the compact kink increases. In some situations, damping and forcing involve bending the direction of the soliton also.
• Several parameter constraints are applied to exploration of breather waves originating from two-soliton solutions; further, two-breathers emanated from four-soliton solutions. These waves demonstrate some of their fascinating properties, such as soliton interaction and soliton overlapping, through simulations of solution wave profiles.
• The positon solution for the presented bilinear system is also derived, and additionally, damping and forcing terms have a considerable impact. A strong orientation in the direction of propagation of the positon arises for an act of damping, whereas the background of the positon is improved in a periodical structure due to the excitation of the system.
Here, the crashes of a breather with an elevated soliton or a depressed soliton are investigated. The “central valley” is seen to either sustain resonances during the collision phase or become “frozen” based on the direction of the colliding soliton and the physical features of the breather, such as its frequency. Such movements are incredibly time-dependent and will certainly have a significant impact on the physical procedures that this non-autonomous Gardner equation may simulate. Solitons of the plateau type and formations resembling bores appear in this domain. A few of the consequences of the present experiment is the following claim, which may be shown by the dynamics of the flow and concentration perturbations in a growing interior marine tidal. The temporal and spatial variation that has been observed in coastal internal soliton patterns is most likely caused by breathers and their contact with solitons.

This research has shown that the bilinear Bäcklund transformation in combination with Hirota’s bilinear approach is an effective analytical tool for solving a more general class of nonlinear evolution equations in the domains of engineering and various scientific fields. Our findings might help to clarify the dynamic behaviour of positions and many other complicated non-autonomous solutions. The non-autonomous Gardner equation is discussed and the findings are presented for the first time in this paper. Its physical underpinning can be used to describe a nonlinear phenomenon that depends on it.

CRediT authorship contribution statement

Santanu Raut: Writing – original draft, Visualization, Writing – review & editing. Wen-Xiu Ma: Conceptualization, Supervision, Writing – review & editing. Ranjan Barman: Software, Visualization. Subrata Roy: Investigation, Methodology, Software.

Declaration of competing interest

The authors declare that there is no conflict of interest between the authors.

Data availability

No data was used for the research described in the article.

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