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A non-autonomous Gardner equation and its integrability: Solitons, positons and breathers

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ABSTRACT

The work studies some integrable properties and soliton type solutions of a non-autonomous Gardner equation with damping and forcing terms. A bilinear form, a bilinear Bäcklund transformation and a Lax pair are derived for the considered Gardner equation explicitly. K-Soliton solution with proper existence condition, smooth positons, breathers and their interaction solutions are presented via the bilinear form. Moreover, the amplitude as well as velocity of the soliton solutions are derived, and a first-order breather solution and a second-order smooth positon are generated from the two-soliton solution. The interaction between a single-breather solution and the single-soliton solution and the interaction of a second-order smooth positon and the single-soliton solution are studied analytically, based on the three-soliton solution. Profiles of various types of the obtained solutions and their interactions are illustrated graphically.

1. Introduction

The study of the dispersion of solitary waves in diverse nonlinear systems has recently attracted a lot of attention. Soliton approaches are widely applicable in many physics and engineering disciplines. In a number of physical systems, the Korteweg–de Vries (KdV) equation, or some of its relatives, has come to be recognized as a classic model for the characterization of long waves with weak nonlinearity and weak dispersion [1,2]. For example, internal waves of gravity in canals with altering section widths [3,4], ion-acoustic waves in plasmas [5,6], Bose–Einstein condensates in weakly interaction molecular gases, and shallow water flows in canals and seas have all been studied using the KdV equation and its variants with the quadratic nonlinearity [7,8]. Again, the modified KdV-type equations along with cubic nonlinearity have emerged in areas like interfacial waves in a different-layer liquid with changing depths [9] and Alfven waves in different plasma environment [10,11]. One particular type of extended KdV equation, also known as the Gardner equation, was created with the KdV quadratic nonlinearity as well as the modified KdV cubic nonlinearity. It can refer to charecterize the dust-acoustic waves in a dusty plasma [12], the internal waves in organized shear flows in the sea or atmosphere [13], and the propagation of wave in a plasma environment consisting with negative ions [14]. The Gardner model shows the rivalry between cubic and quadratic nonlinearities, and dispersion. Based on the asymptotic derivation, the Gardnar equation, defined as follows,

$$\chi_t + A\chi\chi_x + B\chi^2\chi_x + C\chi_{xxx} = 0 \tag{1}$$

can describe various events of fluid dynamics in different environments well [1,2]. Here, the study of the dynamics of the basic localized travelling waves of the Gardner model is the primary objective of the current research. A lot of studies show how the extended KdV equation has lately gained popularity as a framework for the explanation of internal solitary waves in shallow waters [15–17].

Again, it is generally known that particle interactions produce a damping impact to increase in any physical environment. There are numerous more events that can result in dissipation in a dynamical system, such as the resonant energy transfer between molecules and an electrostatic wave in a plasma atmosphere. Investigations conducted on space plasma revealed a considerable impact of various types of outwardly induced damping on wave transmission in plasma environments [18–26]. Additionally, external forces may manifest themselves

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in certain circumstances, such as when flowing water crosses a bottom topography or waves are produced by going ships [27,28]. In accordance with the aforementioned factors, in this essay, we focus on the subsequent non-autonomous Gardner having with external forcing and damping which is presented as,

$$\chi_t + A\chi\chi_x + B\chi^2\chi_x + C\chi_{xxx} + L\chi = \Lambda(t). \tag{2}$$

Here, the function $\chi(x,t)$ incorporates the space and time variables x and t. The coefficients, A, B and C represent the coefficients of quadratic nonlinearity, cubic nonlinearity, and dispersion respectively, whereas the damping and forcing coefficients are, respectively represented by L and $\Lambda(t)$.

The most efficient method for locating various soliton solutions using the dependent variable conversion and the traditional parameter expansion is Hirota's bilinear method [29,30]. On the grounds of Bell polynomials [31], Lambert showed a compact and constructive approach [32,33] for generating Lax pairs and bilinear Bäcklund transformations (BTs) of some integrable systems. As a result of bilinear BTs, it is possible to formulate new NLEE solutions based on the existing ones [34]. Many times, the integrability of a nonlinear partial differential equation can be determined in consequence of the Painlevé property [35,36], Lax pair [37], and symmetries [38]. By virtue of a Lax pair for a nonlinear system, a chain of integrable properties viz. Hamiltonian structures [39], infinitely many conserved quantities [40,41], bi-Hamiltonian structures [42], and Darboux transformations [43], can be shown. Hence, the AKNS scheme [44] is exercised to fabricate the Lax pair of Eq. (2), and the integrability of the said system is claimed under some constraints.

Recently, there has been increasing interest in observing more complex nonlinear coherent structures, such as multi-solitons, multi-shocks, breathers, lumps, and rogue waves, etc. [45–52] to nonlinear evolution equations. For the nonlinear Maccari system, Ma et al. [53] investigated the soliton resonances, soliton molecules, especially the V-type and Y-type soliton molecules. In the Caudrey-Dodd-Gibbon equation, Li and Ma [54] also show breathers, soliton molecules, soliton fusions and fissions, and lump waves under constrained conditions. Furthermore, they examined some parametric constraints of a (3 + 1)-dimensional Geng equation [55] in consideration of hybrid soliton and breather waves, solution molecules, and breather molecules. There is no doubt that breathers as a localized periodic wave are of great significance in water wave dynamics, ion-acoustic wave theory in plasma environment, optics, and biophysics, etc. In general, a breather is an unsteady wave that travels in one direction [56,57]. It is additionally established that the modified KdV and Gardner equations possess breather solutions that correspond with breathing wave packets [58–62] in the event of a positive cubic nonlinearity. Solitons and breathers work together to define the asymptotics of the wave field. The breather solutions have the same polarity as this family of solitons and have densities that range from zero to the previously stated algebraic soliton. While the interactions of two solitons with the same polarization were substantially comparable to the instance for the KdV equation, Slyunyaev [61] got the two-soliton solution for this case under some conditions, using the Darboux transformation, and demonstrated that the interaction of two solitons with opposing directions generated a few virtually distinct characteristics. The main objective of this paper is to find explicit features of the interaction between solitons and breathers.

In the year 1992, Matveev introduced the positon solution for the KdV equation, an actual singular real solution of the KdV equation that has remarkable relevance for quantum physical systems containing supertransparent potentials [63]. As Matveev explained in the literature [64], the positons are analogous to solitons in that they are slowly decreasing and oscillating solutions. Positons are in general weakly localized solutions as opposed to solitons, which decay exponentially after mutual collisions, and two positons are stable even after mutual collisions [65,66]. Recently, positons are studied purposefully in diverse nonlinear equations or systems, viz. the extended KdV equation [67], the Hirota-Satsuma coupled KdV system [68], etc. It is interesting to note that, during a soliton-positon collision, the soliton keeps its same shape, while a positon's carrier wave and its envelope both exhibit finite phase shifts [65,69]. It has the property of being super reflectionless that positons slowly decay and are oscillating solutions which arise in many completely integrable nonlinear partial differential equations [70]. Spectral problems associated with positon solutions have positive eigenvalues, embedded in continuous spectrums. Positon solutions many times remain singular ones for various models, viz. the defocusing mKdV equation [71,72], the SG equation [73], and the Toda lattice [74]. Again, relations among solitons, positons, and breathers are illustrated and studied in [75]. A number of examples of n-pole solutions contain a smooth positon solution of nth-order [76,77]. The finding of breather positons, which are essentially a transition phase from higher-order breather waves to rogue waves, is a highly important development [78].

To the very best of our understanding, there is no information in the currently available research about multi-breathers, interacting between breathers and solitons, or positon solutions in the occurrence of damped and forced terms. The issue at hand is how to easily and swiftly extract higher-level soften positons and breather positons through the general N-soliton solution. Specifically, we are attempting to obtain explicit expressions for solitons and breathers' interaction in this paper. We address an extremely inventive limit method that gives second-order smooth positons, higher-order smooth positons and breather positons for Eq. (2). The article is arranged as follows:

In Section 2, a bilinear form, a bilinear Bäcklund transformation, a Lax pair are constructed to check the integrability of Eq. (2) under some constraint. Section 3 presents K-solitons which are derived directly from the bilinear form of the said equation. The interactions between breathers and solitons are derived from the K-soliton solution in Section 4. In Section 5, using the K-soliton solution, smooth positons, breathers positons, and soliton positons are achieved using an inventive limit method. In Section 6, the interacting natures of breathers and solitons, and the propagating properties of smooth positons, are illustrated numerically, and significant effects of damping and forcing terms are illustrated numerically with sincere care. Finally, the article is concluded in Section 7.

2. Bilinear form, bilinear BT, and Lax pair

It is crucial to look for many soliton solutions to the non-autonomous Gardner equation in order to comprehend many nonlinear elements in various scientific domains. There are various methods for locating numerous soliton solutions to nonlinear evolution equation problems. Hirota's approach is particularly appealing, because it is both elegant and straightforward. It can also be used to obtain phase changes. Here, we use Hirota's method to determine exact solutions to the non-autonomous Gardner equation [29].

2.1. Bilinear form

Using the transformation

$$\chi = R(t)\left(\left[\ln\frac{\mathcal{H}}{\mathcal{G}}\right]_x + \chi_0\right) + M(t), \tag{3}$$

we get the following bilinear form, resulting from an application of the transformation to Eq. (2).

$$D_x^2\mathcal{H}\cdot\mathcal{G} = 0, \tag{4a}$$

$$\left[D_t + CD_x^3 + P(t)D_x\right]\mathcal{H}\cdot\mathcal{G} = 0, \tag{4b}$$

which satisfies the conditions

$$BR(t)^2 = -6C,\ A = -2B(\chi_0 R(t) + M(t)), \tag{5}$$

where $R(t)$, $M(t)$ and $P(t)$ are given by

$$R(t) = s_0 e^{-Lt},\ M(t) = e^{-Lt}\int e^{Lt}\Lambda(t)dt,$$
$$P(t) = A[M(t) + \chi_0 R(t)] + B[\chi_0 R(t) + M(t)]^2. \tag{6}$$

Here s_0 is chosen as an integrating constant. Further, if $L = 0$ and $\Delta(t) = 0$, it is clear note that $\chi = \chi_0$ is a seed solution of Eq. (1), in which χ_0 is a free real disturbance parameters. Again, if $L \neq 0$ and $\Delta(t) \neq 0$, the seed solution of Eq. (2) can be presented as $\chi = s_0 e^{-Lt}\chi_0 + e^{-Lt}\int e^{Lt}\Lambda(t)dt$. Now, a different form can be used to describe the bilinear equations (4a)–(4b) of Eq. (2) as below,

$$\mathcal{K}_1(D_x, D_t, D_x^2, D_x^3, ..)\mathcal{H}\cdot\mathcal{G} = 0, \tag{7a}$$

$$\mathcal{K}_2(D_x, D_t, D_x^2, D_x^3, ..)\mathcal{H}\cdot\mathcal{G} = 0, \tag{7b}$$

where $\mathcal{K}_1$ in a function of D_x, D_t, D_x^3 and $\mathcal{K}_2$ in a function of D_x^2 without constant term. The remaining works dealt with conclusions about 1, 2, 3, and K-soliton solution conditions that were formed utilizing these bilinear equations (7a)–(7b).

2.2. Bilinear BT

Now let us present a bilinear BT of the non-autonomous Gardner equation. Assuming that $(\mathcal{H}', \mathcal{G}')$ and $(\mathcal{H}, \mathcal{G})$ are two different solutions of Eq. (2), we will consider the following:

$$Q_1 = [D_x^2(\mathcal{H}'\cdot\mathcal{G}')]\mathcal{H}G - [D_x^2(\mathcal{H}.\mathcal{G})]\mathcal{H}'\mathcal{G}' = 0, \tag{8a}$$

$$Q_2 = [(D_t + P(t)D_x + CD_x^3)\mathcal{H}'.\mathcal{G}']\mathcal{H}\mathcal{G} - [(D_t + P(t)D_x + CD_x^3)\mathcal{H}.G]\mathcal{H}'\mathcal{G}' = 0. \tag{8b}$$

By considering

$$D_x\mathcal{H}'\cdot\mathcal{G} = \lambda_1\mathcal{H}\mathcal{G}', \; D_x\mathcal{H}\cdot\mathcal{G}' = \lambda_2\mathcal{H}'\mathcal{G} \tag{9}$$

where λ_1, λ_2 are arbitrary constants, we notice that they satisfy the first equation Q_1 and from the second equation Q_2, we find

$$(D_t + (P(t) + 3\lambda_1\lambda_2)D_x + D_x^3)\mathcal{H}'\cdot\mathcal{H} = 0, \tag{10a}$$

$$(D_t + (P(t) + 3\lambda_1\lambda_2)D_x + D_x^3)\mathcal{G}'\cdot\mathcal{G} = 0. \tag{10b}$$

Therefore, a bilinear BT for Eq. (2) becomes

$$D_x\mathcal{H}'\cdot\mathcal{G} = \lambda_1\mathcal{H}\mathcal{G}', \tag{11a}$$

$$D_x\mathcal{H}\cdot\mathcal{G}' = \lambda_2\mathcal{H}'\mathcal{G}, \tag{11b}$$

$$(D_t + (P(t) + 3\lambda_1\lambda_2)D_x + D_x^3)\mathcal{H}'\cdot\mathcal{H} = 0, \tag{11c}$$

$$(D_t + (P(t) + 3\lambda_1\lambda_2)D_x + D_x^3)\mathcal{G}'\cdot\mathcal{G} = 0. \tag{11d}$$

2.3. Lax pair

A Lax pair is often considered to ensure a kind of integrability of an NLEE. The unique Laurent series solution to a partial differential equation can frequently be found when the equation is Painlevé integrable. For a fully integrable system, a Hamiltonian structure, a Lax pair, and an N-soliton solution always exist. In the current study, we assert that Eq. (2) is completely integrable under the constraint:

$$A = -2Be^{-Lt}\int e^{Lt}\Lambda(t)dt, \; C = -\frac{1}{6}B\left(s_0e^{-Lt}\right)^2, \; \chi_0 = 0 \tag{12}$$

The integrability of the system is claimed through the existence of a Lax pair with a nonautonomous term, where Γ will be used to denote the spectral eigenvalue. To build a Lax pair of Eq. (2), we launch the functions $\mathcal{P}(x,t,\Gamma)$, $\mathcal{Q}(x,t,\Gamma)$, $\mathcal{N}(x,t,\Gamma)$ in the AKNS [44] system, and then a Lax pair of Eq. (2) can be presented as

$$\phi_x = \mathcal{U}\phi \quad \text{and} \quad \phi_t = \mathcal{V}\phi, \tag{13}$$

where

$$\mathcal{U} = \begin{pmatrix} \Gamma & \frac{\chi(x,t)-M(t)}{R(t)} \\ \frac{\chi(x,t)-M(t)}{R(t)} & -\Gamma \end{pmatrix} \quad \text{and} \quad \mathcal{V} = \begin{pmatrix} \mathcal{P}(x,t,\Gamma) & \mathcal{Q}(x,t,\Gamma) \\ \mathcal{N}(x,t,\Gamma) & -\mathcal{P}(x,t,\Gamma) \end{pmatrix} \tag{14}$$

Here $\mathcal{U}$ and $\mathcal{V}$ are two 2×2 null-trace matrices, and the eigenvalue Γ is independent of x and t. In this equation, the function $\phi = (\phi_1, \phi_2)^T$, specifying the transpose of a matrix. $\mathcal{P}(x,t,\Gamma)$, $\mathcal{Q}(x,t,\Gamma)$, $\mathcal{N}(x,t,\Gamma)$, are three classes that are simultaneously enlarged with respect to Γ as follows:

$$\mathcal{P}(x,t,\Gamma) = \mathcal{A}_0(x,t) + \mathcal{A}_1(x,t)\Gamma + \mathcal{A}_2(x,t)\Gamma^2 + \mathcal{A}_3(x,t)\Gamma^3 \tag{15a}$$

$$\mathcal{Q}(x,t,\Gamma) = \mathcal{B}_0(x,t) + \mathcal{B}_1(x,t)\Gamma + \mathcal{B}_2(x,t)\Gamma^2 \tag{15b}$$

$$\mathcal{N}(x,t,\Gamma) = \mathcal{C}_0(x,t) + \mathcal{C}_1(x,t)\Gamma + \mathcal{C}_2(x,t)\Gamma^2 \tag{15c}$$

where

$$\mathcal{A}_0(x,t) = 0, \; \mathcal{A}_1(x,t) = 2C\left(\frac{\chi(x,t)-M}{s_0e^{-Lt}}\right)^2 - 6C\frac{M^2}{(s_0e^{-Lt})^2},$$
$$\mathcal{A}_2(x,t) = 0, \; \mathcal{A}_3(x,t) = -4C,$$
$$\mathcal{B}_0(x,t) = 2C\left(\frac{\chi(x,t)-M}{s_0e^{-Lt}}\right)^3 - 6C\frac{M^2}{(s_0e^{-Lt})^2}\left(\frac{\chi(x,t)-M}{s_0e^{-Lt}}\right)$$
$$-C\frac{\chi_{xx}(x,t)}{s_0e^{-Lt}}, \; \mathcal{B}_1(x,t) = -2C$$
$$\frac{\chi_x(x,t)}{s_0e^{-Lt}}, \; \mathcal{B}_2(x,t) = -4C\left(\frac{\chi(x,t)-M}{s_0e^{-Lt}}\right),$$
$$\mathcal{C}_0(x,t) = 2C\left(\frac{\chi(x,t)-M}{s_0e^{-Lt}}\right)^3 - 6C\frac{M^2}{(s_0e^{-Lt})^2}$$
$$\left(\frac{\chi(x,t)-M}{s_0e^{-Lt}}\right) - C\frac{\chi_{xx}(x,t)}{s_0e^{-Lt}}, \; \mathcal{C}_1(x,t) = 2C\frac{\chi_x(x,t)}{s_0e^{-Lt}},$$
$$\mathcal{C}_2(x,t) = -4C\left(\frac{\chi(x,t)-M}{s_0e^{-Lt}}\right).$$

It is straightforward to prove that the zero curvature equation holds:

$$\mathcal{U}_t - \mathcal{V}_x + \mathcal{U}\mathcal{V} - \mathcal{V}\mathcal{U} = 0, \tag{16}$$

with the chosen matrices, $\mathcal{U}$ and $\mathcal{V}$. Because Eq. (16) yields three equations if we substitute $\mathcal{U}$ and $\mathcal{V}$ there,

$$\frac{\partial\Gamma}{\partial t} - \frac{\partial\mathcal{P}}{\partial x} + p\mathcal{N} - p\mathcal{Q} = 0, \tag{17}$$

$$\frac{\partial p}{\partial t} - \frac{\partial\mathcal{Q}}{\partial x} + 2\Gamma\mathcal{Q} - 2p\mathcal{P} = 0, \tag{18}$$

$$\frac{\partial p}{\partial t} - \frac{\partial\mathcal{N}}{\partial x} + 2p\mathcal{P} - 2\Gamma\mathcal{N} = 0, \tag{19}$$

where $p = \frac{\chi(x,t)-M}{s_0e^{-Lt}}$ and $M = M(t) = e^{-Lt}\int e^{Lt}\Lambda(t)dt$.

Eq. (17) is satisfied identically and the presence of non-autonomous Gardner equation (2), follows directly from Eqs. (18) and (19). So in the sense of existence of lax pairs Eq. (2) becomes integrable. In the meantime, it should be noted that Eq. (2) is integrable only if the constraint (12) is met. This implies non-autonomous Gardner equation with constant coefficients are inherently integrable, unlike variable coefficient Gardner equation [79] and variable coefficient extended forced KdV equation [80], which have restrictive constraints.

3. Multi-soliton solutions

The propagating characteristics of solitons are expressed in three segments in this section. At first, a multi-soliton solution is derived for the present system employing Hirota's bilinear method. Recently, Ma in Refs. [30,81] provided a generalized algorithm to demonstrate the Hirota N-soliton condition of bilinear equations in (1+1), (2+1)-dimensions. Also, the N-soliton solution existence requirements for the m-KdV equation are included in Ref. [82]. The authors' goal in proving the K-soliton condition for the non-autonomous Gardner equation is the focus of the current section. We introduce the following expansion to find the K-soliton solutions for Eq. (2),

$$\mathcal{H} = 1 + \epsilon\mathcal{H}_1 + \epsilon^2\mathcal{H}_2 + \epsilon^3\mathcal{H}_3 + \cdots + \epsilon^k\mathcal{H}_k, \tag{20a}$$

$$\mathcal{G} = 1 + \epsilon\mathcal{G}_1 + \epsilon^2\mathcal{G}_2 + \epsilon^3\mathcal{G}_3 + \cdots + \epsilon^k\mathcal{G}_k \tag{20b}$$

$\mathcal{H}_k, \mathcal{G}_k$ $(k = 1,2,3,\dots,K)$ are the real functions of x, t and ϵ is a real constant. When expansion (20) is substituted into the bilinear equation (4), and the coefficients on each power order of ϵ are allowed to disappear, we obtain,

$$\epsilon^0 : (D_x^2)1.1 = 0 \tag{21a}$$

$$\left(D_t + CD_x^3 + P(t)D_x\right) 1.1 = 0 \tag{21b}$$

$$\epsilon^1 : (D_x^2)(\mathcal{H}_1.1 + 1.\mathcal{G}_1) = 0 \tag{21c}$$

$$\left(D_t + CD_x^3 + P(t)D_x\right)(\mathcal{H}_1.1 + 1.\mathcal{G}_1) = 0 \tag{21d}$$

$$\epsilon^2 : (D_x^2)(\mathcal{H}_2.1 + \mathcal{H}_1.\mathcal{G}_1 + 1.\mathcal{G}_2) = 0 \tag{21e}$$

$$\left(D_t + CD_x^3 + P(t)D_x\right)(\mathcal{H}_2.1 + \mathcal{H}_1.\mathcal{G}_1 + 1.\mathcal{G}_2) = 0 \tag{21f}$$

$$\epsilon^3 : (D_x^2)(\mathcal{H}_3.1 + \mathcal{H}_2.\mathcal{G}_1 + \mathcal{H}_1.\mathcal{G}_2 + 1.\mathcal{G}_3) = 0 \tag{21g}$$

$$\left(D_t + CD_x^3 + P(t)D_x\right)(\mathcal{H}_3.1 + \mathcal{H}_2.\mathcal{G}_1 + \mathcal{H}_1.\mathcal{G}_2 + 1.\mathcal{G}_3) = 0 \tag{21h}$$

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3.1. 1-soliton solution

To obtain the one-solution for Eq. (2), we truncate the formulas (20) to $\mathcal{H}_1$ and $\mathcal{G}_1$ based on the differential equation theory and Hirota's D-operators properties. Now, setting

$$\mathcal{H}_1 = e^{\psi_1}, \mathcal{G}_1 = r_1 e^{\psi_1}, \psi_1 = A_1 x + w_1 t - i\frac{\pi}{2} + \xi_1^0 \tag{22}$$

where r_1, A_1, w_1, and ξ_1^0 are all nonzero constants. Replacing expressions (22) to Eqs. (21c)–(21d), we obtain

$$r_1 = -1, w_1 = -\left[P(t)A_1 + CA_1^3\right]. \tag{23}$$

By making this decision and taking into account that $\mathcal{H}_2 = 0, \mathcal{G}_2 = 0$, the coefficient of ϵ^2 is satisfied automatically by using the dispersion relation (23). We derive the first-order solution to Eq. (2) by setting $\epsilon = 1$, without losing generality as,

$$\chi(x,t) = s_0 e^{-Lt}\left[\frac{\partial}{\partial x} ln\left(\frac{1+e^{\psi_1}}{1-e^{\psi_1}}\right)\right] + e^{-Lt}\int e^{Lt}A(t)dt, \tag{24}$$

where $\psi_1 = A_1 x - \left[P(t)A_1 + CA_1^3\right] t - i\frac{\pi}{2} + \xi_1^0$ and $P(t) = A[M(t) + \chi_0 R(t)] + B[\chi_0 R(t) + M(t)]^2$. If the parameters A_1, ξ_1^0 are taken as real constants, the first-order solution is called a one-soliton solution. The dispersion relation $\mathcal{K}_1(A_1) = 0$ i.e., relation (23) is all that the one-soliton requirement requires as well.

3.1.1. Amplitude and velocity of solitons

Now, we define the amplitude of the soliton solution as

$$Amp_s = s_0 A_1 e^{-Lt} + M(t). \tag{25}$$

The amplitude is almost unchanged during its propagation and the typical face of the solitary waves is depicted as

$$A_1 x = \left[P(t)A_1 + CA_1^3\right] t + \xi_1^0. \tag{26}$$

In the spatial direction, the wave velocity is given as follows:

$$V_s = \left[P(t) + CA_1^2\right] + tP'(t). \tag{27}$$

A particular case

In consideration of $A(t) = g_0\cos(\Omega t)$, the amplitude of the one-soliton solution is derived as

$$Amp_s = s_0 A_1 e^{-Lt} + g_0\frac{L\cos(\Omega t) + \Omega\sin(\Omega t)}{L^2+\Omega^2}, \tag{28}$$

and the velocity reads

$$V_s = B\left[\chi_0^2(s_0e^{-Lt})^2 + 2s_0g_0e^{-Lt}\frac{L\cos(\Omega t)+\Omega\sin(\Omega t)}{L^2+\Omega^2}\chi_0 + g_0^2\left(\frac{L\cos(\Omega t)+\Omega\sin(\Omega t)}{L^2+\Omega^2}\right)^2\right] + Ag_0\frac{L\cos(\Omega t)+\Omega\sin(\Omega t)}{L^2+\Omega^2} + CA_1^2 + tP'(t), \tag{29}$$

where

$$P(t) = B\left[\chi_0^2(s_0e^{-Lt})^2 + 2s_0g_0e^{-Lt}\frac{L\cos(\Omega t)+\Omega\sin(\Omega t)}{L^2+\Omega^2}\chi_0 + g_0^2\left(\frac{L\cos(\Omega t)+\Omega\sin(\Omega t)}{L^2+\Omega^2}\right)^2\right] + A[g_0\frac{L\cos(\Omega t)+\Omega\sin(\Omega t)}{L^2+\Omega^2} + \chi_0 s_0 e^{-Lt}]. \tag{30}$$

Here, g_0 and Ω respectively designate the amplitude and frequency of an external force. Finally, the velocity of a solitary wave is designated as V_s that presents the size and the direction of the spatial direction's propagation. The magnitude as well as the sign of the velocity contribute important roles to govern the wave dynamics by fixing the speed and direction of the wave.

Figs. 1(a) and 1(b) show that the magnitudes of the forcing (g_0) and damping (L) terms have a significant impact on the velocities of the solitons. As expected, faster wave velocities are produced by higher values of g_0, whereas slower wave velocities are produced by higher levels of the damping component (see, Fig. 1(c)). The damped and forced soliton amplitudes (Fig. 1(d)–1(f)) play a similar role to the soliton velocity.

3.2. 2-soliton solution

We trim the formulas (20) to $\mathcal{H}_2$ and $\mathcal{G}_2$ and the set given below to create the 2-soliton solution for Eq. (2),

$$\mathcal{H}_1 = e^{\psi_1} + e^{\psi_2}, \mathcal{H}_2 = c_{12}e^{\psi_1+\psi_2}, \tag{31a}$$

$$\mathcal{G}_1 = r_1e^{\psi_1} + r_2e^{\psi_2}, \mathcal{G}_2 = c_{12}e^{\psi_1+\psi_2}, \tag{31b}$$

$$\psi_1 = A_1x + w_1t - i\frac{\pi}{2} + \xi_1^0, \tag{31c}$$

$$\psi_2 = A_2x + w_2t - i\frac{\pi}{2} + \xi_2^0. \tag{31d}$$

By replacing Eqs. (31) in Eqs. (21c)–(21f), and considering the coefficient of ϵ and ϵ^2, provides the results

$$r_i = -1, w_i = -\left[P(t)A_i + CA_i^3\right], c_{12} = \frac{(A_1-A_2)^2}{(A_1+A_2)^2} i = 1,2. \tag{32}$$

The 2-soliton requirement is given by the coefficient ϵ^3 as,

$$\sum_{\sigma=\pm1}(\prod_{r=1}^{2}\sigma_r)\mathcal{K}_2(\sigma_1w_1+\sigma_2w_2,\sigma_1A_1+\sigma_2A_2)\mathcal{K}_2(\sigma_1A_1-\sigma_2A_2) = 0 \tag{33}$$

where

$$\mathcal{K}_2(\sigma_1w_1+\sigma_2w_2,\sigma_1A_1+\sigma_2A_2) = (\sigma_1A_1+\sigma_2A_2)^2, \tag{34}$$

$$\mathcal{K}_2(\sigma_1A_1-\sigma_2A_2) = (\sigma_1A_1-\sigma_2A_2)^2. \tag{35}$$

Thus, it is confirmed there exist always a 2-soliton solution for the non-autonomous Gardner equation (2) under the condition (33). Thus, the second-order solution can be gained (when $\epsilon = 1$) by substituting

$$\mathcal{H} = 1 + e^{\psi_1} + e^{\psi_2} + \frac{(A_1-A_2)^2}{(A_1+A_2)^2}e^{\psi_1+\psi_2}, \tag{36a}$$

$$\mathcal{G} = 1 - (e^{\psi_1} + e^{\psi_2}) + \frac{(A_1-A_2)^2}{(A_1+A_2)^2}e^{\psi_1+\psi_2}, \tag{36b}$$

$$\psi_j = A_jx - \left[P(t)A_j + CA_j^3\right]t - i\frac{\pi}{2} + \xi_j^0, \; j = 1,2, \tag{36c}$$

into $\chi(x,t) = s_0e^{-Lt}\left[\frac{\partial}{\partial x}ln(\frac{\mathcal{H}(x,t)}{\mathcal{G}(x,t)})\right] + e^{-Lt}\int e^{Lt}A(t)dt$.

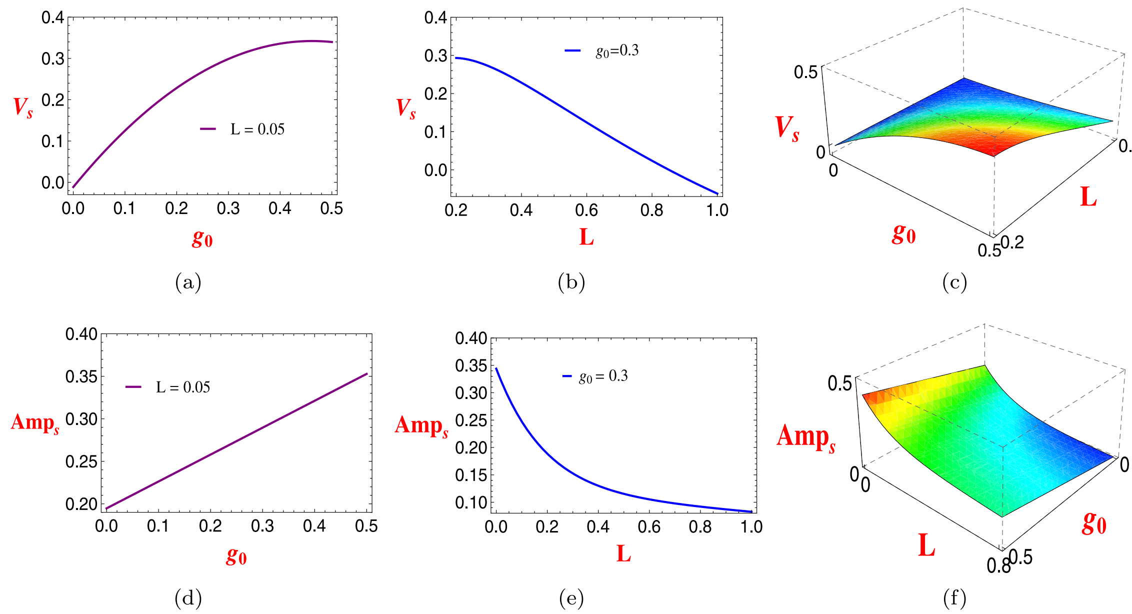

Fig. 1. Profiles of single soliton's velocity, when $\Lambda(t) = g_0\cos(\Omega t)$, $\chi_0 = 0.1$, $B = 3$, $A_1 = 0.5$, $t = 5$, $s_0 = 0.5$, $\Omega = 1.5$.

3.3. 3-soliton solution

Here, our objective is to obtain a 3-soliton for Eq. (2) by truncating the expression (20) to $\mathcal{H}_3$ and $\mathcal{G}_3$. The third-order auxiliary function $\mathcal{H}$ and $\mathcal{G}$ yields the following form when $\epsilon = 1$ is taken into account,

$$\mathcal{H} = 1 + e^{\psi_1} + e^{\psi_2} + e^{\psi_3} + c_{12}e^{\psi_1+\psi_2} + c_{13}e^{\psi_1+\psi_3} + c_{23}e^{\psi_2+\psi_3} + c_{12}c_{13}c_{23}e^{\psi_1+\psi_2+\psi_3}, \tag{37a}$$

$$\mathcal{G} = 1 - (e^{\psi_1} + e^{\psi_2} + e^{\psi_3}) + c_{12}e^{\psi_1+\psi_2} + c_{13}e^{\psi_1+\psi_3} + c_{23}e^{\psi_2+\psi_3} - c_{12}c_{13}c_{23}e^{\psi_1+\psi_2+\psi_3}, \tag{37b}$$

$$\psi_j = A_j x - \left[P(t)A_j + CA_j^3\right]t - i\frac{\pi}{2} + \xi_j^0,$$
$$c_{ij} = \frac{(A_i - A_j)^2}{(A_i + A_j)^2},\ i < j,\ j = 1, 2, 3. \tag{37c}$$

Generally, for the case of (37) with the required conditions (5) and (6), we have discovered several sorts of interaction structures between three riple-solitons. In order to achieve a third-order soliton solution, equations from (37) can be substituted into $\chi(x,t) = s_0e^{-Lt}\left[\frac{\partial}{\partial x}ln(\frac{\mathcal{H}(x,t)}{\mathcal{G}(x,t)})\right] + e^{-Lt}\int e^{Lt}\Lambda(t)dt$. If all the parameters A_1, A_2, A_3 and ξ_1^0, ξ_2^0, ξ_3^0 are taken as real constants, the corresponding third-order solution becomes a 3-soliton in addition to the three soliton condition:

$$\sum_{\sigma=\pm1}\mathcal{K}_1(\sigma_1w_1 + \sigma_2w_2 + \sigma_3w_3, \sigma_1A_1 + \sigma_2A_2 + \sigma_3A_3)\mathcal{K}_2(\sigma_1A_1 - \sigma_2A_2)\times \mathcal{K}_2(\sigma_2A_2 - \sigma_3A_3)\mathcal{K}_2(\sigma_1A_1 - \sigma_3A_3) = 0 \tag{38}$$

where

$$\mathcal{K}_2(\sigma_1A_1 - \sigma_2A_2) = (\sigma_1A_1 - \sigma_2A_2)^2, \mathcal{K}_2(\sigma_2A_2 - \sigma_3A_3) = (\sigma_2A_2 - \sigma_3A_3)^2,\ \mathcal{K}_2(\sigma_1A_1 - \sigma_3A_3) = (\sigma_1A_1 - \sigma_3A_3)^2,\ \mathcal{K}_1(\sigma_1w_1 + \sigma_2w_2 + \sigma_3w_3, \sigma_1A_1 + \sigma_2A_2 + \sigma_3A_3) = (\sigma_1w_1 + \sigma_2w_2 + \sigma_3w_3) + C(\sigma_1A_1 + \sigma_2A_2 + \sigma_3A_3)^3 + P(t)(\sigma_1A_1 + \sigma_2A_2 + \sigma_3A_3). \tag{39}$$

3.4. K-soliton solution

Similarly, in accordance with Hirota's bilinear method using the bilinear equations (4a)–(4b), the K-soliton solution of Eq. (2) is the following:

$$\chi_K(x,t) = \chi(x,t) = s_0e^{-Lt}\left[\frac{\partial}{\partial x}ln\left(\frac{\mathcal{H}(x,t)}{\mathcal{G}(x,t)}\right)\right] + e^{-Lt}\int e^{Lt}\Lambda(t)dt, \tag{40}$$

where

$$\mathcal{H} = \sum_{d=0,1} exp\left(\sum_{r<s}^{K} d_rd_sM_{rs} + \sum_{r=1}^{K} d_r\psi_r\right), \tag{41a}$$

$$\mathcal{G} = \sum_{d=0,1}(-1)^{\sum_r d_r}exp\left(\sum_{r<s}^{K} d_rd_sM_{rs} + \sum_{r=1}^{K} d_r\psi_r\right), \tag{41b}$$

$$\psi_r = A_rx - \left[P(t)A_r + CA_r^3\right]t - i\frac{\pi}{2} + \xi_r^0, \tag{41c}$$

$$c_{rs} = e^{\mathcal{M}_{rs}} = \frac{(A_r - A_s)^2}{(A_r + A_s)^2},\ 1 \le r < s \le K, \tag{41d}$$

with

$$P(t) = A[M(t) + \chi_0R(t)] + B[\chi_0R(t) + M(t)]^2. \tag{42}$$

In this soliton, r, s are assumed to have values of $1, 2, \ldots, K$, where K denotes the soliton number and ξ_r^0 the phase constants. Moreover, $\sum_{d=0,1}$ and $\sum_{r<s}$ express the summation to the conceivable combinations of $d = 0, 1\,(r, s = 1, 2, \ldots, K)$. The real constants $\xi_r^0\,(r = 1, 2, \ldots, K)$ are arbitrarily taken.

We shall demonstrate that a K-soliton solution (41a)–(41b) exists for the Hirota bilinear equation (7a)–(7b), if and only if,

$$\sum_{\sigma=\pm1}\mathcal{K}_1(\sum_{r=1}^{n}\sigma_r\boldsymbol{w}_r, \sum_{r=1}^{n}\sigma_r\boldsymbol{A}_r)\prod_{r<s}^{n}\mathcal{K}_2(\sigma_r\boldsymbol{A}_r - \sigma_s\boldsymbol{A}_s) = 0,\ \textbf{for}\ n = 1, 3, 5, \ldots \le K \tag{43}$$

and

$$\sum_{\sigma=\pm1}(\prod_{r=1}^{n}\sigma_r)\mathcal{K}_2(\sum_{r=1}^{n}\sigma_r\boldsymbol{w}_r, \sum_{r=1}^{n}\sigma_r\boldsymbol{A}_r)\prod_{r<s}^{n}\mathcal{K}_2(\sigma_r\boldsymbol{A}_r - \sigma_s\boldsymbol{A}_s) = 0,\ \textbf{for}\ n = 2, 4, 6, \ldots \le K \tag{44}$$

where

$$\mathcal{K}_1(\sum_{r=1}^{n}\sigma_r\boldsymbol{w}_r, \sum_{r=1}^{n}\sigma_r\boldsymbol{A}_r) = (\sigma_1w_1 + \sigma_2w_2 + \cdots + \sigma_nw_n) + C(\sigma_1A_1 + \sigma_2A_2 + \cdots + \sigma_nA_n)^3 + P(t)(\sigma_1A_1 + \sigma_2A_2 + \cdots + \sigma_nA_n), \tag{45}$$

$$\mathcal{K}_2(\sigma_r\boldsymbol{A}_r - \sigma_s\boldsymbol{A}_s) = (\sigma_rA_r^2 - \sigma_sA_s^2)^2, \tag{46}$$

$$\mathcal{K}_2(\sum_{r=1}^{n}\sigma_r \boldsymbol{w}_r, \sum_{r=1}^{n}\sigma_r \boldsymbol{A}_r) = (\sigma_1 A_1 + \sigma_2 A_2 + \cdots + \sigma_n A_n)^2, \tag{47}$$

where $\sum_{\sigma=\pm 1}$ is the sum of all types of conceivable combinations of σ_i (each σ_i takes 1 or −1), and $\prod_{r<s}^{n}$ is the product of all types of probable combinations of the n elements.

It is evident that for $n = 1$, the identity (43) clearly holds and for $n = 2$, the identity (44) holds also. We will now demonstrate these identities, (43) and (44). Let us consider the left-hand side of Eq. (43), and Eq. (44) as $H_1(A_1, A_2, \dots, A_n)$ and $H_2(A_1, A_2, \dots, A_n)$ respectively. The terms H_1 and H_2 are discovered to have the following characteristics:

(i) The polynomial H_1 is symmetric as well as homogeneous.

(ii) if $A_1 = A_2$ then $H_1(A_1, A_2, \dots, A_n) = 2(2A_1)^2 \prod_{r=3}^{n}(A_1^2 - A_r^2)^2 H_1(A_3, A_4, \dots, A_n)$.

(iii) The polynomial H_2 is also symmetric and homogeneous.

(iv) if $A_1 = 0$, $H_2 = 0$.

(v) if $A_1 = A_2$ then $H_2(A_1, A_2, \dots, A_n) = -2(2A_1)^2 \prod_{r=3}^{n}(A_1^2 - A_r^2)^2 H_2(A_3, A_4, \dots, A_n)$.

For $n = 1$, the identity (43) is simply verified. Assume now that $n - 1$ is the limit of the identity. After that, it is shown that using the properties (i), (ii), H_1 can be factored by a homogeneous polynomial of degree $2n(n - 1)$ as,

$$\prod_{r<s}^{n}(A_r^2 - A_s^2)^2, \tag{48}$$

which, is symmetric too. However, Eq. (43) demonstrates that the degree of H_1 is $n(n - 1) + 3$ (which is less than $2n(n - 1)$ for $n > 1$). The identity has been already established and H_1 must be zero for n.

Now, for $n = 2$, the identity (44) can be easily confirmed. Assume that the identity is valid for $n-2$. Eventually, we discover that H_2 may be factored by a symmetric homogeneous polynomial

$$\prod_{r=1}^{n} A_r \prod_{r<s}^{n}(A_r^2 - A_s^2)^2. \tag{49}$$

of degree n^2 by utilizing the properties (iv), (v), and (vi). In contrast, Eq. (44) reveals that the degree of H_2 is $n(n - 1) + 2$. Therefore, the identity has been established and H_2 must be zero for n. It follows from this that the Hirota bilinear equation (41a)–(41b) has the K-soliton solution, suggesting that the non-autonomous Gardner equation does as well.

4. Breather, breather-soliton interaction solution from K-soliton solution

From the K-soliton solution (40), we explore breathers and breather-soliton; finally, their complicated interacting behaviour is demonstrated through numerical graphs and figures.

4.1. 1-order breather solution from a two-soliton solution

To find a 1-order breather solution from the previous 2-soliton solution, we consider

$$A_1 = p_1 + iq_1,\ A_2 = p_1 - iq_1,\ \xi_1^0 = \xi_{11}^0 + i\xi_{12}^0,\ \xi_2^0 = \xi_{11}^0 - i\xi_{12}^0. \tag{50}$$

Now, the functions $\mathcal{H}$ and $\mathcal{G}$ in Eq. (36) are expressed as

$$\mathcal{H} = 1 + 2e^{\psi_{11} - i\frac{\pi}{2}}\cos(\psi_{12}) - \frac{q_1^2}{p_1^2}e^{2\psi_{11} - i\pi}, \tag{51a}$$

$$\mathcal{G} = 1 - 2e^{\psi_{11} - i\frac{\pi}{2}}\cos(\psi_{12}) - \frac{q_1^2}{p_1^2}e^{2\psi_{11} - i\pi}, \tag{51b}$$

$$\psi_{11} = p_1 x - (P(t)p_1 + C(p_1^3 - 3p_1 q_1^2))t + \xi_{11}^0, \tag{51c}$$

$$\psi_{12} = q_1 x - (P(t)q_1 + C(3p_1^2 q_1 - q_1^3))t + \xi_{12}^0, \tag{51d}$$

where ξ_{11}^0, ξ_{12}^0 being real constants. Then, we have the first-order breather solution:

$$\chi_{br} = s_0 e^{-Lt}\left[\ln\frac{\mathcal{H}}{\mathcal{G}}\right]_x + e^{-Lt}\int e^{Lt}\Lambda(t)dt, \tag{52}$$

where $\mathcal{H}$, $\mathcal{G}$ are determined by (51a), (51b), respectively.

The parameters are chosen in a realistic manner, i.e., $p_1 \neq 0$, and $\exp(c_{12}) > 1$, and thus, the one-breather solution can be generated. Similarly, a general breather can also be obtained; in particular, the period of the one-breather in the x-direction is $\frac{2\pi}{q_1}$. On the other hand, the breather propagating in the (x, t)-plane is localized t-direction and periodically occurs in the x-direction. Accordingly, we assert the breather solution of Eq. (52) has a periodic oscillating localized wave profile moving with the speed

$$V_{br} = (P(t) + C(3p_1^2 - q_1^2))t + tP'(t). \tag{53}$$

4.2. Interaction between 1-order breather and 1-soliton

The interaction solution between a 1-order breather and a 1-soliton to the non-autonomous Gardner equation can be derived from a 3-soliton solution by setting the parameters as follows:

$$A_1 = p_1 + iq_1,\ A_2 = p_1 - iq_1,\ \xi_1^0 = \xi_{11}^0 + i\xi_{12}^0,\ \xi_2^0 = \xi_{11}^0 - i\xi_{12}^0 \text{ and } A_3 = (\text{a constant}). \tag{54}$$

Substituting these into Eq. (37), one can obtain the corresponding interaction solution to the non-autonomous Gardner equation:

$$\chi_{br-s} = s_0 e^{-Lt}\left[\ln\frac{\mathcal{H}}{\mathcal{G}}\right]_x + e^{-Lt}\int e^{Lt}\Lambda(t)dt, \tag{55}$$

where

$$\begin{aligned}\mathcal{H} = {} & 1 + 2e^{\psi_{11} - i\frac{\pi}{2}}\cos(\psi_{12}) - \frac{q_1^2}{p_1^2}e^{2\psi_{11} - i\pi} + e^{\psi_3 - i\frac{\pi}{2}} \\ & - \frac{q_1^2}{p_1^2}\frac{\left[(p_1^2 - A_3^2 + q_1^2)^2 + 4q_1^2 A_3^2\right]^2}{\left[(p_1 + A_3)^2 + q_1^2\right]^4}e^{2\psi_{11} + \psi_3 - i\frac{3\pi}{2}} + \\ & 2\frac{\left((p_1^2 - A_3^2 + q_1^2)^2 - 4q_1^2 A_3^2\right)}{\left[(p_1 + A_3)^2 + q_1^2\right]^2}\cos(\psi_{12})e^{\psi_3 + \psi_{11} - i\pi} \\ & - \frac{8q_1 A_3(p_1^2 - A_3^2 + q_1^2)}{\left[(p_1 + A_3)^2 + q_1^2\right]^2}\sin(\psi_{12})e^{\psi_3 + \psi_{11} - i\pi},\end{aligned} \tag{56a}$$

$$\begin{aligned}\mathcal{G} = {} & 1 - 2e^{\psi_{11} - i\frac{\pi}{2}}\cos(\psi_{12}) - \frac{q_1^2}{p_1^2}e^{2\psi_{11} - i\pi} - e^{\psi_3 - i\frac{\pi}{2}} \\ & + \frac{q_1^2}{p_1^2}\frac{\left[(p_1^2 - A_3^2 + q_1^2)^2 + 4q_1^2 A_3^2\right]^2}{\left[(p_1 + A_3)^2 + q_1^2\right]^4}e^{2\psi_{11} + \psi_3 - i\frac{3\pi}{2}} + \\ & 2\frac{\left((p_1^2 - A_3^2 + q_1^2)^2 - 4q_1^2 A_3^2\right)}{\left[(p_1 + A_3)^2 + q_1^2\right]^2}\cos(\psi_{12})e^{\psi_3 + \psi_{11} - i\pi} \\ & - \frac{8q_1 A_3(p_1^2 - A_3^2 + q_1^2)}{\left[(p_1 + A_3)^2 + q_1^2\right]^2}\sin(\psi_{12})e^{\psi_3 + \psi_{11} - i\pi},\end{aligned} \tag{56b}$$

with

$$\psi_{11} = p_1 x - (P(t)p_1 + C(p_1^3 - 3p_1 q_1^2))t + \xi_{11}^0, \tag{57a}$$

$$\psi_{12} = q_1 x - (P(t)q_1 + C(3p_1^2 q_1 - q_1^3))t + \xi_{12}^0, \tag{57b}$$

$$\psi_3 = A_3 x - \left(P(t)A_3 + CA_3^3\right)t + \xi_3^0. \tag{57c}$$

and ξ_{11}^0, ξ_{12}^0 being constants.

4.3. Interaction structures between solitons and breathers

In this section, from the solution (41) with Eq. (42) and $K = 4$, using various types of breather-solitons interacting with solitons, we are able

$$\mathcal{H} = 1 + 2e^{\psi_{11}-i\frac{\pi}{2}}\cos(\psi_{12}) + 2e^{\psi_{31}-i\frac{\pi}{2}}\cos(\psi_{32}) - \frac{q_1^2}{p_1^2}e^{2\psi_{11}-i\pi} - \frac{q_2^2}{p_2^2}e^{2\psi_{31}-i\pi} + c_{12}c_{34}\frac{(l^2+l_2^2)^2(l^2+l_1^2)^2}{(kk_1)^4}$$
$$e^{2\psi_{11}+2\psi_{31}-i2\pi} + \frac{l^2-l_2^2}{k^2}2\cos(\psi_{12}+\psi_{31})e^{\psi_{11}+\psi_{31}-i\pi} + \frac{l^2-l_1^2}{k_1^2}2\cos(\psi_{12}-\psi_{32})e^{\psi_{11}+\psi_{31}-i\pi} +$$
$$c_{12}\left[2\frac{(l^2+l_1l_2)^2-(ll_2-ll_1)^2}{(kk_1)^2}\cos(\psi_{32}) - 4\frac{(l^2+l_1l_2)(ll_2-ll_1)}{(kk_1)^2}\sin(\psi_{32})\right]e^{2\psi_{11}+\psi_{31}-i\frac{3\pi}{2}} +$$
$$c_{34}\left[2\frac{(l^2+l_1l_2)^2-(ll_2+ll_1)^2}{(kk_1)^2}\cos(\psi_{12}) - 4\frac{(l^2-l_1l_2)(ll_2+ll_1)}{(kk_1)^2}\sin(\psi_{12})\right]e^{\psi_{11}+2\psi_{31}-i\frac{3\pi}{2}} -$$
$$4\frac{ll_2}{k^2}\sin(\psi_{12}+\psi_{32})e^{\psi_{11}+\psi_{31}-i\pi} - 4\frac{ll_1}{k_1^2}\sin(\psi_{12}-\psi_{32})e^{\psi_{11}+\psi_{31}-i\pi}, \tag{61a}$$

$$\mathcal{G} = 1 - 2e^{\psi_{11}-i\frac{\pi}{2}}\cos(\psi_{12}) - 2e^{\psi_{31}-i\frac{\pi}{2}}\cos(\psi_{32}) - \frac{q_1^2}{p_1^2}e^{2\psi_{11}-i\pi} - \frac{q_2^2}{p_2^2}e^{2\psi_{31}-i\pi} + c_{12}c_{34}\frac{(l^2+l_2^2)^2(l^2+l_1^2)^2}{(kk_1)^4}$$
$$e^{2\psi_{11}+2\psi_{31}-i2\pi} + \frac{l^2-l_2^2}{k^2}2\cos(\psi_{12}+\psi_{31})e^{\psi_{11}+\psi_{31}-i\pi} + \frac{l^2-l_1^2}{k_1^2}2\cos(\psi_{12}-\psi_{32})e^{\psi_{11}+\psi_{31}-i\pi} -$$
$$c_{12}\left[2\frac{(l^2+l_1l_2)^2-(ll_2-ll_1)^2}{(kk_1)^2}\cos(\psi_{32}) - 4\frac{(l^2+l_1l_2)(ll_2-ll_1)}{(kk_1)^2}\sin(\psi_{32})\right]e^{2\psi_{11}+\psi_{31}-i\frac{3\pi}{2}} -$$
$$c_{34}\left[2\frac{(l^2+l_1l_2)^2-(ll_2+ll_1)^2}{(kk_1)^2}\cos(\psi_{12}) - 4\frac{(l^2-l_1l_2)(ll_2+ll_1)}{(kk_1)^2}\sin(\psi_{12})\right]e^{\psi_{11}+2\psi_{31}-i\frac{3\pi}{2}} -$$
$$4\frac{ll_2}{k^2}\sin(\psi_{12}+\psi_{32})e^{\psi_{11}+\psi_{31}-i\pi} - 4\frac{ll_1}{k_1^2}\sin(\psi_{12}-\psi_{32})e^{\psi_{11}+\psi_{31}-i\pi}, \tag{61b}$$

Box I.

to generate several brand-new interaction structures. We examine how breather-solitons interact with different types of solitons, by applying a complex conjugate condition technique. We select the fourth-order auxiliary function $\mathcal{H}$ and $\mathcal{G}$ as

$$\mathcal{H} = 1 + \sum_{i=1}^{4} e^{\psi_i} + \sum_{1\le i<j\le 4} c_{ij}e^{\psi_i+\psi_j} + \sum_{1\le i<j<k\le 4} c_{ijk}e^{\psi_i+\psi_j+\psi_k} + c_{1234}e^{\sum_{i=1}^{4}\psi_i}, \tag{58a}$$

$$\mathcal{G} = 1 - \sum_{i=1}^{4} e^{\psi_i} + \sum_{1\le i<j\le 4} c_{ij}e^{\psi_i+\psi_j} - \sum_{1\le i<j<k\le 4} c_{ijk}e^{\psi_i+\psi_j+\psi_k} + c_{1234}e^{\sum_{i=1}^{4}\psi_i}, \tag{58b}$$

where, c_{ij}, $j = 1,2,3$; $c_{123} = c_{12}c_{13}c_{23}$, and $c_{1234} = c_{12}c_{13}c_{14}c_{23}c_{24}c_{34}$ obey the results (41d), respectively. By substituting (58) into Eq. (40), the fourth-order solution is obtained for Eq. (2). For the fourth-order solution, there are two different sorts of combinations. Below, we discuss two typical problems: (i) the presentation of a 2-breather through 4.3.1, and (ii) 4.3.2 exploring interaction properties between a 1-breather and a 2-soliton.

4.3.1. 2-Order breather solution

The 2-order breather type wave can be directly constructed from a 4-soliton solution by (42) with $K = 4$. The fixed restrictive conditions can be fulfilled. Similarly to Eq. (50) by taking advantage of the substitution

$$A_1 = p_1 + iq_1,\ A_2 = p_1 - iq_1,\ A_3 = p_2 + iq_2, A_4 = p_2 - iq_2,$$
$$\xi_1^0 = \xi_{11}^0 + i\xi_{12}^0, \xi_2^0 = \xi_{11}^0 - i\xi_{12}^0, \xi_3^0 = \xi_{31}^0 + i\xi_{32}^0,\ \xi_4^0 = \xi_{31}^0 - i\xi_{32}^0, \tag{59}$$

we obtain the corresponding 2-breather solution:

$$\chi_{2br} = s_0e^{-Lt}\left[\ln\frac{\mathcal{H}}{\mathcal{G}}\right]_x + e^{-Lt}\int e^{Lt}\Lambda(t)dt, \tag{60}$$

with (see Box I) where

$$\psi_{11} = p_1x - (P(t)p_1 + C(p_1^3 - 3p_1q_1^2))t + \xi_{11}^0, \tag{62a}$$
$$\psi_{12} = q_1x - (P(t)q_1 + C(3p_1^2q_1 - q_1^3))t + \xi_{12}^0, \tag{62b}$$
$$\psi_{31} = p_2x - (P(t)p_2 + C(p_2^3 - 3p_2q_2^2))t + \xi_{31}^0, \tag{62c}$$
$$\psi_{32} = q_2x - (P(t)q_2 + C(3p_2^2q_2 - q_2^3))t + \xi_{32}^0, \tag{62d}$$

and

$$c_{12} = -\frac{q_1^2}{p_1^2},\ c_{34} = -\frac{q_2^2}{p_2^2},\ c_{13} = \frac{(l+il_2)^2}{k^2},\ c_{14} = \frac{(l+il_1)^2}{k_1^2},$$
$$c_{24} = \frac{(l-il_2)^2}{k^2},\ c_{23} = \frac{(l-il_1)^2}{k_1^2},$$
$$l = p_1^2 - p_2^2 + q_1^2 - q_2^2,\ l_1 = 2(q_1p_2 + p_1q_2), l_2 = 2(q_1p_2 - p_1q_2),$$
$$k = (p_1 + p_2)^2 + (q_1 + q_2)^2,$$
$$k_1 = (p_1 + p_2)^2 + (q_1 - q_2)^2. \tag{63}$$

Also, $\xi_{11}^0, \xi_{12}^0, \xi_{31}^0, \xi_{32}^0, p_1,\ q_1,\ p_2,\ q_2$ are arbitrary real constants.

4.3.2. Interaction between a 1-order breather and 2-soliton

An interaction solution between a 1-breather and 2-soliton is obtained by choosing $A_1 = p_1 + iq_1$, $A_2 = p_1 - iq_1$. Now, we determine the 1-order breather and 2-soliton solution by substituting $\mathcal{H}$, $\mathcal{G}$ in the equation given below,

$$\chi_{1bsp} = s_0e^{-Lt}\left[\ln\frac{\mathcal{H}}{\mathcal{G}}\right]_x + e^{-Lt}\int e^{Lt}\Lambda(t)dt, \tag{64}$$

where (see Box II) with

$$\psi_{11} = p_1x - \left[P(t)p_1 + C(p_1^3 - 3p_1q_1^2)\right]t + \xi_{11}^0, \tag{66a}$$
$$\psi_{12} = q_1x - \left[P(t)q_1 + C(-q_1^3 + 3p_1^2q_1)\right]t + \xi_{12}^0, \tag{66b}$$
$$\psi_3 = A_3x - \left[P(t)A_3 + CA_3^3\right]t + \xi_3^0, \tag{66c}$$
$$\psi_4 = A_4x - \left[P(t)A_4 + CA_4^3\right]t + \xi_4^0, \tag{66d}$$

$$\mathcal{H} = 1 + 2e^{\psi_{11}-i\frac{\pi}{2}}\cos(\psi_{12}) + e^{\psi_3-i\frac{\pi}{2}} + e^{\psi_4-i\frac{\pi}{2}} - \frac{q_1^2}{p_1^2}e^{2\psi_{11}-i\pi} + c_{34}e^{\psi_3+\psi_4-i\pi} + e^{\psi_{11}+\psi_3-i\pi}[2c_{131}\cos(\psi_{12})- 2c_{132}\sin(\psi_{12})] + e^{\psi_{11}+\psi_4-i\pi}\left[2c_{141}\cos(\psi_{12}) - 2c_{142}\sin(\psi_{12})\right] + c_{12}[(c_{131}^2 + c_{132}^2)e^{2\psi_{11}+\psi_3-i\frac{3\pi}{2}} + (c_{141}^2 + c_{142}^2)e^{2\psi_{11}+\psi_4-i\frac{3\pi}{2}}] + c_{34}e^{\psi_{11}+\psi_3+\psi_4-i\frac{3\pi}{2}}\left[2(c_{131}c_{141} - c_{132}c_{142})\cos(\psi_{12}) - 2(c_{131}c_{142} + c_{132}c_{141})\sin(\psi_{12})\right] + c_{12}c_{34}(c_{131}^2 + c_{132}^2)(c_{141}^2 + c_{142}^2)e^{2\psi_{11}+\psi_3+\psi_4-i2\pi}, \tag{65a}$$

$$\mathcal{G} = 1 - 2e^{\psi_{11}-i\frac{\pi}{2}}\cos(\psi_{12}) - e^{\psi_3-i\frac{\pi}{2}} - e^{\psi_4-i\frac{\pi}{2}} - \frac{q_1^2}{p_1^2}e^{2\psi_{11}-i\pi} + c_{34}e^{\psi_3+\psi_4-i\pi} + e^{\psi_{11}+\psi_3-i\pi}[2c_{131}\cos(\psi_{12})- 2c_{132}\sin(\psi_{12})] + e^{\psi_{11}+\psi_4-i\pi}\left[2c_{141}\cos(\psi_{12}) - 2c_{142}\sin(\psi_{12})\right] - c_{12}[(c_{131}^2 + c_{132}^2)e^{2\psi_{11}+\psi_3-i\frac{3\pi}{2}} + (c_{141}^2 + c_{142}^2)e^{2\psi_{11}+\psi_4-i\frac{3\pi}{2}}] - c_{34}e^{\psi_{11}+\psi_3+\psi_4-i\frac{3\pi}{2}}\left[2(c_{131}c_{141} - c_{132}c_{142})\cos(\psi_{12}) - 2(c_{131}c_{142} + c_{132}c_{141})\sin(\psi_{12})\right] + c_{12}c_{34}(c_{131}^2 + c_{132}^2)(c_{141}^2 + c_{142}^2)e^{2\psi_{11}+\psi_3+\psi_4-i2\pi}, \tag{65b}$$

Box II.

and

$$c_{12} = -\frac{q_1^2}{p_1^2},\ c_{34} = \frac{(A_3 - A_4)^2}{(A_3 + A_4)^2},\ c_{131} = \frac{(p_1^2 - A_3^2 + q_1^2)^2 - 4q_1^2A_3^2}{[(p_1 + A_3)^2 + q_1^2]^2},$$
$$c_{141} = \frac{(p_1^2 - A_4^2 + q_1^2)^2 - 4q_1^2A_4^2}{[(p_1 + A_3)^2 + q_1^2]^2},$$
$$c_{132} = \frac{4q_1A_3(p_1^2 - A_3^2 + q_1^2)}{[(p_1 + A_3)^2 + q_1^2]^2},\ c_{142} = \frac{4q_1A_4(p_1^2 - A_4^2 + q_1^2)}{[(p_1 + A_4)^2 + q_1^2]^2}. \tag{67}$$

We conclude through mathematical induction that the following is true based on the above results:

Inference 1. *By maintaining the parameters in complex conjugate relations, it is possible to derive the high-order breather solution from the K-soliton solution (40). The m-order breather solution is specified by the K-soliton solution (40), if the parameters fulfil the following constraint conditions.*

$$K = 2\,m,\ A_i = A_{i+1}^*,\ \xi_i^0 = \xi_{i+1}^{0*}, i = 1, \dots, K. \tag{68}$$

The K-soliton solution then becomes the m-order breather solution.

Inference 2. *It is possible to derive interactive solutions between m-order breather and k-soliton by letting*

$$K = 2\,m + n, A_{2m} = A_{2m-1}^*,\ \xi_{2m}^0 = \xi_{2m-1}^{0*}. \tag{69}$$

In the above declaration, A_{2m+j}, ξ_{2m+j}^0, $j = 1, 2, \dots, k$ are arbitrarily chosen real constants, in the expression of K-soliton solution (40).

5. Positons, breather-positon interaction, positon-soliton interaction solutions from K-soliton solution

This section explores smooth positons, positon-soliton and breather-positons using the K-soliton solution (40); finally, their complicated interplay is illustrated numerically.

5.1. 2nd-order smooth positon from a two-soliton solution

Now, we utilize the following proposition for finding a second-order smooth positon from a two-soliton solution,

Proposition 1. *If we set some of the parameters of Eq. (36) to be*

$$\xi_1^0 = \ln(-\frac{\alpha}{\delta}) + \eta_1^0,\ \xi_2^0 = \ln\left(\frac{\alpha}{\delta}\right) + \eta_1^0,\ A_2 = A_1 + \delta, \tag{70}$$

then taking limit as $\delta \to 0$ yields a 2nd-order smooth positon solution to the non-autonomous Gardner equation (2):

$$\chi_{2sp} = s_0e^{-Lt}\left[\ln\frac{\mathcal{H}}{\mathcal{G}}\right]_x + e^{-Lt}\int e^{Lt}\Lambda(t)dt, \tag{71a}$$

$$\mathcal{H} = 1 + \alpha\partial_{A_1}e^{\theta_1-i\frac{\pi}{2}} - \frac{\alpha^2}{(2A_1)^2}e^{2\theta_1-i\pi}, \tag{71b}$$

$$\mathcal{G} = 1 - \alpha\partial_{A_1}e^{\theta_1-i\frac{\pi}{2}} - \frac{\alpha^2}{(2A_1)^2}e^{2\theta_1-i\pi}, \tag{71c}$$

with $\theta_1 = A_1x - (P(t)A_1 + CA_1^3)t + \eta_1^0$.

Proof. Utilizing Eq. (70), $\mathcal{G}$ in Eq. (36) is presented as

$$\mathcal{G} = 1 - \frac{\alpha}{\delta}(-e^{\theta_1-i\frac{\pi}{2}} + e^{\theta_2-i\frac{\pi}{2}}) - c_{12}\frac{\alpha^2}{\delta^2}e^{\theta_1+\theta_2-i\pi}, \tag{72}$$

where $\theta_i = A_ix - (P(t)A_i + CA_i^3)t + \eta_1^0$, $i = 1, 2$, which represents the semi-rational expression, when $\delta \to 0$:

$$\mathcal{G} = 1 - \alpha\partial_{A_1}e^{\theta_1-i\frac{\pi}{2}} - \frac{\alpha^2}{(2A_1)^2}e^{2\theta_1-i\pi}. \tag{73}$$

In a similar manner, $\mathcal{H}$ in Eq. (36) is transformed to

$$\mathcal{H} = 1 + \alpha\partial_{A_1}e^{\theta_1-i\frac{\pi}{2}} - \frac{\alpha^2}{(2A_1)^2}e^{2\theta_1-i\pi}. \tag{74}$$

Hence, Eq. (71) is simply verified.

5.2. 3rd-order smooth positon from a three-soliton solution

To obtain a third-order smooth positon, we show the following proposition by a similar argument to Proposition 1.

Proposition 2. *If we choose the parameters of (37) as $\xi_1^0 = \eta_1^0 + \ln(\frac{\alpha}{\delta^2})$, $\xi_2^0 = \eta_1^0 + \ln(\frac{-2\alpha}{\delta^2})$, $\xi_3^0 = \eta_1^0 + \ln(\frac{\alpha}{\delta^2})$, $A_2 = A_1 + \delta$, $A_3 = A_1 + 2\delta$, then taking limit as $\delta \to 0$ yields a smooth third-order position solution:*

$$\chi_{3SP} = s_0e^{-Lt}\left[\ln\frac{\mathcal{H}}{\mathcal{G}}\right]_x + e^{-Lt}\int e^{Lt}\Lambda(t)dt \tag{75}$$

where

$$\mathcal{H} = 1 + \alpha\left(\left(\frac{\partial\theta_1}{\partial A_1}\right)^2 + \frac{\partial^2\theta_1}{\partial A_1^2}\right)e^{\theta_1-i\frac{\pi}{2}} - 8\frac{\alpha^3}{(2A_1)^6}e^{3\theta_1-i\frac{3\pi}{2}} + \frac{\alpha^2}{2}\left(4\frac{\partial^2c_{11}}{\partial A_1^2} - 8\frac{\partial c_{11}}{\partial A_1}\frac{\partial\theta_1}{\partial A_1} - 4c_{11}\left(\frac{\partial\theta_1}{\partial A_1}\right)^2 + c_{11}\frac{\partial^2\theta_1}{\partial A_1^2}\right)e^{2\theta_1-i\pi}, \tag{76a}$$

$$\mathcal{G} = 1 - \alpha\left(\left(\frac{\partial\theta_1}{\partial A_1}\right)^2 + \frac{\partial^2\theta_1}{\partial A_1^2}\right)e^{\theta_1-i\frac{\pi}{2}} + 8\frac{\alpha^3}{(2A_1)^6}e^{3\theta_1-i\frac{3\pi}{2}} +$$

$$\frac{\alpha^2}{2}\left(4\frac{\partial^2 c_{11}}{\partial A_1^2}-8\frac{\partial c_{11}}{\partial A_1}\frac{\partial\theta_1}{\partial A_1}-4c_{11}\left(\frac{\partial\theta_1}{\partial A_1}\right)^2+c_{11}\frac{\partial^2\theta_1}{\partial A_1^2}\right)e^{2\theta_1-i\pi},\quad (76b)$$

with $c_{11}=\frac{1}{4A_1^2}$.

5.3. Interaction between a smooth positon and a soliton

Proposition 3. *By setting the parameters* $\xi_1^0=\eta_1^0+\ln(\frac{-\alpha}{\delta})$, $\xi_2^0=\eta_1^0+\ln(\frac{\alpha}{\delta})$, $A_2=A_1+\delta$ *in Eq. (37) and using the limit technique method as* $\delta\to 0$, *we obtain the corresponding interaction solution between a smooth 2-positon and a 1-soliton:*

$$\chi_{2sp-1sol}=s_0e^{-Lt}\left[\ln\frac{\mathcal{H}}{\mathcal{G}}\right]_x+e^{-Lt}\int e^{Lt}\Lambda(t)dt,\quad (77)$$

where

$$\mathcal{H}=1+\alpha e^{\theta_1-i\frac{\pi}{2}}\frac{\partial\theta_1}{\partial A_1}+e^{\psi_3-i\frac{\pi}{2}}-\frac{\alpha^2}{4A_1^2}e^{2\theta_1-i\pi}$$
$$+\alpha e^{\psi_3+\theta_1-i\pi}\left[\frac{\partial c_{13}}{\partial A_1}+c_{13}\frac{\partial\theta_1}{\partial A_1}\right]-\frac{\alpha^2}{4A_1^2}c_{13}^2e^{2\theta_1+\psi_3-i\frac{3\pi}{2}},\quad (78a)$$

$$\mathcal{G}=1-\alpha e^{\theta_1-i\frac{\pi}{2}}\frac{\partial\theta_1}{\partial A_1}-e^{\psi_3-i\frac{\pi}{2}}-\frac{\alpha^2}{4A_1^2}e^{2\theta_1-i\pi}$$
$$+\alpha e^{\psi_3+\theta_1-i\pi}\left[\frac{\partial c_{13}}{\partial A_1}+c_{13}\frac{\partial\theta_1}{\partial A_1}\right]+\frac{\alpha^2}{4A_1^2}c_{13}^2e^{2\theta_1+\psi_3-i\frac{3\pi}{2}},\quad (78b)$$

with $\theta_1=A_1x-\left[P(t)A_1+CA_1^3\right]t+\eta_1^0$, $\psi_3=A_3x-\left[P(t)A_3+CA_3^3\right]t+\xi_3^0$, $c_{13}=\frac{(A_1-A_3)^2}{(A_1+A_3)^2}$.

The terms η_i^0 and c_{ij} have already been defined in Proposition 1. Again, with the exception of the computation step, the argument for Proposition 3 is nearly identical to that for Proposition 1.

Inference 3. *If the parameters given in Eq. (40) are assigned as follows:*

$$K=2\,m,\ A_2=A_1+\delta,\ A_4=A_3+\delta,\dots,A_{2m}=A_{2m-1}+\delta,$$
$$\xi_1^0=\eta_1^0+\ln\left(-\frac{\alpha}{\delta}\right),\ \xi_2^0=\eta_1^0+\ln\left(\frac{\alpha}{\delta}\right),\dots,\xi_{2m}^0=\eta_{2m-1}^0+\ln\left(\frac{\alpha}{\delta}\right),\quad (79)$$

then, the K*-soliton solution will yield a* m *second-order smooth position when* $\delta\to 0$.

5.4. 2-order breather-positon solution

To obtain a second-order breather-positon solution from the 4-soliton solution, we use the following proposition:

Proposition 4. *If we consider some certain parameters of Eq. (58) as*

$$A_2=A_1+\delta,\ A_4=A_3+\delta,\ \xi_1^0=\eta_1^0+\ln\left(-\frac{\alpha}{\delta}\right),\ \xi_2^0=\eta_1^0+\ln\left(\frac{\alpha}{\delta}\right),$$
$$\xi_3^0=\eta_3^0+ln\left(-\frac{\alpha}{\delta}\right),\ \xi_4^0=\eta_3^0+\ln\left(\frac{\alpha}{\delta}\right),\quad (80)$$

and take a limit as $\delta\to 0$, *we can obtain a second-order breather-positon solution to the non-autonomous Gardner equation as follows:*

$$\chi_{2br-sp}=s_0e^{-Lt}\left[\ln\frac{\mathcal{H}}{\mathcal{G}}\right]_x+e^{-Lt}\int e^{Lt}\Lambda(t)dt,\quad (81)$$

where

$$\mathcal{H}=1+\alpha e^{\theta_1-i\frac{\pi}{2}}\frac{\partial\theta_1}{\partial A_1}+\alpha e^{\theta_3-i\frac{\pi}{2}}\frac{\partial\theta_3}{\partial A_3}-\beta^2c_{11}e^{2\theta_1-i\pi}-\beta^2c_{33}e^{2\theta_3-i\pi}+$$
$$\alpha^2\left[\frac{\partial}{\partial A_1}\frac{\partial c_{13}}{\partial A_3}+\frac{\partial c_{13}}{\partial A_1}\frac{\partial\theta_1}{\partial A_1}+\frac{\partial c_{13}}{\partial A_1}\frac{\partial\theta_3}{\partial A_3}+c_{13}\frac{\partial\theta_1}{\partial A_1}\frac{\partial\theta_{13}}{\partial A_3}\right]e^{\theta_1+\theta_3-i\pi}-$$
$$\alpha^3c_{11}\left(2c_{13}\frac{\partial c_{13}}{\partial A_3}+c_{13}^2\frac{\partial\theta_3}{\partial A_3}\right)e^{2\theta_1+\theta_3-i\frac{3\pi}{2}}-$$
$$\alpha^3c_{33}\left(2c_{13}\frac{\partial c_{13}}{\partial A_1}+c_{13}^2\frac{\partial\theta_3}{\partial A_3}\right)e^{\theta_1+2\theta_3-i\frac{3\pi}{2}}+$$
$$\alpha^4c_{11}c_{33}c_{13}^4e^{2\theta_1+2\theta_3-i2\pi},\quad (82a)$$

$$\mathcal{G}=1-\alpha e^{\theta_1-i\frac{\pi}{2}}\frac{\partial\theta_1}{\partial A_1}-\alpha e^{\theta_3-i\frac{\pi}{2}}\frac{\partial\theta_3}{\partial A_3}-\beta^2c_{11}e^{2\theta_1-i\pi}-\beta^2c_{33}e^{2\theta_3-i\pi}+$$
$$\alpha^2\left[\frac{\partial}{\partial A_1}\frac{\partial c_{13}}{\partial A_3}+\frac{\partial c_{13}}{\partial A_1}\frac{\partial\theta_1}{\partial A_1}+\frac{\partial c_{13}}{\partial A_1}\frac{\partial\theta_3}{\partial A_3}+c_{13}\frac{\partial\theta_1}{\partial A_1}\frac{\partial\theta_{13}}{\partial A_3}\right]e^{\theta_1+\theta_3-i\pi}+$$
$$\alpha^3c_{11}\left(2c_{13}\frac{\partial c_{13}}{\partial A_3}+c_{13}^2\frac{\partial\theta_3}{\partial A_3}\right)e^{2\theta_1+\theta_3-i\frac{3\pi}{2}}+$$
$$\alpha^3c_{33}\left(2c_{13}\frac{\partial c_{13}}{\partial A_1}+c_{13}^2\frac{\partial\theta_3}{\partial A_3}\right)e^{\theta_1+2\theta_3-i\frac{3\pi}{2}}+$$
$$\alpha^4c_{11}c_{33}c_{13}^4e^{2\theta_1+2\theta_3-i2\pi},\quad (82b)$$

with

$$\theta_1=A_1x-(P(t)A_1+CA_1^3)t+\eta_1^0,\ \theta_3=A_3x-(P(t)A_3+CA_3^3)t+\eta_3^0,$$
$$c_{11}=\frac{1}{4A_1^2},\ c_{33}=\frac{1}{4A_3^2},\ c_{13}=\frac{(A_1-A_3)^2}{(A_1+A_3)^2},\ A_3^*=A_1,\ \eta_1^0=(\eta_3^0)^*.$$

The terms A_i and η_i^0 and the complex parameters η_i^0 and c_{ij} are already given in Proposition 1. It is pointed out that the proof for Proposition 4 is almost identical to that of Proposition 1, and is omitted here.

6. Results and discussions

We will now talk about a few intriguing characteristics of the presented 1-order breather, soliton-breather, and 2-order breather and positons of the non-autonomous Gardner equation, by employing Hirota's bilinear approach. The properties that are discovered above can be expressed physically in the following ways.

In Fig. 2(a), we see that χ_{br} evolves across a line which makes a constant angle with the x and t axes in the absence of damping and forcing ($L=0,\Lambda(t)=0$). Fig. 2(b) and 2(c) clearly demonstrate the impact of damping and forcing terms (periodic forcing component ($\Lambda(t)=g_0\cos(\Omega t)$)). From the visual presentation in Fig. 2(b), it is clear that the breather has lost its structure for damping effect as time passes. On the other hand, a periodic nature is found on the background of the wave structure in Fig. 2(c) because of acting of external forces, which also leads to destruct the smooth form of the breather. Fig. 3(a)–3(c) exhibit the propagation properties of a breather under an action of hyperbolic forcing and constant damping. Externally applied hyperbolic forcing leads to formation of a kinky-breather type wave. In Fig. 3(b), we see the significant impact of damping in the formation of a breather on the hyperbolic background. To exhibit a clear visual presentation of acting of external excitation, contour plots are drawn in Fig. 3(c).

We will further separate our discussion into two examples, solitons of being depressed along with those of elevation, in order to provide an even greater overview of the phases of interactions that occur between a soliton and a breather. We select the first scenario (Fig. 4) as a representative illustration of a breather heading to the left slamming into a depression soliton moving to the right. The breather can be visualized immediately as two small hills on either side of a large valley, with polarity switching after half a cycle. The soliton will initially crash with the 'small hill' on the left, leaving the central valley nearly frozen. Despite each of the valleys properly merging, the soliton interactions in the middle valley occur via transaction identity after traversing the minor hill on the left. When the centre valley ultimately separates, it moves slowly to the right as a depression soliton, while the entire remaining structure exhibits clearly defined oscillating aspects and moves to the left for a little respite. With the exception of a few specific phase alterations, the breather and the soliton maintain their original identities through all these intermediary phases. In the absence of damping and any kind of external excitation, the interaction of a breather and a rarefactive soliton is presented in Fig. 4(a). However, the structure is significantly modified in Fig. 4(b), due to an act of damping. On the other hand, combined effects of damping forcing terms are found in Fig. 4(b). A kinky-breather-soliton type wave is formed in Fig. 4(c) when a hyperbolic forcing term is acted. Additionally, we see a

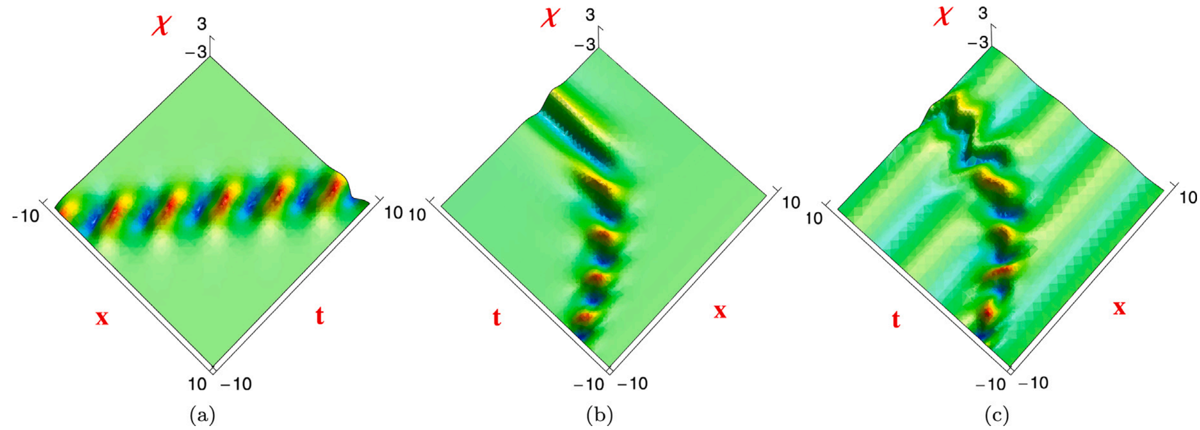

Fig. 2. The three dimensional graphs of 1-breather via solution (52), by considering $\Lambda(t) = g_0\cos(\Omega\tau)$, $A_1 = 1+i$, $A_2 = 1-i$, $B = 2.5$, $\xi_{11}^0 = 0, \xi_{12}^0 = 0$, $\chi_0 = 0.1$, and (a) when $g_0 = 0.0$, $L = 0$, (b) when $g_0 = 0$, $L = 0.05$, $\Omega = 1$, (c) when $g_0 = 0.3$, $L = 0.05$, $\Omega = 1$.

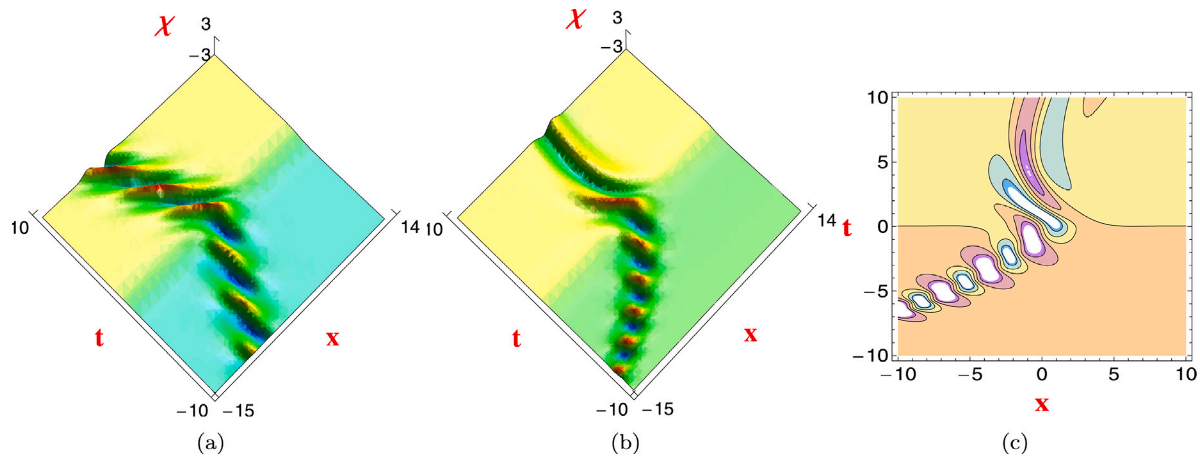

Fig. 3. The three dimensional graphs of 1-breather via solution (52), by considering $\Lambda(t) = g_0\mathrm{sech}^2(\Omega t)$, $A_1 = 1+i$, $A_2 = 1-i$, $B = 2.5$, $\xi_{11}^0 = 0, \xi_{12}^0 = 0$, $\chi_0 = 0.1$, and (a) when $g_0 = 0.3$, $L = 0$, (b) when $g_0 = 0.3$, $L = 0.05$, $\Omega = 1$, (c) Contour plots of the corresponding Fig. 3(a).

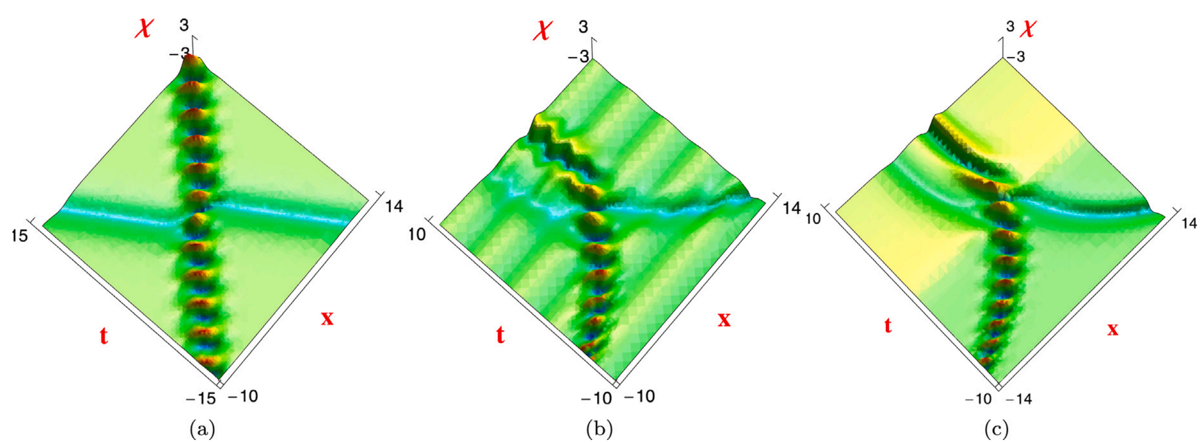

Fig. 4. The three dimensional graphs of 1-breather-1-soliton solution, by considering : $A_1 = 1.2+i$, $A_2 = 1.2-i$, $A_3 = 1$, $B = 2.5$, $\xi_{11}^0 = 0, \xi_{12}^0 = 0$, $\xi_3^0 = 0$, $\chi_0 = 0.1$, and (a) when $g_0 = 0.0$, $L = 0$, (b) when $\Lambda(t) = g_0\cos(\Omega\tau)$, and $g_0 = 0.3$, $L = 0.05$, $\Omega = 1.5$, (c) when $\Lambda(t) = g_0\mathrm{sech}^2(\Omega\tau)$, and $g_0 = 0.3$, $L = 0.05$, $\Omega = 1.5$.

diminishing tendency in the evolution of the breather structure because of damping.

In Fig. 5(a)–5(c), a special type of periodic breather is studied. Purely, an X-type breather is formed when a smooth space (in the absence of damping and forcing terms) is considered (Fig. 5(a)). As before, the breather loses its periodic structure in Fig. 5(b) and 5(c) when an external excitation is performed along with a damping.

In Fig. 6(a)–6(c), we see the interaction of a breather and a rarefactive X-shaped soliton in the smooth as well as non smooth spaces. Fig. 6(a)–6(c) show how the propagating properties vary when external excitations in the form of trigonometric and hyperbolic forces act. Under the action of trigonometric forces, the constant background changes to be a periodic base whereas a kink type background appears when hyperbolic forcing is proceed. Additionally, there is a significant bend in the characteristic line of the soliton as well as the breather by reason of the external excitations and dampings (Fig. 6(b) and 6(c)).

Fig. 7(a)–7(c) exhibit the 3D Profiles of 2nd-order smooth positon solution of Eq. (71), under consideration of an external periodic force

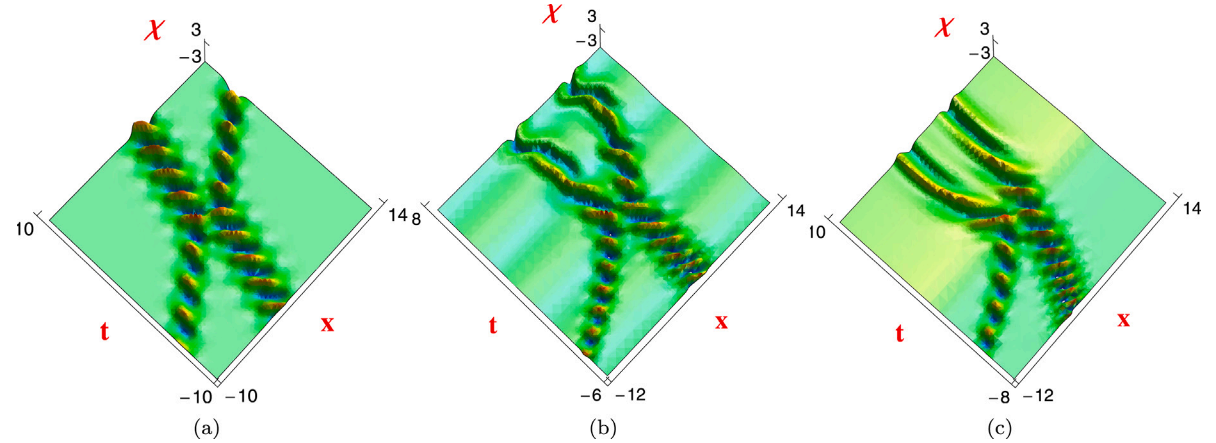

Fig. 5. The three dimensional graphs of 2-breather solution, by considering : $A_1 = 1.6 + 1.2i$, $A_2 = 1.6 - 1.2i$, $A_3 = 1.3 + 1.5i$, $A_4 = 1.3 - 1.5i$, $B = 2.5$, $\xi_{11}^0 = 0, \xi_{12}^0 = 0, \xi_{31}^0 = 0, \xi_{32}^0 = 0$, $\chi_0 = 0.1$, and (a) when $g_0 = 0.0$, $L = 0$, (b) when $\Lambda(t) = g_0\cos(\Omega\tau)$ and $g_0 = 0.15$, $L = 0.05$, $\Omega = 1.5$, (c) when $\Lambda(t) = g_0\text{sech}^2(\Omega\tau)$ and $g_0 = 0.3$, $L = 0.05$, $\Omega = 1.5$.

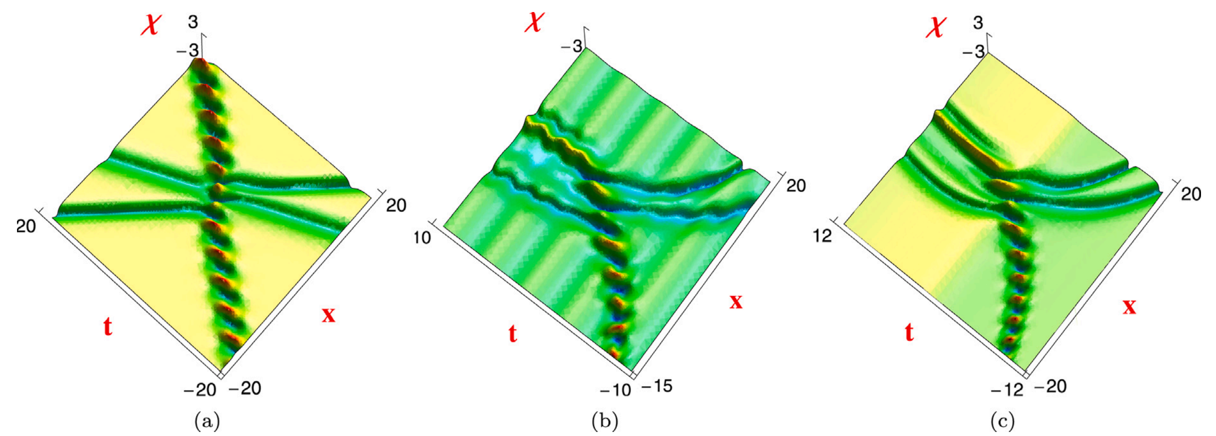

Fig. 6. The three dimensional graphs of 1-breather-2-soliton solution, by considering : $A_1 = 1 + i$, $A_2 = 1 - i$, $A_3 = 1$, $A_4 = 1.5$, $B = 2.5$, $\xi_{11}^0 = 0, \xi_{12}^0 = 0, \xi_3^0 = 0$, $\xi_4^0 = 0$, $\chi_0 = 0.1$, and (a) when $g_0 = 0.0$, $L = 0$, (b) when $\Lambda(t) = g_0\cos(\Omega\tau)$ and $g_0 = 0.3$, $L = 0.05$, $\Omega = 1.5$, (c) when $\Lambda(t) = g_0\text{sech}^2(\Omega\tau)$ and $g_0 = 0.3$, $L = 0.05$, $\Omega = 1.5$.

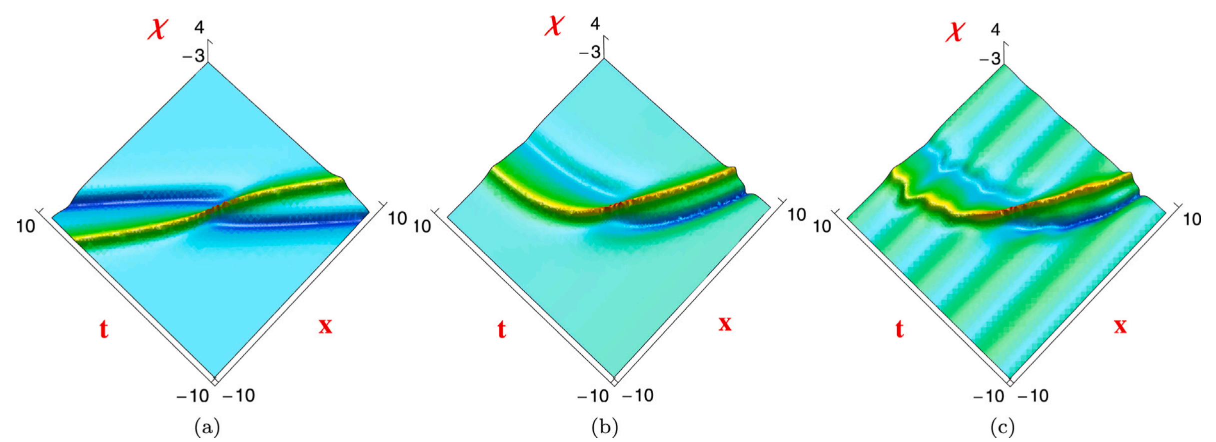

Fig. 7. 3D Profiles of two order smooth positon solution given by (71), by considering : $\Lambda(t) = g_0\cos(\Omega t)$, $A_1 = 1.5$, $B = 2.5$, $\chi_0 = 0.1$, $\alpha = 1$, $\eta_1^0 = 0.5$, and (a) when $g_0 = 0.0$, $L = 0$, (b) when $g_0 = 0$, $L = 0.05$, $\Omega = 1$, (c) when $g_0 = 0.3$, $L = 0.05$, $\Omega = 1$.

($\Lambda(t) = g_0\cos(\Omega t)$) with the adjacency of a damping. Fig. 7(a) presents a smooth positon in a smooth background when damping and forcing are all absent. In Fig. 7(b), the smoothness of the positon is affected due to an act of damping and asymptotically the positon dies out. Additionally, the direction of the positon remarkably changed because of a damping. On the other hand, under the influence of periodic forces, a periodic type of wave backdrop appears in Fig. 7(c). In an approximate 'bound state', a set of upward and downward waves move to the right (Fig. 8). In the left far field, the upward wave follows the depressed wave, but in the opposite right afar field, the order is inverted. Physically, it is comparable to a breather with almost zero frequency. A dual pole/two-positon solution and a 2-soliton are fundamentally different from one another since the former's peaks are spaced apart like the logarithm of time t, making the separation distance almost constant.

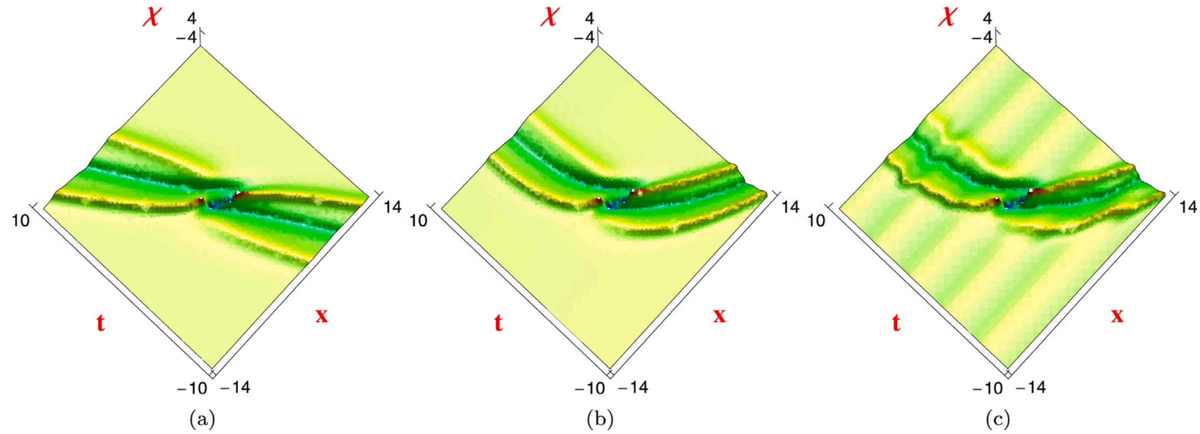

Fig. 8. 3D Profiles of third-order smooth positon solution given by (75), by considering : $A(t) = g_0\cos(\Omega t)$, $A_1 = 1.5$, $\alpha = 1$, $\eta_1^0 = 0.5$, $B = 2.5$, $\omega = 1.5$, $\chi_0 = 0.1$, and (a) when $g_0 = 0.0$, $L = 0$, (b) when $g_0 = 0$, $L = 0.05$, (c) when $g_0 = 0.3$, $L = 0.05$.

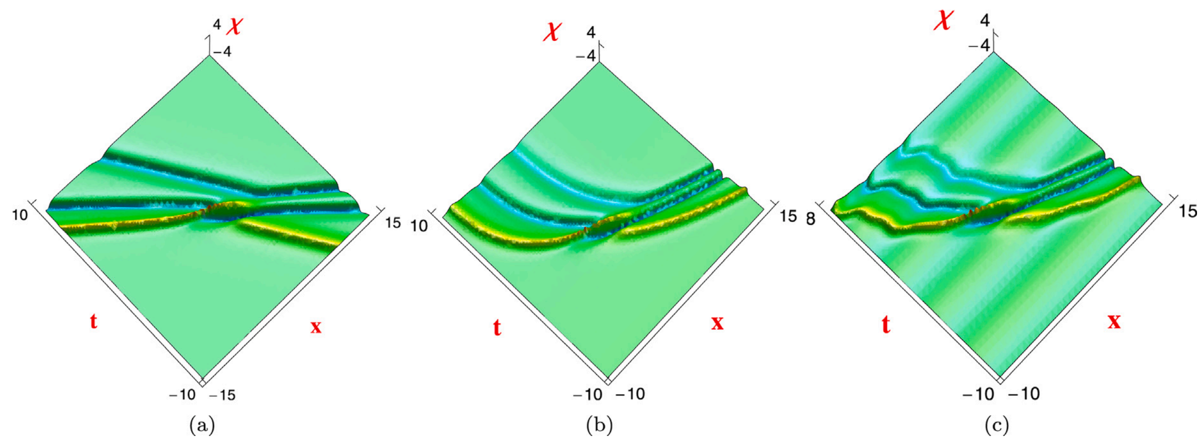

Fig. 9. Interaction profiles of two smooth positon and one soliton solution, by considering : $A(t) = g_0\cos(\Omega t)$, $A_1 = 2$, $A_3 = 1.5$, $\alpha = 1$, $\eta_1^0 = 2$, $B = 2.5$, $\omega = 1.5$, $\chi_0 = 0.1$, and (a) when $g_0 = 0.0$, $L = 0$, (b) when $g_0 = 0$, $L = 0.05$, (c) when $g_0 = 0.3$, $L = 0.05$.

The peaks of a 2-soliton arrangement will differ as the product of the velocity difference and time. A similar pattern of obsession also appears for the 3rd-order smooth position solution in Fig. 8(a)–8(c). It is significant to note that a damping plays a significant part in both of the aforementioned examples in bending the direction of positon. The interactive wave profiles of a smooth 2-positons and a 1-soliton solution are shown in Fig. 9(a)–9(c). The dampening effect in Fig. 9(b) dulled the positon's structural details. It is noteworthy to observe that a dampening also aids in positioning the positon's orientation. Fig. 9(c) depicts the propagation of the positon on a periodic backdrop due to the application of periodic forcing.

7. Conclusion

This article uses the traditional Hirota's bilinear approach to present a number of new analytic multi-solitonic, breather, and positon solutions for the non-autonomous Eq. (2). Moreover, the integrability of the non-autonomous Gardner equation is judged by means of its Bäcklund transformation and Lax pair. The final outcomes can be briefly summarized below,

- The integrability of the non-autonomous Gardner equation is checked via the existence of Bäcklund transformation and Lax pair under some constraint; and new integrability conditions are derived.
- The K-soliton conditions are examined and a set of new analytic K-soliton solution for the presented bilinear Gardner system is derived.
- The corresponding breathers are straightforwardly constructed. For a visual outlook of breathers and positons in the presented bilinear system, some 3D graphs are presented.
- The numerical graphs demonstrate that a damping in the non-autonomous Gardner framework leads to diminish the amplitude as well as speed of the soliton, and in the asymptotic states the soliton may dies out finally.
- Various external forces affect the foundation of a wave background; for instance, a trigonometric force can generate periodic background, and the periodicity increases as the external force strength increases. However, applying the hyperbolic forcing term results in kink-type grounding, and as the magnitude of forcing continues to increase, the height of the compact kink increases. In some situations, damping and forcing involve bending the direction of the soliton also.
- Several parameter constraints are applied to exploration of breather waves originating from two-soliton solutions; further, two-breathers emanated from four-soliton solutions. These waves demonstrate some of their fascinating properties, such as soliton interaction and soliton overlapping, through simulations of solution wave profiles.
- The positon solution for the presented bilinear system is also derived, and additionally, damping and forcing terms have a considerable impact. A strong orientation in the direction of propagation of the positon arises for an act of damping, whereas the background of the positon is improved in a periodical structure due to the excitation of the system.

- Here, the crashes of a breather with an elevated soliton or a depressed soliton are investigated. The "central valley" is seen to either sustain resonances during the collision phase or become "frozen" based on the direction of the colliding soliton and the physical features of the breather, such as its frequency. Such movements are incredibly time-dependent and will certainly have a significant impact on the physical procedures that this non-autonomous Gardner equation may simulate. Solitons of the plateau type and formations resembling bores appear in this domain. A few of the consequences of the present experiment is the following claim, which may be shown by the dynamics of the flow and concentration perturbations in a growing interior marine tidal. The temporal and spatial variation that has been observed in coastal internal soliton patterns is most likely caused by breathers and their contact with solitons.

This research has shown that the bilinear Bäcklund transformation in combination with Hirota's bilinear approach is an effective analytical tool for solving a more general class of nonlinear evolution equations in the domains of engineering and various scientific fields. Our findings might help to clarify the dynamic behaviour of positons and many other complicated non-autonomous solutions. The non-autonomous Gardner equation is discussed and the findings are presented for the first time in this paper. Its physical underpinning can be used to describe a nonlinear phenomenon that depends on it.

CRediT authorship contribution statement

Santanu Raut: Writing – original draft, Visualization, Writing – review & editing. **Wen-Xiu Ma:** Conceptualization, Supervision, Writing – review & editing. **Ranjan Barman:** Software, Visualization. **Subrata Roy:** Investigation, Methodology, Software.

Declaration of competing interest

The authors declare that there is no conflict of interest between the authors.

Data availability

No data was used for the research described in the article.

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Declarations
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