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Further advanced investigation of the complex Hirota-dynamical model to extract soliton solutions

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The complex Hirota-dynamical model (CHDM) has applications in the study of plasma physics, the investigation of fusion energy, astrophysical research, and space studies. The CHDM may also be used to investigate turbulent flows to study shocks and other nonlinear phenomena, and light waves venturing through the fibers. Nowadays, plasma physics, fusion energy, astrophysical research, and space studies are very interesting topics in the modern research. So, we need to shed light on this model as a good application in these fields. For deep investigation of these physical problems, we need to find their analytical solutions. In this study, we explore a variety of soliton solutions with different geometrical structures for the CHDM via the double variable expansion method. By means of this method, we have obtained three types of soliton solutions, namely, hyperbolic, trigonometric, and rational function solutions. The graphical interpretation of these solutions gives us some popular shapes such as singular-periodic, kink,

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bell, and singular shapes. The performed method is an efficient technique to execute and provides reliable analytical soliton solutions which are very important to further advanced investigation of the mention equation.

Keywords: The double variable expansion method; the complex Hirota-dynamical model; traveling wave solutions; soliton solutions.

1. Introduction

The majority of incidences of our natural world are expressed by nonlinear partial differential equations (NLPDEs) or nonlinear ordinary differential equations (NLODEs).[1–15] For instance, the CHDM can be utilized to investigate the plasma physics, the fusion energy, astrophysical research, space studies, turbulent flows to study shocks, the behavior of wavelengths of light advancing through the optical fiber and other nonlinear phenomena.[16–30] Furthermore, the CHDM can be used to study gaseous ionization in plasma physics. Also singular solutions of the CHDM have a plethora of physical repercussions. Indeed, they can be adopted to explore the formation of solitons, which are waves that preserve their form and momentum over longer distances due to their capacity to transmit data over long distances with no considerable deformation. In turbulent flows, singular solutions of the CHDM can be used for studying the process of generating shocks and other nonlinear characteristics. Since these mathematical equations with exact solutions (ESs) of the system have improved our understanding of their functioning, application, and development. Consequently, numerous researchers[31–54] have utilized a variety of numerical techniques to obtain precise solutions for NLPDEs and NLODEs over the course of many years. Beside these references, the Hirota bilinear method[45,48] is used to examine such kinds of NLPDEs.

Researchers use a wide variety of strategies in their pursuit of a solution to the CHDM because of the problem's significance. For instance, Ali *et al.*[7] used the unified auxiliary equation method, Bekir and Zahran[9] used the solitary wave ansatz and extended simple equation methods, Seadawy and Abdullah[42] implemented the extended mapping technique. Also, Sugati *et al.*[46] used the variational principle and computational techniques to acquire new solutions, such as chirp optical and numerical wave solutions, and to study the existence, uniqueness, and stability of these solutions. Another study[8] examined rogue waves and rational solutions for this model.

Li *et al.*,[28] first proposed the double variable expansion algorithms and applied to find traveling wave solutions of Zakharov equation. After that, many researchers also implemented this method to find the exact traveling wave solutions of some well known NLPDEs and NLEEs. Miah *et al.* implemented the mention method successfully in Refs. 32–34 to find the exact traveling wave solutions and the abundant exact traveling wave solutions for some well known NLEEs. Also, Ali *et al.* applied the method to find the abundant wave solutions of Burgers equation, Bogoyavlenskii equation and negative Gardner-KP equation in Ref. 6, to find the solitary wave solutions of nonlinear fractional evolution equations in Ref. 4.

Recently, Iqbal *et al.*[19] explored the soliton solutions of the mZK equation and the Gerdjikov–Ivanov equation, also explored the exact dynamic wave solutions of Date–Jimbo–Kashiwara–Miwa equation with conformable derivative dependent on time parameter in Ref. 20 via aforementioned method. After all, this method is being used by several researchers[19,28,32–35,52] to identify the precise travelling wave and solition solutions of NLEEs and some other higher order NLPDEs.

Although the double variable expansion algorithms is reliable, easy to implement and mathematically well established, as far as we know, no one considered the aforementioned method to find the traveling wave solutions for the CHDM. In this paper, analytical traveling wave solutions to the CHDM have been obtained using double variable expansion algorithms.

This paper is organized as follows. The background for this study is given in Sec. 1. In Sec. 2, we have given the short introduction of the double variable expansion method. In Sec. 3, we provide the formulation of the soliton solutions to the mention problem. In Sec. 4, we have shown the graphical interpretation of some particular solutions. Finally, results and discussion are given in Sec. 5.

2. Short Review of the Double Variable Expansion Method

In this part, we give a short review for the double variable expansion method to find the ESs of NLPDEs. First of all, let us consider the following second order linear ordinary differential equation:

$$G''(\zeta) + \eta G(\zeta) = \beta \tag{2.1}$$

and set,

$$\gamma(\zeta) = \frac{G'}{G}, \quad \theta(\zeta) = \frac{1}{G}. \tag{2.2}$$

By differentiating γ and θ with respect to ζ, we get

$$\gamma' = \frac{G''}{G} - \frac{(G')^2}{G^2}, \quad \theta' = -\frac{G'}{G^2}. \tag{2.3}$$

By a direct substitution from (2.1) and (2.2) in Eq. (2.3), we reach

$$\gamma' = -\gamma^2 + \beta\theta - \eta \quad \text{and} \quad \theta' = -\gamma\theta. \tag{2.4}$$

Let us solve Eq. (2.1), then we will have the three cases for different values of η.

Case 1: When $\eta > 0$, the general solutions (GSs) of Eq. (2.1)

$$G(\zeta) = \mathcal{C}_1 \sin(\zeta\sqrt{\eta}) + \mathcal{C}_2 \cos(\zeta\sqrt{\eta}) + \frac{\beta}{\eta}, \tag{2.5}$$

with

$$\theta^2 = \frac{\eta(\gamma^2 - 2\beta\theta + \eta)}{\eta^2 c_1 - \beta^2} \tag{2.6}$$

and $c_1 = {\mathcal{C}_1}^2 + {\mathcal{C}_2}^2$.

Case 2: For $\eta < 0$, the GSs of Eq. (2.1) are

$$G(\zeta) = \mathcal{C}_1 \sinh(\zeta\sqrt{-\eta}) + \mathcal{C}_2 \cosh(\zeta\sqrt{-\eta}) + \frac{\beta}{\eta}, \tag{2.7}$$

where

$$\theta^2 = \frac{-\eta(\gamma^2 - 2\beta\theta + \eta)}{\eta^2 c_2 + \beta^2} \tag{2.8}$$

and $c_2 = \mathcal{C}_1{}^2 - \mathcal{C}_2{}^2$.

Case 3: If $\eta = 0$, the GSs of Eq. (2.1) are

$$G(\zeta) = \frac{\beta}{2}\zeta^2 + \mathcal{C}_1\zeta + \mathcal{C}_2, \tag{2.9}$$

with,

$$\theta^2 = \frac{(\gamma^2 - 2\beta\theta)}{\mathcal{C}_1{}^2 - 2\beta\mathcal{C}_2}. \tag{2.10}$$

Let us consider the NLPDE as

$$\mathcal{N}(\mathcal{V}, \mathcal{V}_x, \mathcal{V}_t, \mathcal{V}_{xx}, \mathcal{V}_{tt}, \mathcal{V}_{tx}, \ldots) = 0, \tag{2.11}$$

where $\mathcal{N}$ is a polynomial in $\mathcal{V}$ and its partial derivatives. Let us consider the following wave transformation:

$$\mathcal{V}(x,t) = \mathcal{V}(\zeta), \quad \zeta = x - \rho t, \tag{2.12}$$

where ρ is the wave velocity. Now, using Eq. (2.12) in Eq. (2.11), we have the following ordinary differential equation:

$$\mathcal{M}(\mathcal{V}, \mathcal{V}', -\rho\mathcal{V}', \mathcal{V}'', \rho^2\mathcal{V}'', -\rho\mathcal{V}'', \ldots) = 0. \tag{2.13}$$

Finally, let us consider that solution of Eq. (2.13) by $\left(\frac{G'}{G}, \frac{1}{G}\right)$-expansion method that is given by

$$\mathcal{V}(\zeta) = a_0 + \sum_{i=1}^{K} a_i\gamma^i(\zeta) + \sum_{i=1}^{K} b_i\gamma^{i-1}(\zeta)\theta^i(\zeta), \tag{2.14}$$

where γ and θ are given in Eq. (2.2) and $a_0,\ a_i,\ b_i\ (i = 1, 2, 3, \ldots, K)$ are arbitrary constants. After computing the value of balance number K by using the homogeneous balance property, we have the exact form of the solution given in Eq. (2.14). Then, substituting reformed equation (2.14) into Eq. (2.13), for each of the three cases discussed earlier, the left-hand side of Eq. (2.13) transforms into a polynomial in γ and θ with arbitrary constants, where the power of γ is always less or equal unity. Then we obtain a set of algebraic equations in arbitrary constants a_0, a_1, b_1, ρ, and β if we set the coefficient of the variables equal to zero. After that, by using a computer-aided package software like Mathematica to solve the resulting system of equations, we receive the values for arbitrary constants. Finally using these values of the arbitrary constants together Eqs. (2.2), (2.12) and (2.14), we obtain the respective solutions of our mentioned equation for three cases, respectively.

3. The Formulation of the Solutions to the CHDM

The CHDM is given by the following equation:

$$i\mathcal{V}_t + \mathcal{V}_{xx} + 2|\mathcal{V}|^2\mathcal{V} + i\alpha\mathcal{V}_{xxx} + 6i\alpha|\mathcal{V}|^2\mathcal{V}_x = 0, \tag{3.1}$$

where $\alpha \in \mathbb{R}$. We consider the a complex valued wave solution followed by Ref. 7

$$\mathcal{V}(x,t) = \Psi(\zeta)e^{i\xi(x,t)}, \quad \xi(x,t) = -kx + wt + \Theta, \quad \zeta = x - \rho t, \tag{3.2}$$

which specify the movement of a wave by means of both space and time. While the complex phase function $\xi(x,t)$ describes how the wave's phase changes as it travels and ρ is the wave velocity. The complex amplitude function $\Psi(\zeta)$ only depends on the wavefront's separation from the observer. The wave number, angular velocity, and beginning phase of a propagating wave are represented by the parameters k, w, and Θ, respectively. Now, applying the transformation from Eq. (3.2) in Eq. (3.1), we get the following ordinary differential equation to determine the traveling wave solution of the CHDM equation:

$$\begin{aligned}&(-k^2 - w - k^3\alpha)\Psi + (2 + 6k\alpha)\Psi^3 - i(k(2 + 3k\alpha) + \rho)\Psi' \\ &\quad + 6i\alpha\Psi^2\Psi' + (1 + 3k\alpha)\Psi'' + i\alpha\Psi^{(3)} = 0.\end{aligned} \tag{3.3}$$

The following equations are obtained by equating both the imaginary and the real components of Eq. (3.3):

$$(-k^2 - w - k^3\alpha)\Psi + (2 + 6k\alpha)\Psi^3 + (1 + 3k\alpha)\Psi'' = 0 \tag{3.4}$$

and

$$-(k(2 + 3k\alpha) + \rho)\Psi' + 6\alpha\Psi^2\Psi' + \alpha\Psi^{(3)} = 0. \tag{3.5}$$

We get $k = -\frac{1}{3\alpha}$, and $w = -\frac{2}{27\alpha^2}$ if we set the coefficient in Eq. (3.4) to zero. Now, we have these values together with the following equation:

$$-(k(2 + 3k\alpha) + \rho)\Psi' + 6\alpha\Psi^2\Psi' + \alpha\Psi^{(3)} = 0. \tag{3.6}$$

By using homogeneous balance property in Eq. (3.6), we have the balance number $K = 1$ and for this balance number the solution of (3.6) will be given in the following form:

$$\Psi(\zeta) = a_0 + a_1\gamma(\zeta) + b_1\theta(\zeta), \tag{3.7}$$

where a_0, a_1, b_1 are arbitrary constants. Now, implementing the double variable expansion method discussed in Sec. 2, the unknown values of the arbitrary constant can be found easily for each of the three cases. The detail of the mentioned method is given in Refs. 19, 28, 32–35 and 52.

Now, we employ the values of the arbitrary constants to find the solutions to the CHDM by the implementation of the double variable expansion method for the three cases as follows:

Case 1: For $\eta > 0$, substituting Eq. (3.7) into Eq. (3.6) and using Eqs. (2.4) and (2.6), we will have the following three sets of solutions corresponding to three sets of values of arbitrary constants.

Set 1: $a_0 = 0,\ a_1 = 0,\ b_1 = \pm\sqrt{-\alpha}\sqrt{\eta}\sqrt{c_2},\ \beta = 0,$ and $\rho = \frac{1-3\alpha^2\eta}{3\alpha}$. Now using the values of $a_0, a_1, b_1, \rho, \beta$ in Eq. (3.7), we have the following:

$$\Psi(\zeta) = \pm\sqrt{-\alpha}\sqrt{\eta}\sqrt{c_2}\theta(\zeta). \tag{3.8}$$

Then, from Eqs. (2.2), (2.5) and (3.8), we have

$$\Psi(\zeta) = \frac{\pm\sqrt{-\alpha}\sqrt{\eta}\sqrt{c_2}}{\mathcal{C}_1\sin(\zeta\sqrt{\eta}) + \mathcal{C}_2\cos(\zeta\sqrt{\eta})}. \tag{3.9}$$

Now, from Eq. (2.12), the solution of the CHDM is obtained as follows:

$$\mathcal{V}(x,t) = \frac{\pm\ e^{i(-kx+wt+\Theta)}\sqrt{-\alpha}\sqrt{\eta}\sqrt{c_2}}{\mathcal{C}_1\sin(\zeta\sqrt{\eta}) + \mathcal{C}_2\cos(\zeta\sqrt{\eta})}. \tag{3.10}$$

Inserting the value of ζ in Eq. (3.10), we get

$$\mathcal{V}(x,t) = \frac{\pm\sqrt{-\alpha}\sqrt{\eta}\sqrt{c_2}\ e^{i(-kx+wt+\Theta)}}{\mathcal{C}_1\sin\left(\left(x - \left(\frac{1-3\alpha^2\eta}{3\alpha}\right)t\right)\sqrt{\eta}\right) + \mathcal{C}_2\cos\left(\left(x - \left(\frac{1-3\alpha^2\eta}{3\alpha}\right)t\right)\sqrt{\eta}\right)} \tag{3.11}$$

for $\alpha < 0, k = -\frac{1}{3\alpha}, w = -\frac{2}{27\alpha^2}$ and the value of Θ is known from initial phase of wave.

Set 2: $a_0 = 0,\ a_1 = \pm\frac{\sqrt{-\alpha}}{2},\ b_1 = \pm\frac{1}{2}\sqrt{-\alpha}\sqrt{\eta}\sqrt{c_2},\ \beta = 0$ and $\rho = \frac{2+3\alpha^2\eta}{6\alpha}$. Now using the values of $a_0, a_1, b_1, \rho, \beta$ in Eq. (3.7), we have the following:

$$\Psi(\zeta) = \pm\frac{\sqrt{-\alpha}}{2}\gamma(\zeta) \pm \frac{1}{2}\sqrt{-\alpha}\sqrt{\eta}\sqrt{c_1}\theta(\zeta). \tag{3.12}$$

Again, from Eqs. (2.2), (2.7) and (3.12), we have

$$\begin{aligned}\Psi(\zeta) = &\pm\frac{\sqrt{-\alpha}\sqrt{\eta}\{\mathcal{C}_1\cos(\zeta\sqrt{\eta}) - \mathcal{C}_2\sin(\zeta\sqrt{\eta})\}}{2(\mathcal{C}_1\sin(\zeta\sqrt{\eta}) + \mathcal{C}_2\cos(\zeta\sqrt{\eta}))}\\ &\pm\frac{\sqrt{-\alpha}\sqrt{\eta}\sqrt{c_2}}{2(\mathcal{C}_1\sin(\zeta\sqrt{\eta}) + \mathcal{C}_2\cos(\zeta\sqrt{\eta}))}.\end{aligned} \tag{3.13}$$

Thus, from Eq. (2.12), the solution of the CHDM is

$$\begin{aligned}\mathcal{V}(x,t) = &\left[\pm\frac{\sqrt{-\alpha}\sqrt{\eta}\{\mathcal{C}_1\cos(\zeta\sqrt{\eta}) - \mathcal{C}_2\sin(\zeta\sqrt{\eta})\}}{2(\mathcal{C}_1\sin(\zeta\sqrt{\eta}) + \mathcal{C}_2\cos(\zeta\sqrt{\eta}))}\right.\\ &\left.\pm\frac{\sqrt{-\alpha}\sqrt{\eta}\sqrt{c_2}}{2(\mathcal{C}_1\sin(\zeta\sqrt{\eta}) + \mathcal{C}_2\cos(\zeta\sqrt{\eta}))}\right]e^{i(-kx+wt+\Theta)}.\end{aligned} \tag{3.14}$$

By putting the value of ζ in Eq. (3.14), we have

$$\therefore \mathcal{V}(x,t) = \left[\pm \frac{\sqrt{-\alpha}\sqrt{\eta}\left\{\mathcal{C}_1 \cos\left(\left(x-\left(\frac{2+3\alpha^2\eta}{6\alpha}\right)t\right)\sqrt{\eta}\right) - \mathcal{C}_2 \sin\left(\left(x-\left(\frac{2+3\alpha^2\eta}{6\alpha}\right)t\right)\sqrt{\eta}\right)\right\}}{2\left(\mathcal{C}_1 \sin\left(\left(x-\left(\frac{2+3\alpha^2\eta}{6\alpha}\right)t\right)\sqrt{\eta}\right) + \mathcal{C}_2 \cos\left(\left(x-\left(\frac{2+3\alpha^2\eta}{6\alpha}\right)t\right)\sqrt{\eta}\right)\right)} \pm \frac{\sqrt{-\alpha}\sqrt{\eta}\sqrt{c_2}}{2\left(\mathcal{C}_1 \sin\left(\left(x-\left(\frac{2+3\alpha^2\eta}{6\alpha}\right)t\right)\sqrt{\eta}\right) + \mathcal{C}_2 \cos\left(\left(x-\left(\frac{2+3\alpha^2\eta}{6\alpha}\right)t\right)\sqrt{\eta}\right)\right)}\right] e^{i(-kx+wt+\Theta)}, \tag{3.15}$$

where $\alpha < 0$, $k = -\frac{1}{3\alpha}$, $w = -\frac{2}{27\alpha^2}$, and the value of Θ is known from initial phase of wave.

Set 3: $a_0 = 0$, $a_1 = \pm\sqrt{-\alpha}$, $b_1 = 0$, $\beta = 0$, and $\rho = \frac{1+6\alpha^2\eta}{3\alpha}$. Now using the values of $a_0, a_1, b_1, \rho, \beta$ in Eq. (3.7), we have the following:

$$\Psi(\zeta) = \pm\sqrt{-\alpha}\gamma(\zeta). \tag{3.16}$$

So, from Eqs. (2.2), (2.7) and (3.16), we have

$$\Psi(\zeta) = \pm\frac{\sqrt{-\alpha}\{\mathcal{C}_1 \cos(\zeta\sqrt{\eta}) - \mathcal{C}_2 \sin(\zeta\sqrt{\eta})\}}{(\mathcal{C}_1 \sin(\zeta\sqrt{\eta}) + \mathcal{C}_2 \cos(\zeta\sqrt{\eta}))}. \tag{3.17}$$

As a consequence, from Eq. (2.12), the solution of the CHDM is

$$\mathcal{V}(x,t) = \pm\frac{\sqrt{-\alpha}\{\mathcal{C}_1 \cos(\zeta\sqrt{\eta}) - \mathcal{C}_2 \sin(\zeta\sqrt{\eta})\}}{(\mathcal{C}_1 \sin(\zeta\sqrt{\eta}) + \mathcal{C}_2 \cos(\zeta\sqrt{\eta}))} e^{i(-kx+wt+\Theta)}. \tag{3.18}$$

If we replace the given value of ζ, we have the following:

$$\therefore \mathcal{V}(x,t) = \pm\frac{\sqrt{-\alpha}\left\{\mathcal{C}_1 \cos\left(\left(x-\left(\frac{1+6\alpha^2\eta}{3\alpha}\right)t\right)\sqrt{\eta}\right) - \mathcal{C}_2 \sin\left(\left(x-\left(\frac{1+6\alpha^2\eta}{3\alpha}\right)t\right)\sqrt{\eta}\right)\right\} e^{i(-kx+wt+\Theta)}}{\left(\mathcal{C}_1 \sin\left(\left(x-\left(\frac{1+6\alpha^2\eta}{3\alpha}\right)t\right)\sqrt{\eta}\right) + \mathcal{C}_2 \cos\left(\left(x-\left(\frac{1+6\alpha^2\eta}{3\alpha}\right)t\right)\sqrt{\eta}\right)\right)}, \tag{3.19}$$

for $\alpha < 0$, $k = -\frac{1}{3\alpha}$, $w = -\frac{2}{27\alpha^2}$, and the value of Θ is known from initial phase of wave.

Case 2: When we have $\eta < 0$, substituting (3.7) into (3.6) and using (2.4) and (2.8), we will have the following four sets of solutions:

Set 1: $a_0 = 0$, $a_1 = \pm\frac{\sqrt{-\alpha}}{2}$, $b_1 = \pm\frac{\sqrt{-\alpha}}{\sqrt{-\eta}}\sqrt{\left(\frac{\beta^2+\eta^2 c_1}{4}\right)}$, and $\rho = \frac{2+3\alpha^2\eta}{6\alpha}$. Now using the values of a_0, a_1, b_1, and ρ in Eq. (3.7), we have the following:

$$\Psi(\zeta) = \pm\frac{\sqrt{-\alpha}}{2}\gamma(\zeta) + \pm\frac{\sqrt{-\alpha}}{\sqrt{-\eta}}\sqrt{\left(\frac{\beta^2+\eta^2 c_1}{4}\right)}\theta(\zeta). \tag{3.20}$$

Now, Eqs. (2.2), (2.7) and (3.20) imply

$$\begin{aligned}\Psi(\zeta) = &\pm \frac{\sqrt{-\alpha}\sqrt{-\eta}\{\mathcal{C}_1 \cosh(\zeta\sqrt{-\eta}) + \mathcal{C}_2 \sinh(\zeta\sqrt{-\eta})\}}{2(\mathcal{C}_1 \sinh(\zeta\sqrt{-\eta}) + \mathcal{C}_2 \cosh(\zeta\sqrt{-\eta}))}\\ &\pm \frac{\sqrt{-\alpha}\sqrt{(\beta^2 + \eta^2 c_1)}}{2\sqrt{-\eta}(\mathcal{C}_1 \sinh(\zeta\sqrt{-\eta}) + \mathcal{C}_2 \cosh(\zeta\sqrt{-\eta}))}.\end{aligned} \tag{3.21}$$

So, the solution of the CHDM is explored from Eq. (2.12) as follows:

$$\begin{aligned}\mathcal{V}(x,t) = e^{i(-kx+wt+\Theta)}&\left[\pm \frac{\sqrt{-\alpha}\sqrt{-\eta}\{\mathcal{C}_1 \cosh(\zeta\sqrt{-\eta}) + \mathcal{C}_2 \sinh(\zeta\sqrt{-\eta})\}}{2(\mathcal{C}_1 \sinh(\zeta\sqrt{-\eta}) + \mathcal{C}_2 \cosh(\zeta\sqrt{-\eta}))}\right.\\ &\left.\pm \frac{\sqrt{-\alpha}\sqrt{(\beta^2 + \eta^2 c_1)}}{2\sqrt{-\eta}(\mathcal{C}_1 \sinh(\zeta\sqrt{-\eta}) + \mathcal{C}_2 \cosh(\zeta\sqrt{-\eta}))}\right].\end{aligned} \tag{3.22}$$

After inserting the value of ζ, we have

$$\begin{aligned}\therefore \mathcal{V}(x,t) = &\left[\pm \frac{\sqrt{-\alpha}\sqrt{-\eta}\Big\{\mathcal{C}_1 \cosh\left(\left(x - \left(\frac{2+3\alpha^2\eta}{6\alpha}\right)t\right)\sqrt{-\eta}\right) + \mathcal{C}_2 \sinh\left(\left(x - \left(\frac{2+3\alpha^2\eta}{6\alpha}\right)t\right)\sqrt{-\eta}\right)\Big\}}{2\Big(\mathcal{C}_1 \sinh\left(\left(x - \left(\frac{2+3\alpha^2\eta}{6\alpha}\right)t\right)\sqrt{-\eta}\right) + \mathcal{C}_2 \cosh\left(\left(x - \left(\frac{2+3\alpha^2\eta}{6\alpha}\right)t\right)\sqrt{-\eta}\right) + \frac{\beta}{\eta}\Big)}\right.\\ &\left.\pm \frac{\sqrt{-\alpha}\sqrt{(\beta^2 + \eta^2 c_1)}}{2\sqrt{-\eta}\Big(\mathcal{C}_1 \sinh\left(\left(x - \left(\frac{2+3\alpha^2\eta}{6\alpha}\right)t\right)\sqrt{-\eta}\right) + \mathcal{C}_2 \cosh\left(\left(x - \left(\frac{2+3\alpha^2\eta}{6\alpha}\right)t\right)\sqrt{-\eta}\right) + \frac{\beta}{\eta}\Big)}\right] e^{i(-kx+wt+\Theta)},\end{aligned} \tag{3.23}$$

for $\alpha < 0$, $k = -\frac{1}{3\alpha}$, $w = -\frac{2}{27\alpha^2}$, and the value of Θ is known from initial phase of wave.

Set 2: $a_0 = 0$, $a_1 = 0$, $b_1 = \pm\sqrt{-\alpha}\sqrt{-\eta}\sqrt{c_1}$, $\beta = 0$, and $\rho = \frac{1-3\alpha^2\eta}{3\alpha}$. Now using the values of $a_0, a_1, b_1, \rho, \beta$ in Eq. (3.7), we have the following:

$$\Psi(\zeta) = \pm\sqrt{-\alpha}\sqrt{-\eta}\sqrt{c_1}\theta(\zeta). \tag{3.24}$$

Thus, Eqs. (2.2), (2.7) and (3.24) imply

$$\Psi(\zeta) = \pm \frac{\sqrt{-\alpha}\sqrt{-\eta}\sqrt{c_1}}{(\mathcal{C}_1 \sinh(\zeta\sqrt{-\eta}) + \mathcal{C}_2 \cosh(\zeta\sqrt{-\eta}))}. \tag{3.25}$$

Now, from Eq. (2.12), the solution of the CHDM is

$$\mathcal{V}(x,t) = \pm \frac{\sqrt{-\alpha}\sqrt{-\eta}\sqrt{c_1}\ e^{i(-kx+wt+\Theta)}}{(\mathcal{C}_1 \sinh(\zeta\sqrt{-\eta}) + \mathcal{C}_2 \cosh(\zeta\sqrt{-\eta}))}. \tag{3.26}$$

Again, from Eq. (3.26), the given value of ζ transfers into the following form:

$$\therefore \mathcal{V}(x,t) = \pm \frac{\sqrt{-\alpha}\sqrt{-\eta}\sqrt{c_1}\ e^{i(-kx+wt+\Theta)}}{\left(\mathcal{C}_1 \sinh\left(\left(x - \left(\frac{1-3\alpha^2\eta}{3\alpha}\right)t\right)\sqrt{-\eta}\right) + \mathcal{C}_2 \cosh\left(\left(x - \left(\frac{1-3\alpha^2\eta}{3\alpha}\right)t\right)\sqrt{-\eta}\right)\right)}, \tag{3.27}$$

for $\alpha < 0$, $k = -\frac{1}{3\alpha}$, $w = -\frac{2}{27\alpha^2}$, and the value of Θ is known from initial phase of wave.

Set 3: $a_0 = 0$, $a_1 = \pm\frac{\sqrt{-\alpha}}{2}$, $b_1 = \pm\frac{1}{2}\sqrt{-\alpha}\sqrt{-\eta}\sqrt{c_1}$, $\beta = 0$, and $\rho = \frac{2+3\alpha^2\eta}{6\alpha}$. Now using the values of $a_0, a_1, b_1, \rho, \eta$ in Eq. (3.7), we have the following:

$$\Psi(\zeta) = \pm\frac{\sqrt{-\alpha}}{2}\gamma(\zeta) + \pm\frac{1}{2}\sqrt{-\alpha}\sqrt{-\eta}\sqrt{c_1}\theta(\zeta). \tag{3.28}$$

Now, from Eqs. (2.2), (2.7) and (3.28), we have

$$\begin{aligned}\Psi(\zeta) = {} & \pm\frac{\sqrt{-\alpha}\sqrt{-\eta}\{\mathcal{C}_1 \cosh(\zeta\sqrt{-\eta}) + \mathcal{C}_2 \sinh(\zeta\sqrt{-\eta})\}}{2(\mathcal{C}_1 \sinh(\zeta\sqrt{-\eta}) + \mathcal{C}_2 \cosh(\zeta\sqrt{-\eta}))} \\ & \pm \frac{\sqrt{-\alpha}\sqrt{-\eta}\sqrt{c_1}}{2(\mathcal{C}_1 \sinh(\zeta\sqrt{-\eta}) + \mathcal{C}_2 \cosh(\zeta\sqrt{-\eta}))}.\end{aligned} \tag{3.29}$$

So, from Eq. (2.12), the solution of the CHDM is

$$\begin{aligned}\mathcal{V}(x,t) = e^{i(-kx+wt+\Theta)} & \left[\pm\frac{\sqrt{-\alpha}\sqrt{-\eta}\{\mathcal{C}_1 \cosh(\zeta\sqrt{-\eta}) + \mathcal{C}_2 \sinh(\zeta\sqrt{-\eta})\}}{2(\mathcal{C}_1 \sinh(\zeta\sqrt{-\eta}) + \mathcal{C}_2 \cosh(\zeta\sqrt{-\eta}))}\right. \\ & \left. \pm \frac{\sqrt{-\alpha}\sqrt{-\eta}\sqrt{c_1}}{2(\mathcal{C}_1 \sinh(\zeta\sqrt{-\eta}) + \mathcal{C}_2 \cosh(\zeta\sqrt{-\eta}))}\right].\end{aligned} \tag{3.30}$$

Now, using the value of ζ in Eq. (3.30), we have

$$\begin{aligned}\therefore \mathcal{V}(x,t) = & \left[\pm\frac{\sqrt{-\alpha}\sqrt{-\eta}\left\{\mathcal{C}_1 \cosh\left(\left(x - \left(\frac{2+3\alpha^2\eta}{6\alpha}\right)t\right)\sqrt{-\eta}\right) + \mathcal{C}_2 \sinh\left(\left(x - \left(\frac{2+3\alpha^2\eta}{6\alpha}\right)t\right)\sqrt{-\eta}\right)\right\}}{2\left(\mathcal{C}_1 \sinh\left(\left(x - \left(\frac{2+3\alpha^2\eta}{6\alpha}\right)t\right)\sqrt{-\eta}\right) + \mathcal{C}_2 \cosh\left(\left(x - \left(\frac{2+3\alpha^2\eta}{6\alpha}\right)t\right)\sqrt{-\eta}\right)\right)}\right. \\ & \left. \pm \frac{\sqrt{-\alpha}\sqrt{-\eta}\sqrt{c_1}}{2\left(\mathcal{C}_1 \sinh\left(\left(x - \left(\frac{2+3\alpha^2\eta}{6\alpha}\right)t\right)\sqrt{-\eta}\right) + \mathcal{C}_2 \cosh\left(\left(x - \left(\frac{2+3\alpha^2\eta}{6\alpha}\right)t\right)\sqrt{-\eta}\right)\right)}\right] e^{i(-kx+wt+\Theta)},\end{aligned} \tag{3.31}$$

for $\alpha < 0$, $k = -\frac{1}{3\alpha}$, $w = -\frac{2}{27\alpha^2}$, and the value of Θ is known from initial phase of wave.

Set 4: $a_0 = 0$, $a_1 = \pm\sqrt{-\alpha}$, $b_1 = 0$, $\beta = 0$, and $\rho = \frac{1+6\alpha^2\eta}{3\alpha}$. Now using the values of $a_0, a_1, b_1, \rho, \beta$ in Eq. (3.7), we have the following:

$$\Psi(\zeta) = \pm\sqrt{-\alpha}\gamma(\zeta). \tag{3.32}$$

Again, from Eqs. (2.2), (2.7) and (3.28), we get

$$\Psi(\zeta) = \pm\frac{\sqrt{-\alpha}\sqrt{-\eta}\{\mathcal{C}_1\cosh(\zeta\sqrt{-\eta}) + \mathcal{C}_2\sinh(\zeta\sqrt{-\eta})\}}{(\mathcal{C}_1\sinh(\zeta\sqrt{-\eta}) + \mathcal{C}_2\cosh(\zeta\sqrt{-\eta}))}. \tag{3.33}$$

From Eq. (2.12), the solution of the CHDM is

$$\mathcal{V}(x,t) = \pm\frac{\sqrt{-\alpha}\sqrt{-\eta}\{\mathcal{C}_1\cosh(\zeta\sqrt{-\eta}) + \mathcal{C}_2\sinh(\zeta\sqrt{-\eta})\}}{(\mathcal{C}_1\sinh(\zeta\sqrt{-\eta}) + \mathcal{C}_2\cosh(\zeta\sqrt{-\eta}))}e^{i(-kx+wt+\Theta)}. \tag{3.34}$$

Inserting the value of ζ in Eq. (3.34), we have

$$\begin{aligned}\therefore\ &\mathcal{V}(x,t)\\ &= \pm\Bigg[\frac{\sqrt{-\alpha}\sqrt{-\eta}\Big\{\mathcal{C}_1\cosh\Big(\Big(x-\Big(\frac{1+6\alpha^2\eta}{3\alpha}\Big)t\Big)\sqrt{-\eta}\Big)\Big\}\ e^{i(-kx+wt+\Theta)}}{\Big(\mathcal{C}_1\sinh\Big(\Big(x-\Big(\frac{1+6\alpha^2\eta}{3\alpha}\Big)t\Big)\sqrt{-\eta}\Big) + \mathcal{C}_2\cosh\Big(\Big(x-\Big(\frac{1+6\alpha^2\eta}{3\alpha}\Big)t\Big)\sqrt{-\eta}\Big)\Big)}\\ &\quad + \frac{\sqrt{-\alpha}\sqrt{-\eta}\Big\{\mathcal{C}_2\sinh\Big(\Big(x-\Big(\frac{1+6\alpha^2\eta}{3\alpha}\Big)t\Big)\sqrt{-\eta}\Big)\Big\}\ e^{i(-kx+wt+\Theta)}}{\Big(\mathcal{C}_1\sinh\Big(\Big(x-\Big(\frac{1+6\alpha^2\eta}{3\alpha}\Big)t\Big)\sqrt{-\eta}\Big) + \mathcal{C}_2\cosh\Big(\Big(x-\Big(\frac{1+6\alpha^2\eta}{3\alpha}\Big)t\Big)\sqrt{-\eta}\Big)\Big)}\Bigg],\end{aligned} \tag{3.35}$$

for $\alpha < 0$, $k = -\frac{1}{3\alpha}$, $w = -\frac{2}{27\alpha^2}$, and the value of Θ is known from initial phase of wave.

Case 3: When we have $\eta = 0$, substituting (2.14) into (3.6) and using (2.4) and (2.10), we will have another four sets of values of arbitrary constant and corresponding four sets of solutions.

Set 1: $a_0 = 0$, $a_1 = \pm\frac{\sqrt{-\alpha}}{2}$, $b_1 = \pm\frac{1}{2}\sqrt{-\alpha{\mathcal{C}_1}^2 + 2\alpha\nu\mathcal{C}_2}$, and $\rho = \frac{1}{3\alpha}$. Now using the values of $a_0, a_1, b_1, \rho, \beta$ in Eq. (3.7), we have the following:

$$\Psi(\zeta) = \pm\frac{\sqrt{-\alpha}}{2}\gamma(\zeta) \pm \frac{1}{2}\sqrt{-\alpha{\mathcal{C}_1}^2 + 2\alpha\beta\mathcal{C}_2}\theta(\zeta). \tag{3.36}$$

Thus, (2.2), (2.9) and (3.36) imply the following:

$$\Psi(\zeta) = \pm\frac{(\beta\zeta + \mathcal{C}_1)\sqrt{-\alpha}}{2\big(\frac{\beta}{2}\zeta^2 + \mathcal{C}_1\zeta + \mathcal{C}_2\big)} \pm \frac{\sqrt{-\alpha{\mathcal{C}_1}^2 + 2\alpha\zeta\mathcal{C}_2}}{2\big(\frac{\beta}{2}\zeta^2 + \mathcal{C}_1\zeta + \mathcal{C}_2\big)} \tag{3.37}$$

As a result from Eq. (2.12), we have the following:

$$\mathcal{V}(\zeta) = \pm \frac{(\beta(x-\rho t)+\mathcal{C}_1)\sqrt{-\alpha}\ e^{i(-kx+wt+\Theta)}}{2\left(\frac{\beta}{2}(x-\rho t)^2+\mathcal{C}_1(x-\rho t)+\mathcal{C}_2\right)} \pm \frac{\sqrt{-\alpha{\mathcal{C}_1}^2+2\alpha(x-\rho t)\mathcal{C}_2}\ e^{i(-kx+wt+\Theta)}}{2\left(\frac{\beta}{2}(x-\rho t)^2+\mathcal{C}_1(x-\rho t)+\mathcal{C}_2\right)}. \tag{3.38}$$

Now, providing the value of ζ in Eq. (3.38), we have

$$\therefore \mathcal{V}(\zeta) = \pm \frac{(\beta(x-\rho t)+\mathcal{C}_1)\sqrt{-\alpha}\ e^{i(-kx+wt+\Theta)}}{2\left(\frac{\beta}{2}(x-\frac{1}{3\alpha}t)^2+\mathcal{C}_1(x-\frac{1}{3\alpha}t)+\mathcal{C}_2\right)} \pm \frac{\sqrt{-\alpha{\mathcal{C}_1}^2+2\alpha(x-\frac{1}{3\alpha}t)\mathcal{C}_2}\ e^{i(-kx+wt+\Theta)}}{2\left(\frac{\beta}{2}(x-\frac{1}{3\alpha}t)^2+\mathcal{C}_1(x-\frac{1}{3\alpha}t)+\mathcal{C}_2\right)}, \tag{3.39}$$

for $\alpha < 0$, $k = -\frac{1}{3\alpha}$, $w = -\frac{2}{27\alpha^2}$, and the value of Θ is known from initial phase of wave.

Set 2: $a_0 = 0$, $a_1 = \pm\sqrt{-\alpha}$, $b_1 = 0$, $\beta = 0$, and $\rho = \frac{1}{3\alpha}$. Now using the values of $a_0, a_1, b_1, \rho, \beta$ in Eq. (3.7), we have the following:

$$\Psi(\zeta) = \pm\frac{\sqrt{-\alpha}}{2}\gamma(\zeta). \tag{3.40}$$

Hence from Eqs. (2.2), (2.9) and (3.40), we have

$$\Psi(\zeta) = \pm\frac{\mathcal{C}_1\sqrt{-\alpha}}{(\mathcal{C}_1\zeta+\mathcal{C}_2)}. \tag{3.41}$$

Thus, from Eq. (2.12), we have

$$\mathcal{V}(\zeta) = \pm\frac{\mathcal{C}_1\sqrt{-\alpha}}{(\mathcal{C}_1(x-\rho t)+\mathcal{C}_2)}e^{i(-kx+wt+\Theta)}. \tag{3.42}$$

Using the value of ζ in Eq. (3.42), we have

$$\therefore \mathcal{V}(\zeta) = \pm\frac{\mathcal{C}_1\sqrt{-\alpha}}{\left(\mathcal{C}_1(x-\frac{1}{3\alpha}t)+\mathcal{C}_2\right)}\ e^{i(-kx+wt+\Theta)}, \tag{3.43}$$

for $\alpha < 0$, $k = -\frac{1}{3\alpha}$, $w = -\frac{2}{27\alpha^2}$, and the value of Θ is known from initial phase of wave.

Set 3: $a_0 = 0$, $a_1 = 0$, $b_1 = \pm\sqrt{-\alpha}\mathcal{C}_1$, $\beta = 0$, and $\rho = \frac{1}{3\alpha}$. Now using the values of $a_0, a_1, b_1, \rho, \beta$ in Eq. (3.7), we have the following:

$$\Psi(\zeta) = \pm\sqrt{-\alpha}\mathcal{C}_1\theta(\zeta). \tag{3.44}$$

Now, from Eqs. (2.2), (2.9) and (3.44), we have

$$\Psi(\zeta) = \pm\frac{\sqrt{-\alpha}\mathcal{C}_1}{(\mathcal{C}_1\zeta+\mathcal{C}_2)}. \tag{3.45}$$

So, Eq. (2.12) implies

$$\mathcal{V}(\zeta) = \pm \frac{\sqrt{-\alpha}\mathcal{C}_1}{(\mathcal{C}_1(x - \rho t) + \mathcal{C}_2)} e^{i(-kx+wt+\Theta)}. \tag{3.46}$$

Putting the value of ζ, we have

$$\therefore \mathcal{V}(\zeta) = \pm \frac{\sqrt{-\alpha}\mathcal{C}_1}{\left(\mathcal{C}_1\left(x - \frac{1}{3\alpha}t\right) + \mathcal{C}_2\right)} e^{i(-kx+wt+\Theta)}, \tag{3.47}$$

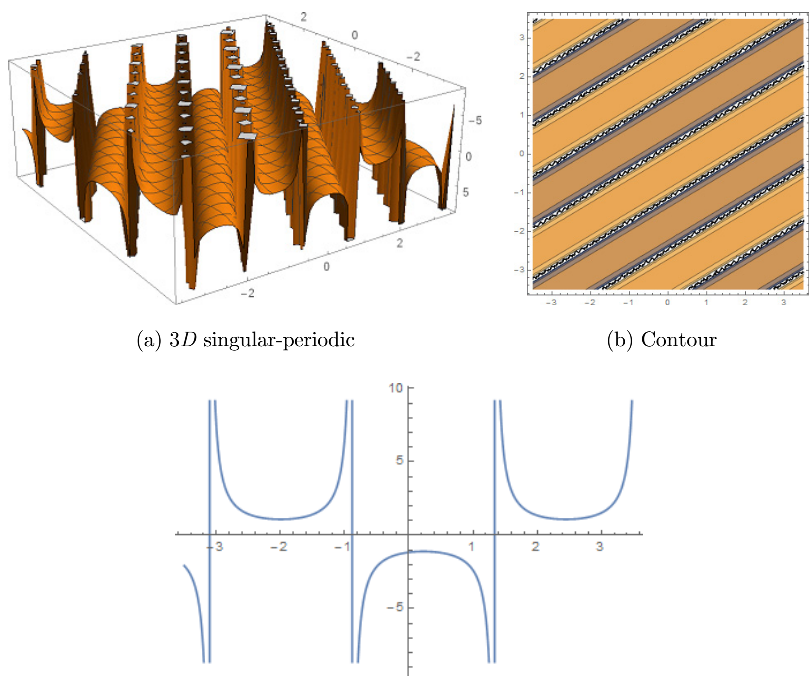
for $\alpha < 0$, $k = -\frac{1}{3\alpha}$, $w = -\frac{2}{27\alpha^2}$, and the value of Θ is known from initial phase of wave.

Set 4: $a_0 = 0$, $a_1 = \pm\frac{\sqrt{-\alpha}}{2}$, $b_1 = \pm\frac{1}{2}\sqrt{-\alpha}\mathcal{C}_1$, and $\rho = \frac{1}{3\alpha}$. Now using the values of $a_0, a_1, b_1, \rho, \beta$ in Eq. (3.7), we have the following:

$$\Psi(\zeta) = \pm\frac{\sqrt{-\alpha}}{2}\gamma(\zeta) \pm \frac{1}{2}\sqrt{-\alpha}\mathcal{C}_1\theta(\zeta). \tag{3.48}$$

Hence from Eqs. (2.2), (2.9) and (3.48), we have

$$\Psi(\zeta) = \left(\pm\frac{(\beta\zeta + \mathcal{C}_1)\sqrt{-\alpha}}{2\left(\frac{\beta}{2}\zeta^2 + \mathcal{C}_1\zeta + \mathcal{C}_2\right)} \pm \frac{\pm\sqrt{-\alpha}\mathcal{C}_1}{2\left(\frac{\beta}{2}\zeta^2 + \mathcal{C}_1\zeta + \mathcal{C}_2\right)}\right). \tag{3.49}$$



(a) $3D$ singular-periodic

(b) Contour

(c) $2D$

Fig. 1. (Color online) Figures of Eq. (3.11) for the following values of the constants, $\alpha = -1.0$, $\eta = 2.0$, $\mathcal{C}_1 = 2.0$, $\mathcal{C}_2 = 1.0$ and $t = 1.0$ (for $2D$).

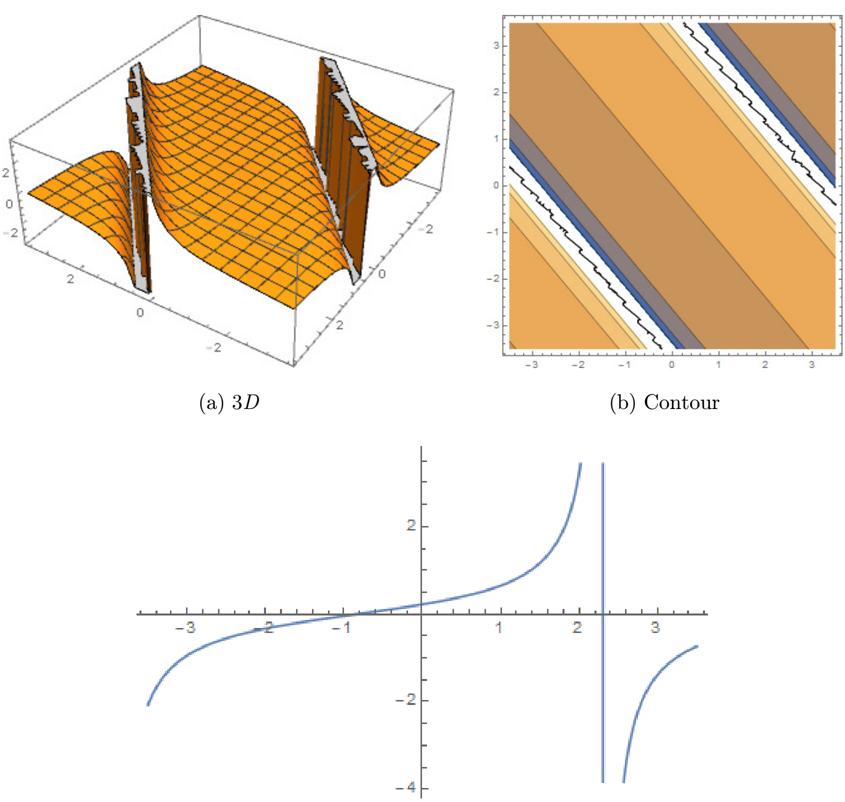
So, from Eq. (2.12), we have

$$\mathcal{V}(\zeta) = \left(\pm \frac{(\beta(x-\rho t)+\mathcal{C}_1)\sqrt{-\alpha}e^{i(-kx+wt+\Theta)}}{2\left(\frac{\beta}{2}(x-\rho t)^2+\mathcal{C}_1(x-\rho t)+\mathcal{C}_2\right)} \pm \frac{\pm\sqrt{-\alpha}\mathcal{C}_1\ e^{i(-kx+wt+\Theta)}}{2\left(\frac{\beta}{2}(x-\rho t)^2+\mathcal{C}_1(x-\rho t)+\mathcal{C}_2\right)} \right). \tag{3.50}$$

Putting the value of ζ in Eq. (3.50), we have

$$\begin{aligned} \therefore \mathcal{V}(\zeta) = & \left(\pm \frac{(\beta(x-\frac{1}{3\alpha}t)+\mathcal{C}_1)\sqrt{-\alpha}\ e^{i(-kx+wt+\Theta)}}{2\left(\frac{\beta}{2}(x-\frac{1}{3\alpha}t)^2+\mathcal{C}_1(x-\frac{1}{3\alpha}t)+\mathcal{C}_2\right)} \right. \\ & \left. \pm \frac{\pm\sqrt{-\alpha}\mathcal{C}_1\ e^{i(-kx+wt+\Theta)}}{2\left(\frac{\beta}{2}(x-\frac{1}{3\alpha}t)^2+\mathcal{C}_1(x-\frac{1}{3\alpha}t)+\mathcal{C}_2\right)} \right), \end{aligned} \tag{3.51}$$

for $\alpha < 0$, $k = -\frac{1}{3\alpha}$, $w = -\frac{2}{27\alpha^2}$, and the value of Θ is known from initial phase of wave.



(a) $3D$

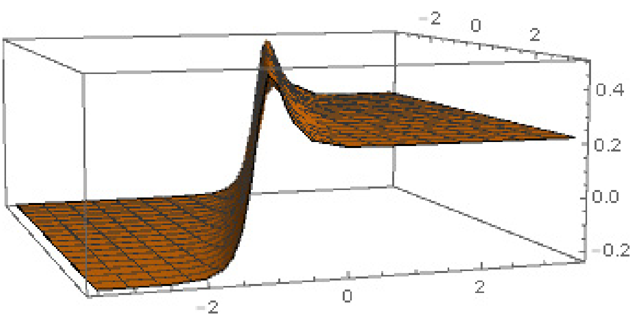
(b) Contour

(c) $2D$

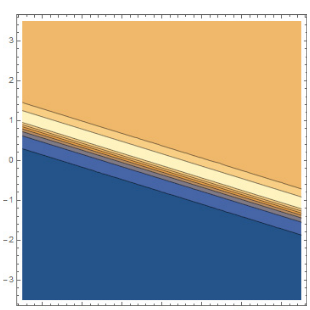
Fig. 2. (Color online) Figures of Eq. (3.15) for the following values of the constants, $\alpha = -1.0,\ \eta = 1.0,\ \mathcal{C}_1 = 1.0,\ \mathcal{C}_2 = 0.0$, and $t = 1.0$ (for $2D$).

4. Graph of Some Particular Solutions of the CHDM and their Physical Interpretations

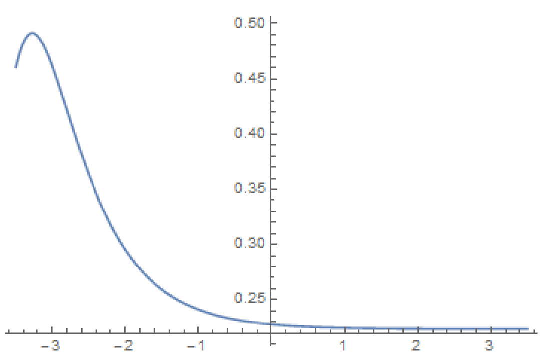
In this section, we have plotted some obtained new solutions of the CHDM for some particular values of the constants. For the sake of simplicity, we have selected only five sets of solutions among our obtained solutions to present graphically. For a clearer explanation, we have shown the graph of 3D, Contour and 2D for the same equation. We have chosen the solutions presented in Eqs. (3.11), (3.15), (3.23), (3.27) and (3.39). The graph for the selected equations is presented in Figs. 1–5, respectively. The 3D, Contour and 2D figures are given in sub-figures (a), (b) and (c), respectively, for each of the same equation. To avoid the complexity, we have to consider the same domain $x \in [-3.5,\ 3.5]$ for all the graph and the values of the constants for the graph are given in the caption of the respective figures. From Figs. 1 and 2 for Eqs. (3.11) and (3.15), we can clearly see that the shape of the graph is singular-periodic solutions (these solutions have rational forms and their denominators contain sums of trigonometric functions). While the graph of equations (3.23) and (3.27) is kink-shaped soliton (they are waves that travel



(a) $3D$

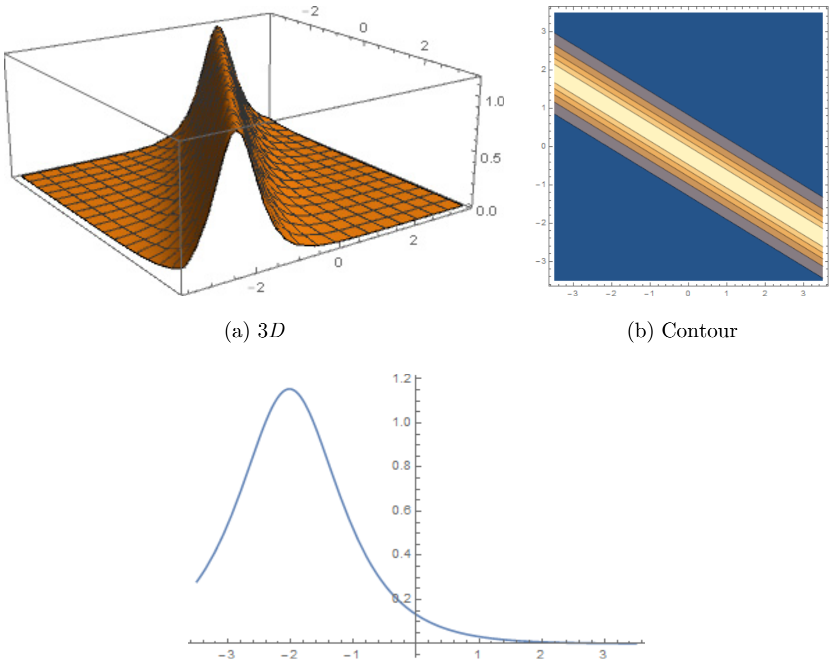


(b) Contour



(c) $2D$

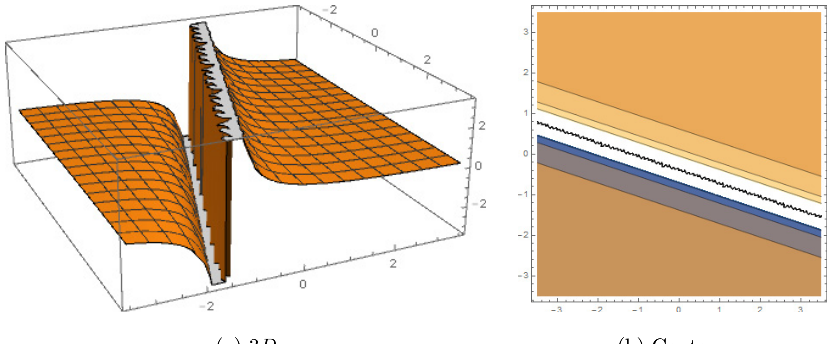
Fig. 3. (Color online) Figures of Eq. (3.23) for the following values of the constants, $\alpha = -0.1$, $\eta = -2.0$, $\beta = 1.0$, $\mathcal{C}_1 = 1.0$, $\mathcal{C}_2 = 2.0$ and $t = 1.0$ (for $2D$).



(a) $3D$ (b) Contour (c) $2D$

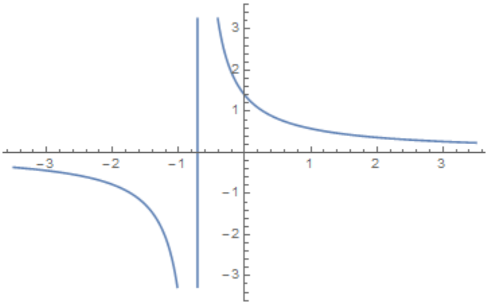
Fig. 4. (Color online) Figures of Eq. (3.27) for the following values of the constants, $\alpha = -0.4$, $\eta = -2.0$, $\mathcal{C}_1 = 1.0$, $\mathcal{C}_2 = 2.0$ and $t = 1.0$ (for $2D$).

from one asymptotic position to another or rise and constant at infinity) and bell-shaped soliton (usually appears as a result of the balance between nonlinearity and dispersion), respectively. For Eq. (3.39), we obtained a singular soliton.



(a) $3D$ (b) Contour

Fig. 5. (Color online) Figures of Eq. (3.39) for the following values of the constants, $\alpha = -1.0$, $\beta = 2.0$, $\mathcal{C}_1 = 3.0$, $\mathcal{C}_2 = 1.0$ and $t = 1.0$ (for $2D$).



(c) $2D$

Fig. 5. (*Continued*)

5. Conclusion

In this work, we attained a variety of traveling wave solutions of the CHDM via the double variable expansion method. By implementing the aforementioned algorithm, we have successfully obtained 11 new sets of results that led to soliton solutions with different shapes of the mentioned equation. Further, we discussed the dynamical behaviors graphically for five different solutions for the three independent cases mentioned earlier. Figures 1–5 classify the shapes of the selected solutions into singular-periodic, kink, bell, and singular solition solutions. These new shape soliton solutions are used to improve the study of plasma physics, the investigation of fusion energy, astrophysical research, and space studies. The obtained solutions in this study may have great importance in solving some real-world problems related to the CHDM in various fields of physics and engineering. The method used in this study is not only effective, but also really well suited to the purpose of looking for soliton solutions for the problem discussed in this study.

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