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Solitary Wave and Quasi-Periodic Wave Solutions to a (3+1)-Dimensional Generalized Calogero-Bogoyavlenskii-Schiff Equation

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Abstract. A (3+1)-dimensional generalized Calogero-Bogoyavlenskii-Schiff equation is considered, which can be used to describe many nonlinear phenomena in plasma physics. By virtue of binary Bell polynomials, a bilinear representation of the equation is succinctly presented. Based on its bilinear formalism, we construct soliton solutions and Riemann theta function periodic wave solutions. The relationships between the soliton solutions and the periodic wave solutions are strictly established and the asymptotic behaviors of the Riemann theta function periodic wave solutions are analyzed with a detailed proof.

AMS subject classifications: 35Q51,35Q53, 35C99, 68W30, 74J35

Key words: A (3+1)-dimensional generalized Calogero-Bogoyavlenskii-Schiff equation, Bell polynomial, solitary wave solution, periodic wave solution, asymptotic behavior.

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1 Introduction

It is well known that the study on exact solutions of nonlinear evolution equations (NLEEs) is always one of the central themes in fluids, fiber optics and other fields [1]. Recently, there has been paying more attention to some generalized NLEEs because of their wide range of applications in various physical fields. It is of importance and practical significance to systematically investigate integrable properties and various exact analytic solutions to those NLEEs, both constant-coefficient and variable-coefficient. In the past decades, different solution methods have been developed in a variety of directions. Various kinds of exact solutions such as solitons, cuspon, positons complexitons and quasi-periodic solutions have been presented for NLEEs. Available solution methods include the inverse scattering transformation [1], the Hirota direct method [2], Lie group method [3], Darboux transformation and Bäcklund transformation [4,5] and the algebro-geometrical approach [6]. The Hirota direct method is one of the most powerful analytic tools for solving soliton problems of NLEEs. If a bilinear representation is known for a given NLEE, one can find its soliton solutions, bilinear BT and some other integrable properties [7–9] directly.

Based on the Bell polynomials, the Hirota bilinear method has also been developed to obtain explicit periodic wave solutions based on the Riemann theta functions. In 1980s, Nakamura proposed a direct method to construct a kind of quasi-periodic wave solutions for nonlinear equations in his essay [10], where the periodic wave solutions of the KdV equation and the Boussinesq equation were obtained by means of the Hirota direct method. The presented method only depends on the existence of Hirota bilinear forms, rather than relies on the Lax pairs and their induced Riemann surfaces for the considered equations. Recently, this method has been extended to investigate the discrete Toda lattice, $(2+1)$-dimensional Bogoyavlenskiss's breaking soliton equation and the asymmetrical Nizhnik-Novikov-Veselov equation by Fan and Hon [11–14]. One of the authors (Ma) constructed one- and two- periodic wave solutions for a class of $(2+1)$-dimensional Hirota bilinear equations and a class of trilinear differential operators used to create trilinear differential equations [15–19]. Zhang et al. [20] constructed periodic wave solutions of the Boussinesq equation. Chen et al. [21, 22] obtained a Maple package to construct bilinear forms, bilinear Bäcklund transformations, Lax pairs and conservation laws for Korteweg-de Vries-type equations. One of our authors (Tian) and his collaborators [23–27] presented soliton solutions, Riemann theta function periodic wave solutions and integrabilities of some nonlinear differential equations, discrete soliton equations and supersymmetric equations, etc.

In this paper, we focus on a (3+1)-dimensional generalized Calogero-Bogoyavlenskii-Schiff (gCBS) equation

$$u_t-h_1(uu_y+u_xv)-h_2uu_z-h_3u_{xxy}-h_4u_{xxz}+h_5u_x+h_6v_y+h_7w_z-h_8u_xw=0, \tag{1.1a}$$

$$u_y=v_x, \tag{1.1b}$$

$$u_z=w_x, \tag{1.1c}$$

where $u=u(x,y,z;t)$ and h_i $(i=1,2,\cdots,8)$ are all arbitrary constants. Under $h_1=0$, $h_3=0$, $h_5=h_6=h_7=0$ and $v=y=0$, Eq. (1.1) can be reduced to the classical $(2+1)$-dimensional CBS equation

$$u_t-h_2uu_z-h_4u_{xxz}-h_8u_xw=0, \tag{1.2a}$$

$$u_z=w_x, \tag{1.2b}$$

which can be written in the potential form,

$$u_t-h_2uu_z-h_4u_{xxz}-h_8u_x\partial_x^{-1}u_z=0. \tag{1.3}$$

Eq. (1.3) admits singular solutions, exact analytical soliton-like solutions, quasi-periodic wave solutions, periodic-like solutions and a Lax representation by taking different values of the coefficients h_2, h_4 and h_8. It is also integrable by the one-dimensional inverse scattering transform and Painléve test [28–31]. Recently, more and more people are interested in studying some generalized nonlinear evolution equations [32–38], resulting from their more widely applications in many physical fields [39–54]. To our knowledge, Riemann theta function periodic wave solutions for Eq. (1.1) have not been studied via binary Bell polynomials.

The main purpose of this paper is to systematically construct a bilinear formalism, soliton solution and some Riemann theta function periodic wave solutions of Eq. (1.1) by means of the Bell polynomials method. Moreover, we present asymptotic behaviors of the periodic wave solutions by establishing two interesting theorems and derive a relationship between the periodic wave solutions and the soliton solutions, which shows that the former solutions tend to the latter solutions under certain conditions.

The rest of the paper is organized as follows. In Section 2, some basic characters of the Hirota bilinear operator and binary Bell polynomials are briefly introduced. Then by virtue of the properties of binary Bell polynomials, we construct a bilinear representation of Eq. (1.1). In Sections 3 and Section 4, the soliton solutions and the Riemann theta function periodic wave solutions of Eq. (1.1) are will investigated, respectively. In Section 5, we further analyze asymptotic behaviors of one-periodic and two-periodic wave solutions to the gCBS equation, by making a limiting procedure, which is used to strictly show that under a small amplitude limit, the periodic wave solutions tend to the known soliton solutions. Finally in Section 6, a few conclusions and remarks are presented.

2 A bilinear representation and its binary Bell polynomials

In this section, a bilinear form of Eq. (1.1) will be constructed. It will be easy to obtain multisoliton solutions when we get a bilinear form of a nonlinear equation. To make our presentation to be easily understood, we give the definition of bilinear operators D_x, D_y,

D_z and D_t as follows:

$$\begin{aligned}&D_x^m D_y^n D_z^s D_t^p f(x,y,z,t)\cdot g(x,y,z,t)\\ =&(\partial_x-\partial_{x'})^m(\partial_y-\partial_{y'})^n(\partial_z-\partial_{z'})^s(\partial_t-\partial_{t'})^p f(x,y,z,t)\cdot g(x',y',z',t')\Big|_{x=x',y=y',z=z',t=t'}.\end{aligned} \tag{2.1}$$

In particular, when the Hirota operators act on exponential functions, we can get a concise formula

$$D_x^m D_y^n D_z^s D_t^p e^{\xi_1}\cdot e^{\xi_2}=(k_1-k_2)^m(\rho_1-\rho_2)^n(l_1-l_2)^s(\omega_1-\omega_2)^p e^{\xi_1+\xi_2}, \tag{2.2}$$

in which $\xi_i=k_ix+\rho_iy+l_iz+\omega_it+\varepsilon_i$, $i=1,2$ with k_i, ρ_i, l_i ω_i and ε_i being constants. Moreover, we have a general formula

$$G(D_x,D_y,D_z,D_t)e^{\xi_1}\cdot e^{\xi_2}=G(k_1-k_2,\rho_1-\rho_2,l_1-l_2,\omega_1-\omega_2)e^{\xi_1+\xi_2}, \tag{2.3}$$

in which $G(D_x,D_y,D_z,D_t)$ is a polynomial of D_x, D_y, D_z and D_t. This formula is very important in constructing one-, two- and N- periodic wave solutions to a given nonlinear differential equation. In what follows, we simply recall some necessary notations on multi-dimensional binary Bell polynomials, for details please refer, for instance, to Lembert and Gilson's work [55–57].

Let f=$f(x_1,x_2,\cdots,x_r)$ be a $\mathbb{C}^\infty$ function in multiple variables. Multi-dimensional Bell polynomials are defined by

$$Y_{n_1x_1,\cdots,n_rx_r}(f)\equiv Y_{n_1,\cdots,n_r}(f_{l_1x_1},\cdots,f_{l_rx_r})=e^{-f}\partial_{x_1}^{n_1}\cdots\partial_{x_r}^{n_r}e^f, \tag{2.4}$$

in which $f_{l_1x_1,\cdots,l_rx_r}=\partial_{x_1}^{l_1}\cdots\partial_{x_r}^{l_r}$ $(0\le l_i\le n_i,\ i=1,2,\cdots,r)$. Taking $r=1$, Bell polynomials read

$$Y_{nx}(f)\equiv Y_n(f_1,\cdots,f_n)=\sum\frac{n!}{s_1!\cdots s_n!(1!)^{s_1}\cdots(n!)^{s_n}}f_1^{s_1}\cdots f_n^{s_n},\quad n=\sum_{k=1}^n ks_k, \tag{2.5a}$$

$$Y_x(f)=f_x,\quad Y_{2x}(f)=f_{2x}+f_x^2,\quad Y_{3x}(f)=f_{3x}+3f_xf_{2x}+f_x^3,\cdots. \tag{2.5b}$$

To make a link between the Bell polynomials and the Hirota D-operators, we need to introduce multi-dimensional binary Bell polynomials [56]:

$$\mathscr{Y}_{n_1x_1,\cdots,n_rx_r}(v,\omega)=Y_{n_1,\cdots,n_r}(f)\Big|_{f_{l_1x_1,\cdots,l_rx_r}}=\begin{cases}v_{l_1x_1,\cdots,l_rx_r},\ l_1+\cdots+l_r\text{ is odd},\\ \omega_{l_1x_1,\cdots,l_rx_r},\ l_1+\cdots+l_r\text{ is even},\end{cases} \tag{2.6a}$$

$$\mathscr{Y}_x(v,\omega)=v_x,\quad \mathscr{Y}_{2x}(v,\omega)=v_x^2+\omega_{2x},\quad \mathscr{Y}_{x,t}(v,\omega)=v_xv_t+\omega_{xt}, \tag{2.6b}$$

$$\mathscr{Y}_{3x}(v,\omega)=v_{3x}+3v_x\omega_{2x}+v_x^3,\cdots, \tag{2.6c}$$

which inherit clear recognizable partial derivative structures of the Bell polynomials.

The link between $\mathscr{Y}$-polynomials and the Hirota bilinear derivatives $D_{x_1}^{n_1}\cdots D_{x_1}^{n_r}F\cdot G$ that we need can be given through the identity [56]:

$$\mathscr{Y}_{n_1x_1,\cdots,n_rx_r}(v=\ln F/G,\ \omega=\ln FG)=(FG)^{-1}D_{x_1}^{n_1}\cdots D_{x_r}^{n_r}F\cdot G, \tag{2.7}$$

where F and G are both the functions of x and t. Taking $F=G$, the identity (2.7) becomes

$$F^{-2}D_{x_1}^{n_1}\cdots D_{x_r}^{n_r}F\cdot F=\mathscr{Y}(0,q=2\ln F)=\begin{cases}0, & n_1+\cdots+n_r \text{ is odd},\\ P_{n_1x_1,\cdots,n_rx_r}(q), & n_1+\cdots+n_r \text{ is even},\end{cases} \tag{2.8}$$

where the P-polynomials can be characterized by an equally recognizable even-part partitional structure

$$P_{2x}(q)=q_{2x},\quad P_{x,t}(q)=q_{xt},\quad P_{4x}(q)=q_{4x}+3q_{2x}^2, \tag{2.9a}$$

$$P_{3x,y}(q)=q_{3x,y}+3q_{2x}q_{xy},\quad P_{6x}(q)=q_{6x}+15q_{2x}q_{4x}+15q_{2x}^3,\cdots. \tag{2.9b}$$

The binary Bell polynomials $\mathscr{Y}_{n_1x_1,\cdots,n_rx_r}(v,\omega)$ can be separated into P-polynomials and Y-polynomials

$$\begin{aligned}&(FG)^{-1}D_{x_1}^{n_1}\cdots D_{x_r}^{n_r}F\cdot G=\mathscr{Y}_{n_1x_1,\cdots,n_rx_r}(v,\omega)|_{v=\ln F/G,\omega=\ln FG}\\ =&\mathscr{Y}_{n_1x_1,\cdots,n_rx_r}(v,v+q)|_{v=\ln F/G,\omega=\ln FG}\\ =&\sum_{n_1+\cdots+n_r=even}\sum_{l_1=0}^{n_1}\cdots\sum_{l_r=0}^{n_r}\prod_{i=0}^{r}\binom{n_i}{l_i}P_{l_1x_1,\cdots,l_rx_r}(q)Y_{(n_1-l_1)x_1,\cdots,(n_r-l_r)x_r}(v).\end{aligned} \tag{2.10}$$

The key property of the multi-dimensional Bell polynomials

$$Y_{n_1x_1,\cdots,n_rx_r}(v)|_{v=\ln\psi}=\psi_{n_1x_1,\cdots,n_rx_r}/\psi, \tag{2.11}$$

implies that the binary Bell polynomials $\mathscr{Y}_{n_1x_1,\cdots,n_rx_r}(v,\omega)$ can still be linearized by means of the Hopf-Cole transformation $v=\ln\psi$, that is, $\psi=F/G$. The formulae (2.10) and (2.11) will then provide the shortest way to the associated Lax system of nonlinear equations.

In the following, we construct a bilinear form of Eq. (1.1) by using an extra auxiliary variable instead of the exchange formulas and then, get multi-soliton solutions to Eq. (1.1).

Theorem 2.1. *Under the following transformation,*

$$u=2(\ln f)_{xx},\quad v=2(\ln f)_{xy},\quad \omega=2(\ln f)_{xz}, \tag{2.12}$$

Eq. (1.1) *is bilinearized into the following bilinear equation*

$$\mathscr{D}(D_t,D_x,D_y,D_z)\equiv(D_xD_t-h_3D_x^3D_y-h_4D_x^3D_z+h_5D_x^2+h_6D_y^2+h_7D_z^2)f\cdot f=0, \tag{2.13}$$

if and only if $h_1=3h_3$, $h_2=h_8=3h_4$.

Proof. In order to detect the existence of a linearizable form of Eq. (1.1), we need to choose an appropriate transformation. Let

$$u=c(t)q_{xx},\quad v=c(t)q_{xy},\quad w=c(t)q_{xz}, \tag{2.14}$$

where $c=c(t)$ is a function to be determined, which makes a connection between Eq. (1.1) and P-polynomials. Combining the transformation (2.14) and Eq. (1.1), we can obtain a new form

$$\begin{aligned}&c_t(t)q_{2x}+c(t)q_{2x,t}-h_1[c^2(t)q_{2x}q_{2x,y}+c^2(t)q_{3x}q_{x,y}]-h_2c^2(t)q_{2x}q_{2x,z}\\&-h_3c(t)q_{4x,y}-h_4c(t)q_{4x,z}+h_5c(t)q_{3x}+h_6c(t)q_{x,2y}+h_7c(t)q_{x,2z}-h_8c^2(t)q_{3x}q_{x,z}=0. \quad (2.15)\end{aligned}$$

By setting $c(t)=1$, $h_2=h_8$ and integrating (2.15) with respect to x, we have

$$E(q)=q_{x,t}-h_1q_{2x}q_{x,y}-h_2q_{2x}q_{x,z}-h_3q_{3x,y}-h_4q_{3x,z}+h_5q_{xx}+h_6q_{yy}+h_7q_{zz}=\vartheta, \quad (2.16)$$

where $\vartheta=\vartheta(y,z,t)$ is an integration constant. Then according to the formula (2.9) and setting $h_1=3h_3$, $h_2=3h_4$, (2.16) can be transformed into a combination form of P-polynomials

$$E(q)\equiv P_{xt}(q)-h_3P_{3x,y}(q)-h_4P_{3x,z}(q)+h_5P_{2x}(q)+h_6P_{2y}(q)+h_7P_{2z}(q)=\vartheta. \quad (2.17)$$

Particularly, when $\vartheta=0$, Eq. (2.17) will be simplified as follows

$$E(q)\equiv P_{xt}(q)-h_3P_{3x,y}(q)-h_4P_{3x,z}(q)+h_5P_{2x}(q)+h_6P_{2y}(q)+h_7P_{2z}(q)=0. \quad (2.18)$$

Referring to the property (2.11) and making use of the change as follows:

$$q=2(\ln f)\Longleftrightarrow u=c(t)q_{xx}=2(\ln f)_{xx}, \quad (2.19a)$$
$$q=2(\ln f)\Longleftrightarrow v=c(t)q_{xy}=2(\ln f)_{xy}, \quad (2.19b)$$
$$q=2(\ln f)\Longleftrightarrow w=c(t)q_{xz}=2(\ln f)_{xz}. \quad (2.19c)$$

Eq. (1.1) can be cast into the bilinear representation as shown in $\mathscr{D}$ (2.13). □

3 Soliton solutions

3.1 Soliton solutions of gCBS equation

In this section, we will consider soliton solutions to Eq. (1.1) through the use of the Hirota bilinear method.

According to the Hirota bilinear theory, Eq. (1.1) has the following one-soliton solution

$$u_1=2[\ln(1+e^{\eta})]_{xx}, \quad v_1=2[\ln(1+e^{\eta})]_{xy}, \quad w_1=2[\ln(1+e^{\eta})]_{xz}, \quad (3.1a)$$
$$\eta=\mu x+\nu y+\sigma z-\frac{1}{\mu}(-h_3\mu^3\nu-h_4\mu^3\sigma+h_5\mu^2+h_6\nu^2+h_7\sigma^2)t+\delta, \quad (3.1b)$$

where μ, ν, σ, δ are all arbitrary real constants.

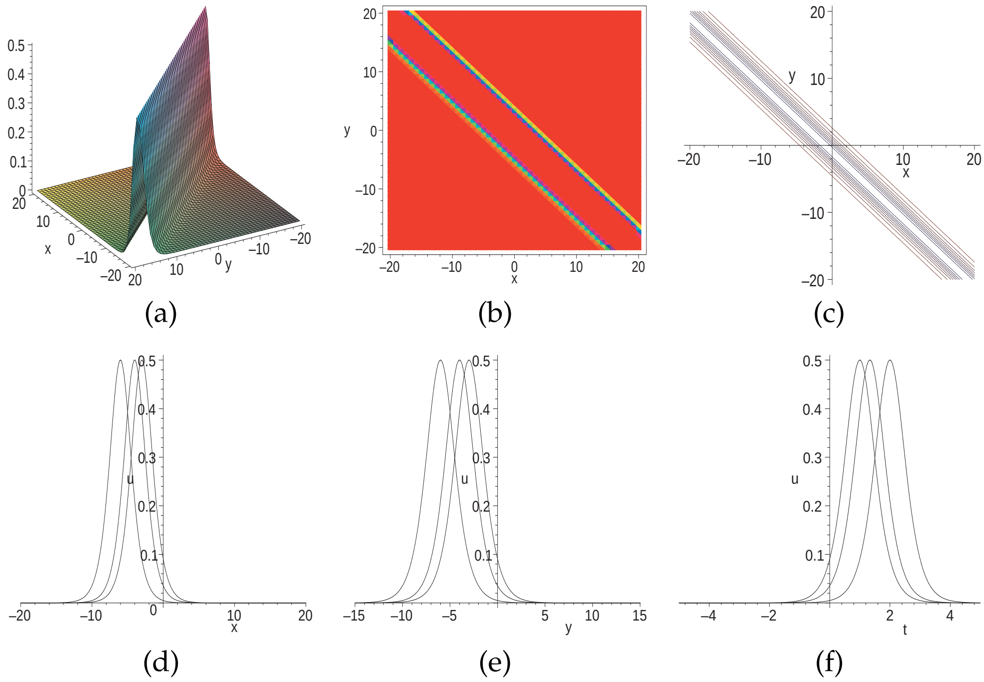

Figure 1: (Color online) Propagation of the one solitary wave for the gCBS (1.1) via expression (3.1) with suitable parameters: $\mu=1$, $\nu=1$, $\sigma=1$, $\delta=1$, $h_3=-1$, $h_4=2$, $h_5=1$, $h_6=2$, $h_7=1$ with $z=t=0$. (a) Perspective view of the wave. (b) The overhead view of the wave. (c) The corresponding contour plot. (d) The wave propagation pattern of the wave along the x-axis. (e) The wave propagation pattern of the wave along the y-axis. (f) The wave propagation pattern of the wave along the t-axis.

Furthermore, in a similar way, the two-soliton solution is given by

$$u_2=2(\ln f)_{xx},\quad v_2=2(\ln f)_{xy},\quad w_2=2(\ln f)_{xz}, \tag{3.2a}$$

$$f=1+e^{\eta_1}+e^{\eta_2}+e^{\eta_1+\eta_2+A_{12}}, \tag{3.2b}$$

$$\eta_i=\mu_i x+\nu_i y+\sigma_i z-\frac{1}{\mu_i}(-h_3\mu_i^3\nu_i-h_4\mu_i^3\sigma_i+h_5\mu_i^2+h_6\nu_i^2+h_7\sigma_i^2)t+\delta_i, \tag{3.2c}$$

where μ_i, ν_i, σ_i, δ_i are all real constants and

$$\exp(A_{12})=-\frac{A_{12}^{numerator}}{A_{12}^{denominator}}, \tag{3.3}$$

where $A_{12}^{numerator}=(\mu_1-\mu_2)(\omega_1-\omega_2)-h_3(\mu_1-\mu_2)^3(\nu_1-\nu_2)-h_4(\mu_1-\mu_2)^3(\sigma_1-\sigma_2)+h_5(\mu_1-\mu_2)^2+h_6(\nu_1-\nu_2)^2+h_7(\sigma_1-\sigma_2)^2$, $A_{12}^{denominator}=(\mu_1+\mu_2)(\omega_1+\omega_2)-h_3(\mu_1+\mu_2)^3(\nu_1+\nu_2)-h_4(\mu_1+\mu_2)^3(\sigma_1+\sigma_2)+h_5(\mu_1+\mu_2)^2+h_6(\nu_1+\nu_2)^2+h_7(\sigma_1+\sigma_2)^2$. The relationships of the coefficients are $\omega_i=h_3\mu_i^2\nu_i+h_4\mu_i^2\sigma_i-h_5\mu_i-h_6\mu_i^{-1}\nu_i^2-h_7\mu_i^{-1}\sigma_i^2$, μ_i, ν_i, σ_i and δ_i $(i=1,2)$ are free constants.

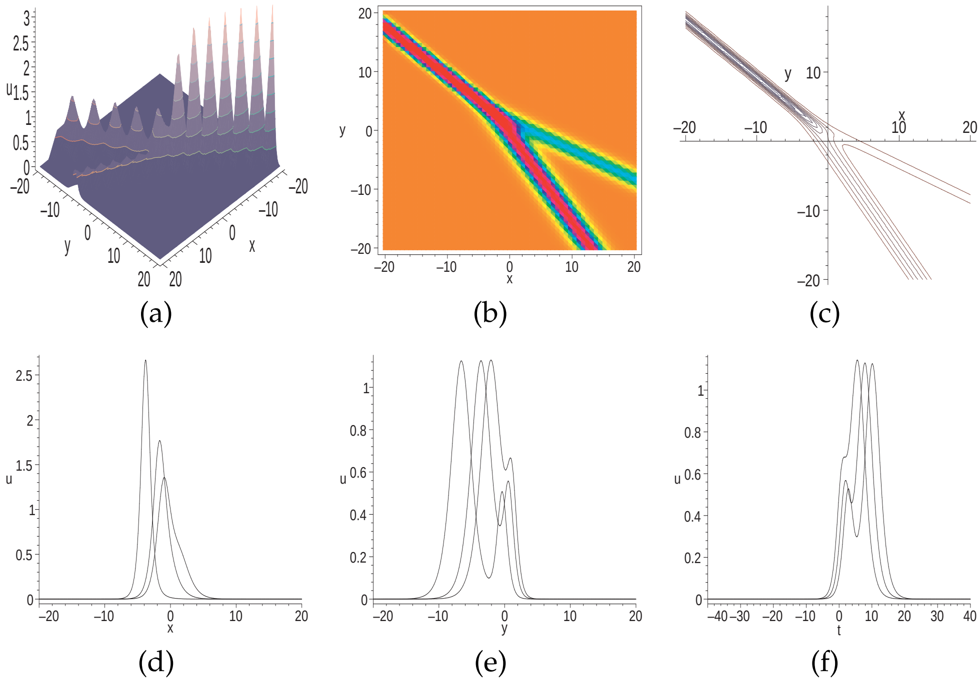

Figure 2: (Color online) Propagation of the two solitary wave for the gCBS (1.1) via expression (3.2) with suitable parameters: $\mu_1=1$, $\mu_2=-1.5$, $\nu_1=2$, $\nu_2=-1$, $\sigma_1=1$, $\sigma_2=2$, $\delta_1=0$, $\delta_2=0$, $h_3=0.5$, $h_4=-1$, $h_5=1$, $h_6=-1$, $h_7=-1.5$ with $z=t=1$. (a) Perspective view of the wave. (b) The overhead view of the wave. (c) The corresponding contour plot. (d) The wave propagation pattern of the wave along the x-axis. (e) The wave propagation pattern of the wave along the y-axis. (f) The wave propagation pattern of the wave along the t-axis.

From the soliton solutions (3.1) and (3.2), we present Figs. 1 and 2 to show the propagation situations of solitary waves.

Fig. 1(a) shows a three-dimensional space graph of one-soliton solution with small excited state. It implies that the amplitude of the excited state is limited. Fig. 1(b) represents a three-dimensional density graph of one soliton solution. It shows that the one-soliton is a line-soliton solution. The Figs. 1(d), (e) and (f) show the wave propagation of the wave along x-axis, y-axis and t-axis, respectively. These figures represent the one-soliton wave propagation with the same amplitude.

Fig. 2(a) shows a three-dimensional space graph of two-soliton solution with small excited state. It implies that the amplitude of the excited state is limited. Fig. 2(b) represents a three-dimensional density graph of two soliton solution with **M**-type. Its surface pattern is two-dimensional, i.e., there are two phase variables η_1 and η_2, which shows that the two-soliton wave admits two-independent spatial one-soliton wave in two independent horizontal directions. The Fig. 2(f) shows the wave propagation of the wave along t-axis. The figure shows the two-soliton wave propagation along t-axis with three wave crests.

3.2 The reduction of soliton solutions of gCBS equation

Next, we construct the soliton solutions of the $(2+1)$-dimensional CBS equation (1.3) by considering the reduction of the soliton solutions of the gCBS equation (1.1).

In the following analysis, we mainly study the relationship between (1.1) and (1.3). Under $h_1=0$, $h_3=0$, $h_5=h_6=h_7=0$, $v=y=0$ and the condition $h_2=h_8=3h_4$ provided in Theorem 2.1, we have the following results:

For the one-soliton solution, we have $\eta=\mu x+\sigma z+h_4\mu^2\sigma t+\delta$ from the solution (3.1) of Eq. (1.1). Then the one-soliton solution of Eq. (1.3) is given by

$$u_1=2(\ln f)_{xx}, \quad f=1+\exp(\eta)=1+\exp(\mu x+\sigma z+h_4\mu^2\sigma t+\delta). \tag{3.4}$$

For the two-soliton solution, we obtain $\eta_i=\mu_i x+\sigma_i z+h_4\mu_i^2\sigma_i t+\delta_i$, and

$$\exp(A_{12})=\frac{(\mu_1-\mu_2)(\omega_1-\omega_2)-(\mu_1-\mu_2)^3(\sigma_1-\sigma_2)}{(\mu_1+\mu_2)(\omega_1+\omega_2)-(\mu_1+\mu_2)^3(\sigma_1+\sigma_2)} \quad \text{with } \omega_i=\mu_i^2\sigma_i, \quad i=1,2. \tag{3.5}$$

From the solution (3.2) of Eq. (1.1), we obtain the two-soliton solution of Eq. (1.3) as follows

$$u_2=2(\ln f)_{xx}, \quad f=1+e^{\eta_1}+e^{\eta_2}+e^{\eta_1+\eta_2+A_{12}}. \tag{3.6}$$

For the three-soliton solution, similarly we obtain $\eta_i=\mu_i x+\sigma_i z+h_4\mu_i^2\sigma_i t+\delta_i$, and

$$\exp(A_{ij})=\frac{(\mu_i-\mu_j)(\omega_i-\omega_j)-(\mu_i-\mu_j)^3(\sigma_i-\sigma_j)}{(\mu_i+\mu_j)(\omega_i+\omega_j)-(\mu_i+\mu_j)^3(\sigma_i+\sigma_j)} \quad \text{with } \omega_i=\mu_i^2\sigma_i, \quad i,j=1,2,3. \tag{3.7}$$

Then, the soliton solutions of the $(2+1)$-dimensional CBS equation (1.3) are the special cases of the soliton solutions for the gCBS equation (1.1).

4 Periodic wave solutions

In order to construct multi-periodic wave solutions to Eq. (1.1), firstly, we introduce the following multi-dimensional Riemann theta function of genus n

$$\vartheta(\xi)=\vartheta(\xi,\tau)=\sum_{n\in\mathbb{Z}^N} e^{\pi \mathrm{i}\langle n\tau,n\rangle+2\pi\mathrm{i}\langle\xi,n\rangle}, \tag{4.1}$$

where $n=(n_1,\cdots,n_N)^T\in\mathbb{Z}^N$ denotes the integer value vector and complex phase variable $\xi=(\xi_1,\cdots,\xi_N)^T\in\mathbb{C}^N$. Moreover, for the two vectors $f=(f_1,\cdots,f_N)^T$ and $g=(g_1,\cdots,g_N)^T$, their inner product is defined by

$$\langle f,g\rangle=f_1g_1+f_2g_2+\cdots+f_Ng_N. \tag{4.2}$$

The $-\mathrm{i}\tau=(-\mathrm{i}\tau_{ij})$ is a positive definite and real-valued symmetric $N\times N$ matrix, which can be called the period matrix of the theta function. The entries τ_{ij} of τ can be seen as free parameters of the theta function (4.1). Under these conditions, the Fourier series (4.1) converges to a real-valued function with an arbitrary vector $\xi\in\mathbb{C}^N$.

Remark 4.1. Constructing periodic wave solutions can use an algebra geometric method, the matrix τ is usually constructed by a compact Riemann surface Γ of genus $N\in\mathbb{N}$. In this paper, taking the matrix τ to be pure imaginary matrix, i.e., the matrix $-i\tau$ real valued, yields the Riemann theta function periodic wave solutions of Eq. (1.1).

4.1 One-periodic wave solutions

To construct the multiperiodic wave solutions of Eq. (1.1), we should consider a more generalized form of the bilinear equation (2.13) by introducing one more widely available. Suppose that Eq. (1.1) satisfies the nonzero asymptotic condition $u\to u_0$ as $|\xi|\to 0$, we can find the solution of Eq. (1.1) of the form as follows

$$u=u_0+2\partial_x^2\ln\vartheta(\xi), \tag{4.3}$$

where u_0 is a constant solution of Eq. (1.1) and phase variable ξ is of the form $\xi=(\xi_1,\cdots,\xi_N)^T$, $\xi_i=k_ix+\rho_iy+l_iz+\omega_it+\varepsilon_i$, $i=1,2,\cdots,N$. Combining Eq. (1.1) and (4.3), we can obtain the bilinear equation by integrating with respect to x as follows

$$\begin{aligned}&\mathscr{L}(D_x,D_y,D_z,D_t)\vartheta(\xi)\cdot\vartheta(\xi)\\ =&\big(D_xD_t-h_3D_x^3D_y-h_3u_0D_x^3D_y-h_4D_x^3D_z-h_4u_0D_x^3D_z\\ &+h_5D_x^2+h_6D_y^2+h_7D_z^2+c\Big)\vartheta(\xi)\cdot\vartheta(\xi)=0,\end{aligned} \tag{4.4}$$

where $c=c(y,z,t)$ is an arbitrary integration constant. For the bilinear equation (4.4), we are interested in its multi-periodic solutions in terms of the Riemann theta function $\vartheta(\xi)$.

Remark 4.2. The constant $c=c(y,z,t)$ may be taken to be zero in the construction of soliton solutions. But in our present periodic case, the nonzero constant c plays an important role and must not be dropped since elliptic functions generally do not satisfy equations with zero integration constants such as (2.13).

In [23], the authors proposed two key theorems to construct Riemann theta function periodic wave solutions for nonlinear equations by virtue of a multi-dimensional Riemann theta function. Now using the results of [23], we can directly obtain some periodic wave solutions for Eq. (1.1).

Theorem 4.1. *Suppose that $\vartheta(\xi,\tau)$ is a Riemann theta function for $N=1$ with $\xi=kx+\rho y+lz+\omega t+\varepsilon$. Eq.* (1.1) *admits a one-periodic wave solution*

$$u=u_0+2\partial_x^2\ln\vartheta(\xi,\tau), \tag{4.5}$$

with

$$\omega=\frac{b_1a_{22}-b_2a_{12}}{a_{11}a_{22}-a_{12}a_{21}},\quad c=\frac{b_1a_{21}-b_2a_{11}}{a_{12}a_{21}-a_{11}a_{22}}, \tag{4.6}$$

where

$$a_{11}=-\sum_{n=-\infty}^{+\infty}16n^2\pi^2k\wp^{2n^2}, \quad a_{12}=\sum_{n=-\infty}^{+\infty}\wp^{2n^2}, \tag{4.7a}$$

$$a_{21}=-\sum_{n=-\infty}^{+\infty}4\pi^2(2n-1)^2k\wp^{2n^2-2n+1}, \quad a_{22}=\sum_{n=-\infty}^{+\infty}\wp^{2n^2-2n+1}, \tag{4.7b}$$

$$\begin{aligned} b_1=&\sum_{n=-\infty}^{+\infty}\Big((256h_3\pi^4n^4k^3\rho+256h_4\pi^4n^4k^3l)(1+u_0)+16h_5n^2\pi^2k^2 \\ &+16h_6n^2\pi^2\rho^2+16h_7n^2\pi^2l^2\Big)\wp^{2n^2}, \qquad (4.7c)\\ b_2=&\sum_{n=-\infty}^{+\infty}\Big((16h_3\pi^4(2n-1)^4k^3\rho+16h_4\pi^4(2n-1)^4k^3l)(1+u_0)+4h_5\pi^2(2n-1)^2k^2 \\ &+4h_6\pi^2(2n-1)^2\rho^2+4h_7\pi^2(2n-1)^2l^2\Big)\wp^{2n^2-2n+1}, \quad \wp=e^{\pi i\tau}, \qquad (4.7d)\end{aligned}$$

and the other parameters k, ρ, l, τ, ε and u_0 are free.

Proof. We consider the following one-Riemann theta function $\vartheta(\xi)$ with N=1 for constructing the one-periodic wave solution of Eq. (1.1),

$$\vartheta(\xi)=\vartheta(\xi,\tau)=\sum_{n=-\infty}^{+\infty}e^{\pi \mathrm{i}n^2\tau+2\pi\mathrm{i}n\xi}, \tag{4.8}$$

where the phase variable $\xi=kx+\rho y+lz+\omega t+\varepsilon$ and the parameter satisfies $\mathrm{Im}(\tau)>0$.

The Riemann theta function (4.8) satisfying the bilinear equation (4.4) yields a sufficient conditions for obtaining periodic wave solutions. By substituting function (4.8) into the left of Eq. (4.4) and using the property (2.3), one can get

$$\begin{aligned} &\mathscr{L}(D_x,D_y,D_z,D_t)\vartheta(\xi)\cdot\vartheta(\xi)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\mathscr{L}(D_x,D_y,D_z,D_t)e^{\pi\mathrm{i}m^2\tau+2\pi\mathrm{i}m\xi}\cdot e^{\pi\mathrm{i}n^2\tau+2\pi\mathrm{i}(m+n)\xi}\\ =&\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\mathscr{L}(2\pi\mathrm{i}(n-m)k,2\pi\mathrm{i}(n-m)\rho,2\pi\mathrm{i}(n-m)l,2\pi\mathrm{i}(n-m)\omega)e^{\pi\mathrm{i}(m^2+n^2)\tau+2\pi\mathrm{i}(m+n)\xi}\\ =&\sum_{m'=-\infty}^{\infty}\left\{\sum_{n=-\infty}^{\infty}\mathscr{L}\left(2\pi\mathrm{i}(2n-m')k,2\pi\mathrm{i}(2n-m')\rho,2\pi\mathrm{i}(2n-m')l,2\pi\mathrm{i}(2n-m')\omega\right)e^{\pi[n^2+(n-m')^2]\tau}\right\}e^{2\pi\mathrm{i}m'\xi}\\ \triangleq&\sum_{m'=-\infty}^{\infty}\widetilde{\mathscr{L}}(m')e^{2\pi\mathrm{i}m'\xi}, \quad m'=m+n. \end{aligned} \tag{4.9}$$

In the following, we compute each series $\widetilde{\mathscr{L}}(m')$ for $m'\in\mathbb{Z}$. By shifting summation index by $n'=n-1$, one has the fact that

$$\begin{aligned}
\widetilde{\mathscr{L}}(m') &= \sum_{n=-\infty}^{\infty} \mathscr{L}\left(2\pi\mathrm{i}(2n-m')k, 2\pi\mathrm{i}(2n-m')\rho, 2\pi\mathrm{i}(2n-m')l, 2\pi\mathrm{i}(2n-m')\omega\right) e^{\pi\mathrm{i}[n^2+(n-m')^2]\tau}\\
&= \sum_{n=-\infty}^{\infty} \mathscr{L}\left(2\pi\mathrm{i}[2n'-(m'-2)]k, 2\pi\mathrm{i}[2n'-(m'-2)]\rho, 2\pi\mathrm{i}[2n'-(m'-2)]l, 2\pi\mathrm{i}(2n'-(m'-2))\omega\right)\\
&\quad\times e^{\pi\mathrm{i}\{n'^2+[n'-(m'-2)]^2\}\tau}\cdot e^{2\pi\mathrm{i}(m'-1)\tau}\\
&=\widetilde{\mathscr{L}}(m'-2)e^{2\pi\mathrm{i}(m'-1)\tau}=\cdots=\begin{cases}\widetilde{\mathscr{L}}(0)e^{\pi\mathrm{i}m'\tau}, & m' \text{ is even},\\ \widetilde{\mathscr{L}}(1)e^{\pi\mathrm{i}(m'+1)\tau}, & m' \text{ is odd},\end{cases} \quad m',n'\in\mathbb{Z}.
\end{aligned} \tag{4.10}$$

This implies that $\widetilde{\mathscr{L}}(m'),m\in\mathbb{Z}$ are completely dominated by $\widetilde{\mathscr{L}}(0)$ and $\widetilde{\mathscr{L}}(1)$. If $\widetilde{\mathscr{L}}(0)=\widetilde{\mathscr{L}}(1)=0$, then it follows that $\widetilde{\mathscr{L}}(m')=0,m'\in\mathbb{Z}$ and thus the theta function (4.8) is an exact solution to Eq. (4.4), i.e., $\mathscr{L}(D_x,D_y,D_z,D_t)\vartheta(\xi)\cdot\vartheta(\xi)=0$. Noticing the specific form of (4.4), a one-periodic wave solution can be obtained, if we require

$$\sum_{n=-\infty}^{+\infty}\mathscr{L}(4n\pi\mathrm{i}k,4n\pi\mathrm{i}\rho,4n\pi\mathrm{i}l,4n\pi\mathrm{i}\omega)e^{2n^2\pi\mathrm{i}\tau}=0, \tag{4.11a}$$

$$\sum_{n=-\infty}^{+\infty}\mathscr{L}(2\pi\mathrm{i}(2n-1)k,2\pi\mathrm{i}(2n-1)\rho,2\pi\mathrm{i}(2n-1)l,2\pi\mathrm{i}(2n-1)\omega)e^{(2n^2-2n+1)\pi\mathrm{i}\tau}=0. \tag{4.11b}$$

Combining (4.4) and (4.11a), (4.11b), we obtain

$$\begin{aligned}
\widetilde{\mathscr{L}}(0)=\sum_{n=-\infty}^{\infty}\Big(&-16\pi^2n^2k\omega-256h_3\pi^4n^4k^3\rho-256h_3u_0\pi^4n^4k^3\rho-256h_4\pi^4n^4k^3l\\
&-256h_4u_0\pi^4n^4k^3l-16h_5\pi^2n^2k^2-16h_6\pi^2n^2\rho^2-16h_7\pi^2n^2l^2+c\Big)e^{2\pi\mathrm{i}n^2\tau}=0,
\end{aligned} \tag{4.12a}$$

$$\begin{aligned}
\widetilde{\mathscr{L}}(1)=\sum_{n=-\infty}^{\infty}\Big(&-4\pi^2(2n-1)^2k\omega-16h_3\pi^4(2n-1)^4k^3\rho-16h_3u_0\pi^4(2n-1)^4k^3\rho\\
&-16h_4\pi^4(2n-1)^4k^3l-16h_4u_0\pi^4(2n-1)^4k^3l-4h_5\pi^2(2n-1)^2k^2-4h_6\pi^2(2n-1)^2\rho^2\\
&-4h_7\pi^2(2n-1)^2l^2+c\Big)e^{\pi\mathrm{i}(2n^2-2n+1)\tau}=0.
\end{aligned} \tag{4.12b}$$

With the same constants in the system (4.7), Eqs. (4.12a) and (4.12b) can be reduced into a linear system about the frequency ω and the integration constant c as follows

$$\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}\omega\\ c\end{pmatrix}=\begin{pmatrix}b_1\\ b_2\end{pmatrix}. \tag{4.13}$$

Now solving this system, we obtain a one-periodic wave solution to Eq. (1.1)

$$u=u_0+2\partial_x^2\ln\vartheta(\xi,\tau), \tag{4.14}$$

from which we can get the vector $(\omega,c)^T$, and the theta function $\vartheta(\xi)$ can be obtained from (4.8). The other parameters k, ρ, l, τ, ε and u_0 are free. $\square$

4.2 Two-periodic wave solutions

Theorem 4.2. *Suppose that $\vartheta(\xi,\tau)$ is a Riemann theta function for $N=2$ with $\xi_i=k_ix+\rho_iy+l_iz+\omega_it+\varepsilon_i$ $(i=1,2)$. Eq.* (1.1) *admits a two-periodic wave solution as follows*

$$u=u_0+2\partial_x^2\ln\vartheta(\xi_1,\xi_2,\tau), \tag{4.15}$$

where ω_1, ω_2, u_0, c satisfy the system as follows

$$H(\omega_1,\omega_2,u_0,c)^T=b, \tag{4.16}$$

in which

$$H=(h_{ij})_{4\times4},\quad b=(b_1,b_2,b_3,b_4)^T, \tag{4.17a}$$

$$h_{i1}=-4\pi^2\sum_{(n_1,n_2)\in\mathbb{Z}^2}\langle 2n-\theta_i,k\rangle(2n_1-\theta_i^1)\Im_i(n), \tag{4.17b}$$

$$h_{i2}=-4\pi^2\sum_{(n_1,n_2)\in\mathbb{Z}^2}\langle 2n-\theta_i,k\rangle(2n_2-\theta_i^2)\Im_i(n), \tag{4.17c}$$

$$\begin{aligned}h_{i3}=&\sum_{(n_1,n_2)\in\mathbb{Z}^2}(-16h_3\pi^4\langle 2n-\theta_i,k\rangle^3\langle 2n-\theta_i,\rho\rangle\\&-16h_4\pi^4\langle 2n-\theta_i,k\rangle^3\langle 2n-\theta_i,l\rangle)\Im_i(n),\end{aligned} \tag{4.17d}$$

$$\begin{aligned}b_i=&\sum_{(n_1,n_2)\in\mathbb{Z}^2}\Big(16h_3\pi^4\langle 2n-\theta_i,k\rangle^3\langle 2n-\theta_i,\rho\rangle+16h_4\pi^4\langle 2n-\theta_i,k\rangle^3\langle 2n-\theta_i,l\rangle\\&+4h_5\pi^2\langle 2n-\theta_i,k\rangle^2+4h_6\pi^2\langle 2n-\theta_i,\rho\rangle^2+4h_7\pi^2\langle 2n-\theta_i,l\rangle^2\Big)\Im_i(n),\end{aligned} \tag{4.17e}$$

$$h_{i4}=\sum_{(n_1,n_2)\in\mathbb{Z}^2}\Im_i(n),\quad \Im_i(n)=\wp_1^{n_1^2+(n_1-\theta_i^1)^2}\wp_2^{n_2^2+(n_2-\theta_i^2)^2}\wp_3^{n_1n_2+(n_1-\theta_i^1)(n_2-\theta_i^2)}, \tag{4.17f}$$

$$\wp_1=e^{\pi i\tau_{11}},\quad \wp_2=e^{\pi i\tau_{22}},\quad \wp_3=e^{2\pi i\tau_{12}},\quad i=1,2,3,4, \tag{4.17g}$$

and $\theta_i=(\theta_i^1,\theta_i^2)^T$, $\theta_1=(0,0)^T$, $\theta_2=(1,0)^T$, $\theta_3=(0,1)^T$, $\theta_4=(1,1)^T$, $i=1,\cdots,4$ and k_i, ρ_i, l_i, τ_{ij}, ε_i (i, j=1, 2) are free parameters.

Proof. By taking N=2, the two-Riemann theta function $\vartheta(\xi_1,\xi_2,\tau)$ is of the form

$$\vartheta(\xi,\tau)=\vartheta(\xi_1,\xi_2,\tau)=\sum_{n\in\mathbb{Z}^2}e^{\pi\mathrm{i}\langle\tau n,n\rangle+2\pi\mathrm{i}\langle\xi,n\rangle}, \tag{4.18}$$

where the variables $n=(n_1,n_2)^T\in\mathbb{Z}^2$, $\xi=(\xi_1,\xi_2)\in\mathbb{C}^2$, $\xi_i=k_ix+\rho_iy+l_iz+\omega_it+\varepsilon_i$, $i=1,2$, and $-\mathrm{i}\tau$ is a real-valued and positive definite symmetric 2×2 matrix, which can be taken of the form

$$\tau=\begin{pmatrix}\tau_{11}&\tau_{12}\\\tau_{12}&\tau_{22}\end{pmatrix},\quad \mathrm{Im}(\tau_{11})>0,\quad \mathrm{Im}(\tau_{22})>0,\quad \tau_{11}\tau_{22}-\tau_{12}^2<0. \tag{4.19}$$

In order to get some sufficient conditions for the theta function (4.18) to satisfy the bilinear equation (4.4), we substitute the function (4.18) into the left of Eq. (4.4) and obtain

$$\begin{aligned}
&\mathscr{L}(D_x,D_y,D_z,D_t)\vartheta(\xi_1,\xi_2,\tau)\cdot\vartheta(\xi_1,\xi_2,\tau)\\
=&\sum_{m,n\in\mathbb{Z}^2}\mathscr{L}(2\pi\mathrm{i}\langle n-m,k\rangle,2\pi\mathrm{i}\langle n-m,\rho\rangle,2\pi\mathrm{i}\langle n-m,l\rangle,2\pi\mathrm{i}\langle n-m,\omega\rangle)\\
&\quad\times e^{2\pi\mathrm{i}\langle\xi,m+n\rangle+\pi\mathrm{i}(\langle\tau m,m\rangle+\langle\tau n,n\rangle)}\\
=&\sum_{m'\in\mathbb{Z}^2}\Big\{\sum_{n\in\mathbb{Z}^2}\mathscr{L}\big(2\pi\mathrm{i}\langle 2n-m',k\rangle,2\pi\mathrm{i}\langle 2n-m',\rho\rangle,2\pi\mathrm{i}\langle 2n-m',l\rangle,2\pi\mathrm{i}\langle 2n-m',\omega\rangle\big)\\
&\quad e^{\pi\mathrm{i}[\langle\tau(n-m'),n-m'\rangle+\langle\tau n,n\rangle]}\Big\}e^{2\pi\mathrm{i}\langle\xi,m'\rangle}\\
\triangleq&\sum_{m'\in\mathbb{Z}^2}\widetilde{\mathscr{L}}(m_1',m_2')e^{2\pi\mathrm{i}\langle\xi,m'\rangle}=\sum_{m'\in\mathbb{Z}^2}\widetilde{\mathscr{L}}(m')e^{2\pi\mathrm{i}\langle\xi,m'\rangle},\quad m'=m+n.
\end{aligned}\tag{4.20}$$

Shifting index n as $n'=n-\delta_{ij}$, $j=1,2$, we can compute that

$$\begin{aligned}
&\widetilde{\mathscr{L}}(m')=\widetilde{\mathscr{L}}(m_1',m_2')\\
=&\sum_{n\in\mathbb{Z}^2}\mathscr{L}\big(2\pi\mathrm{i}\langle 2n-m',k\rangle,2\pi\mathrm{i}\langle 2n-m',\rho\rangle,2\pi\mathrm{i}\langle 2n-m',l\rangle,2\pi\mathrm{i}\langle 2n-m',\omega\rangle\big)\\
&\quad\times e^{\pi\mathrm{i}[\langle\tau(n-m'),n-m'\rangle+\langle\tau n,n\rangle]}\\
=&\sum_{n\in\mathbb{Z}^2}\mathscr{L}\Bigg(2\pi\mathrm{i}\sum_{i=1}^{2}[2n_i'-(m_i'-2\delta_{ij})]k_i,2\pi\mathrm{i}\sum_{i=1}^{2}[2n_i'-(m_i'-2\delta_{ij})]\rho_i,2\pi\mathrm{i}\sum_{i=1}^{2}[2n_i'-(m_i'-2\delta_{ij})]l_i,\\
&\quad 2\pi\mathrm{i}\sum_{i=1}^{2}[2n_i'-(m_i'-2\delta_{ij})]\omega_i\Bigg)e^{\pi\mathrm{i}\sum_{i,k=1}^{2}[(n_i'+\delta_{ij})(n_k'+\delta_{kj})+(m_i'-n_i'-\delta_{ij})(m_k'-n_k'-\delta_{kj})]\tau_{ik}}\\
=&\begin{cases}\widetilde{\mathscr{L}}(m_1'-2,m_2')e^{2\pi\mathrm{i}(m_1'-1)\tau_{11}+2\pi\mathrm{i}m_2'\tau_{12}}, & j=1,\\ \widetilde{\mathscr{L}}(m_1',m_2'-2)e^{2\pi\mathrm{i}(m_2'-1)\tau_{22}+2\pi\mathrm{i}m_1'\tau_{12}}, & j=2,\end{cases}\qquad m',n'\in\mathbb{Z}^2,
\end{aligned}\tag{4.21}$$

with δ_{ij} representing Kronecker's delta. It implies that $\widetilde{\mathscr{L}}(m')$, $m'\in\mathbb{Z}^2$ are completely dominated by four functions $\widetilde{\mathscr{L}}(0,0)$, $\widetilde{\mathscr{L}}(1,0)$, $\widetilde{\mathscr{L}}(0,1)$ and $\widetilde{\mathscr{L}}(1,1)$. We can show that if $\widetilde{\mathscr{L}}(0,0)=\widetilde{\mathscr{L}}(1,0)=\widetilde{\mathscr{L}}(0,1)=\widetilde{\mathscr{L}}(1,1)=0$, then $\widetilde{\mathscr{L}}(m_1',m_2')=0$ for all $m_1',m_2'\in\mathbb{Z}^2$ and thus the theta function (4.18) is an exact solution to Eq. (4.4). Noticing the specific form of Eq. (4.4), two-periodic wave solutions can be obtained if the following system holds

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}^2}\mathscr{L}(2\pi\mathrm{i}\langle 2n-\theta_1,k\rangle,2\pi\mathrm{i}\langle 2n-\theta_1,\rho\rangle,2\pi\mathrm{i}\langle 2n-\theta_1,l\rangle,2\pi\mathrm{i}\langle 2n-\theta_1,\omega\rangle)\\
&\qquad\times e^{\pi\mathrm{i}[\langle\tau(n-\theta_1),n-\theta_1\rangle+\langle\tau n,n\rangle]}=0,
\end{aligned}\tag{4.22a}$$

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}^2}\mathscr{L}(2\pi\mathrm{i}\langle 2n-\theta_2,k\rangle,2\pi\mathrm{i}\langle 2n-\theta_2,\rho\rangle,2\pi\mathrm{i}\langle 2n-\theta_2,l\rangle,2\pi\mathrm{i}\langle 2n-\theta_2,\omega\rangle)\\
&\qquad\times e^{\pi\mathrm{i}[\langle\tau(n-\theta_2),n-\theta_2\rangle+\langle\tau n,n\rangle]}=0,
\end{aligned}\tag{4.22b}$$

$$\sum_{n\in\mathbb{Z}^2}\mathscr{L}(2\pi\mathrm{i}\langle 2n-\theta_3,k\rangle,2\pi\mathrm{i}\langle 2n-\theta_3,\rho\rangle,2\pi\mathrm{i}\langle 2n-\theta_3,l\rangle,2\pi\mathrm{i}\langle 2n-\theta_3,\omega\rangle)$$
$$\times e^{\pi\mathrm{i}[\langle\tau(n-\theta_3),n-\theta_3\rangle+\langle\tau n,n\rangle]}=0, \tag{4.22c}$$
$$\sum_{n\in\mathbb{Z}^2}\mathscr{L}(2\pi\mathrm{i}\langle 2n-\theta_4,k\rangle,2\pi\mathrm{i}\langle 2n-\theta_4,\rho\rangle,2\pi\mathrm{i}\langle 2n-\theta_4,l\rangle,2\pi\mathrm{i}\langle 2n-\theta_4,\omega\rangle)$$
$$\times e^{\pi\mathrm{i}[\langle\tau(n-\theta_4),n-\theta_4\rangle+\langle\tau n,n\rangle]}=0, \tag{4.22d}$$

where $\theta_i=(\theta_i^1,\theta_i^2)^T$, $\theta_1=(0,0)^T$, $\theta_2=(1,0)^T$, $\theta_3=(0,1)^T$, $\theta_4=(1,1)^T$, $i=1,2,3,4$.

Combining Eqs. (4.4) and (4.22a), (4.22b), (4.22c), (4.22d), we obtain

$$\begin{aligned}&\sum_{n\in\mathbb{Z}^2}[-4\pi^2\langle 2n-\theta_i,k\rangle\langle 2n-\theta_i,\omega\rangle-16h_3\pi^4\langle 2n-\theta_i,k\rangle^3\langle 2n-\theta_i,\rho\rangle\\&-16h_3u_0\pi^4\langle 2n-\theta_i,k\rangle^3\langle 2n-\theta_i,\rho\rangle-16h_4\pi^4\langle 2n-\theta_i,k\rangle^3\langle 2n-\theta_i,l\rangle\\&-16h_4u_0\pi^4\langle 2n-\theta_i,k\rangle^3\langle 2n-\theta_i,l\rangle-4h_5\pi^2\langle 2n-\theta_i,k\rangle^2-4h_6\pi^2\langle 2n-\theta_i,\rho\rangle^2\\&-4h_7\pi^2\langle 2n-\theta_i,l\rangle^2+c]e^{\pi\mathrm{i}[\langle\tau(n-\theta_i),n-\theta_i\rangle+\langle\tau n,n\rangle]}=0,\quad i=1,2,3,4.\end{aligned} \tag{4.23}$$

These equations can be written as a new form, under the above notation (4.17), given by

$$\begin{pmatrix} h_{11} & h_{12} & h_{13} & h_{14}\\ h_{21} & h_{22} & h_{23} & h_{24}\\ h_{31} & h_{32} & h_{33} & h_{34}\\ h_{41} & h_{42} & h_{43} & h_{44}\end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\\ u_0\\ c\end{pmatrix}=\begin{pmatrix} b_1\\ b_2\\ b_3\\ b_4\end{pmatrix}. \tag{4.24}$$

Now solving the above system, we can obtain a two-periodic wave solution of Eq. (1.1),

$$u=u_0+2\partial_x^2\ln\vartheta(\xi_1,\xi_2,\tau), \tag{4.25}$$

where $\vartheta(\xi_1,\xi_2,\tau)$ and parameters ω_1,ω_2,u_0,c are determined by Eqs. (4.18) and (4.25), respectively. The other parameters k_i, ρ_i, l_i, ε_i, τ_{ij} are free. $\square$

The figures of the one-periodic wave solution (4.5) and the two-periodic wave solution (4.15) are presented in Figs. 3-6. By taking the appropriate parameters, we can plot different figures, which can help us analyze the properties and propagation of the periodic wave solutions well. The Figs. 3(a), (b) and (c) represent a three-dimensional space graph of the one-periodic wave solution with different labels. The Figs. 3(d), (e) and (f) show the wave propagation of the wave along x-axis, y-axis and t-axis with the same amplitude, respectively. It shows from (d), (e) and (f) that the one-periodic wave solution admits different fundamental periods in x-axis, y-axis and t-axis, respectively.

The propagation of the symmetric and asymmetric two-periodic wave solution (4.15) is presented in Figs. 4-6. The Fig. 4(a) shows a three-dimensional space graph of the two-periodic wave solution and it also shows that the symmetric two-periodic wave solution is periodic in two directions. Furthermore, it implies that the two-periodic wave solution

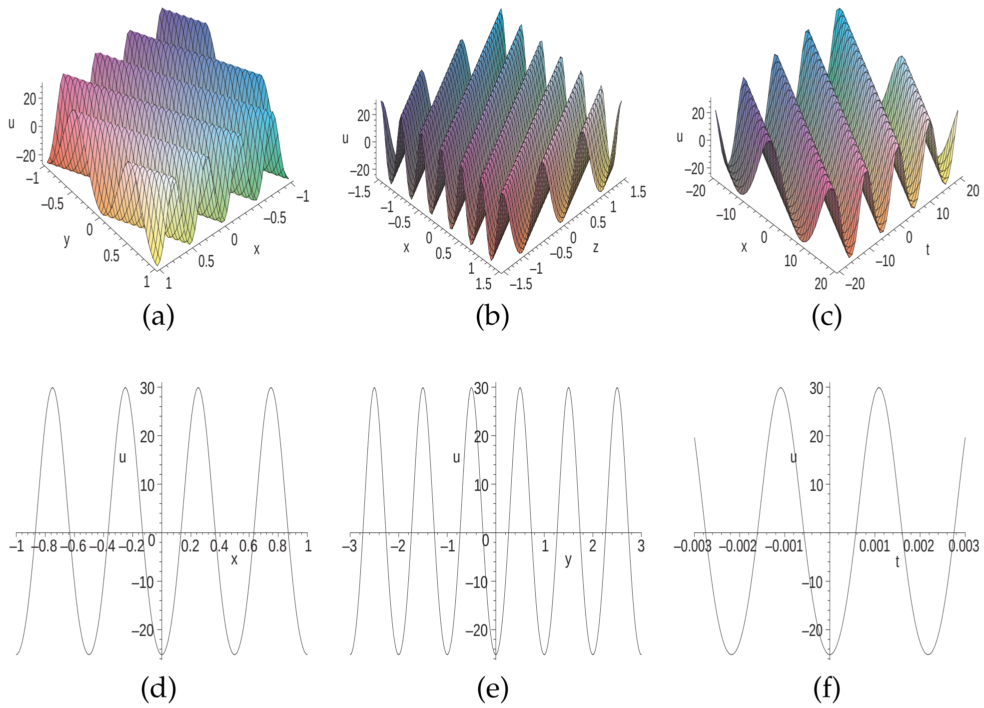

Figure 3: (Color online) A one-periodic wave of the gCBS (1.1) via expression (4.5) with suitable parameters: $k=2$, $\rho=1$, $l=1$, $\tau=\mathrm{i}$, $h_3=2$, $h_4=1$, $h_5=2$, $h_6=3$, $h_7=4$ and $\varepsilon=0$. This figure shows that the one-periodic wave solution is one-dimensional. Perspective view of the real part of the periodic wave $Re(u)$ with: (a) $t=z=0$. (b) $t=y=0$. (c) $y=z=0$. Wave propagation pattern of the wave along with: (d) the x-axis. (e) the y-axis. (f) the t-axis.

is actually one dimensional and it degenerates to a one-periodic wave solution. The Figs. 5 and 6 show that the asymmetric two-periodic wave solution is spatially periodic in two directions, but it need not be periodic in either the x or t directions.

Remark 4.3. Under the condition $h_1=0$, $h_3=0$, $h_5=h_6=h_7=0$, $v=y=0$ and $h_2=h_8=3h_4$, one can obtain the quasi-periodic wave solutions of (1.3) from ones of (1.1). Since the expressions of the solutions are relatively large, we omit them here.

5 Asymptotic properties of the periodic waves

In the following, we analyze relations between the one- and two-periodic wave solutions (4.5), (4.15) and the one- and two-soliton solutions (3.1), (3.2) to Eq. (1.1).

5.1 Feature and asymptotic properties of one-periodic waves

The one-periodic wave solution (4.5) has the following properties.

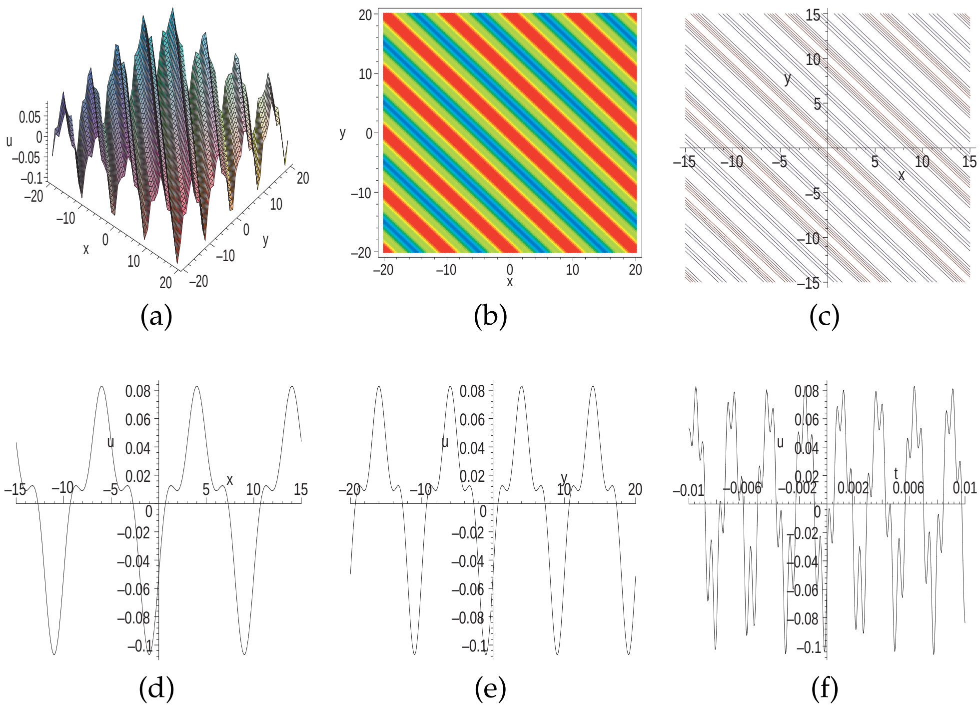

Figure 4: (Color online) A symmetric two-periodic wave of the gCBS (1.1) via expression (4.15) with suitable parameters: $\frac{k_1}{k_2}=\frac{\rho_1}{\rho_2}=\frac{l_1}{l_2}$ and $u_0=0$, $k_1=\rho_1=l_1=0.1$, $k_2=\rho_2=l_2=0.3$, $\tau_{11}=\mathrm{i}$, $\tau_{12}=0.5\mathrm{i}$, $\tau_{22}=2\mathrm{i}$, $h_3=-1$, $h_4=2$, $h_5=4$, $h_6=6$, $h_7=8$, $\varepsilon_1=0$, $\varepsilon_2=0$ with $z=1$. This figure shows that two-periodic wave solution is almost one-dimensional. (a) Perspective view of the real part of the periodic wave $Re(u)$. (b) The overhead view of the wave. (c) The corresponding contour plot. (d) The wave propagation pattern of the wave along the x-axis. (e) The wave propagation pattern of the wave along the y-axis. (f) The wave propagation pattern of the wave along the t-axis.

(i) It has two fundamental periods 1 and τ in the phase variable ξ.

(ii) There is a single phase variable ξ. Its speed parameter is given by

$$\omega=\frac{b_1a_{22}-b_2a_{12}}{a_{11}a_{22}-a_{12}a_{21}}. \tag{5.1}$$

(iii) The one-periodic wave has only one wave pattern and it can be viewed as a parallel superposition of overlapping one-solitary waves, placed one period apart (see Fig. 3).

Now we further study asymptotic properties of the one-periodic wave solution (4.14), we have to use the solutions of the system (4.13). Because both the coefficient matrix and the right-side vector of the system (4.13) are power series about $\wp$, its solution $(\omega,c)^T$ should also be a series about $\wp$. We can solve the system (4.13) via a small parameter expansion method and a general procedure described as follows.

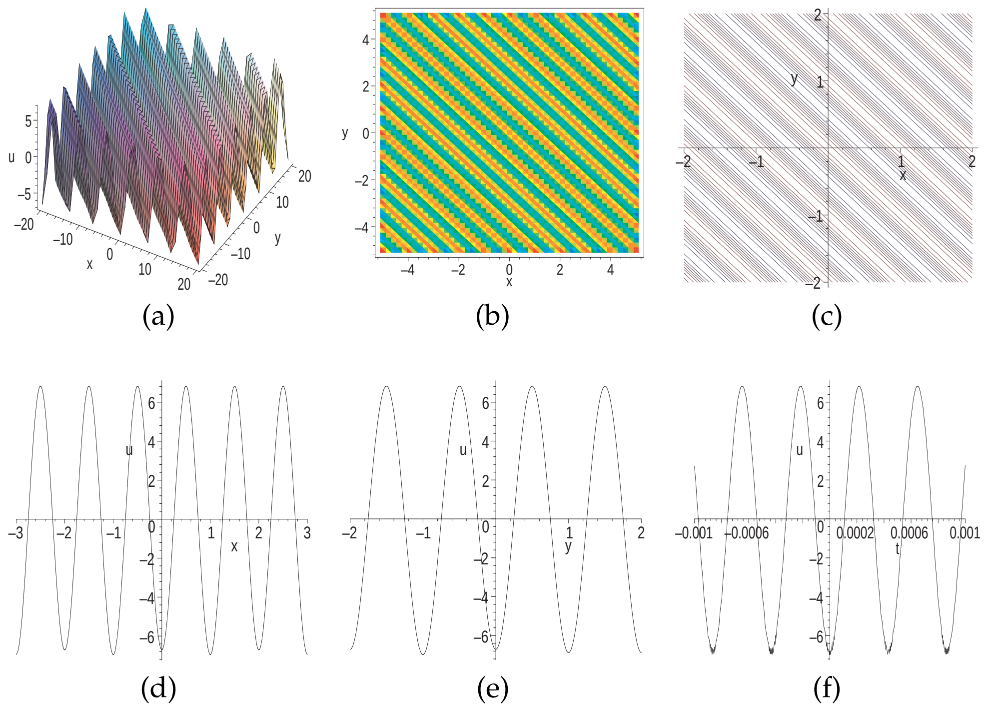

Figure 5: (Color online) A asymmetric two-periodic wave of the gCBS (1.1) via expression (4.15) with suitable parameters: $\frac{k_1}{k_2} \neq \frac{\rho_1}{\rho_2} = \frac{l_1}{l_2}$ and $u_0=0$, $k_1=1$, $k_2=0.5$, $\rho_1=1$, $\rho_2=0.4$, $l_1=1$, $l_2=0.4$, $\tau_{11}=\mathrm{i}$, $\tau_{12}=0.5\mathrm{i}$, $\tau_{22}=2\mathrm{i}$, $h_3=-1$, $h_4=2$, $h_5=4$, $h_6=6$, $h_7=8$, $\varepsilon_1=0$, $\varepsilon_2=0$ with $z=1$. (a) Perspective view of the real part of the periodic wave $Re(u)$. (b) The overhead view of the wave. (c) The corresponding contour plot. (d) The wave propagation pattern of the wave along the x-axis. (e) The wave propagation pattern of the wave along the y-axis. (f) The wave propagation pattern of the wave along the t-axis.

The system (4.13) can be rewritten as the following power series

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = A_0 + A_1\wp + A_2\wp^2 + \cdots, \tag{5.2a}$$

$$\begin{pmatrix} \omega \\ c \end{pmatrix} = X_0 + X_1\wp + X_2\wp^2 + \cdots, \tag{5.2b}$$

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = B_0 + B_1\wp + B_2\wp^2 + \cdots. \tag{5.2c}$$

Substituting Eqs. (5.2a)-(5.2c) into Eq. (4.13), we have the following recursion relations

$$A_0X_0 = B_0, \quad A_0X_n + A_1X_{n-1} + \cdots + A_nX_0 = B_n, \quad n \geq 1, \quad n \in \mathbb{N}. \tag{5.3}$$

Assuming that the matrix A_0 is reversible, one can obtain

$$X_0 = A_0^{-1}B_0, \quad X_n = A_0^{-1}\left(B_n - \sum_{i=1}^{n} A_iB_{n-1}\right), \quad n \geq 1, \quad n \in \mathbb{N}. \tag{5.4}$$

If two matrices A_0 and A_1 are not invertible, and read

$$A_0=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_1=\begin{pmatrix} 0 & 0 \\ -8\pi^2 & 2 \end{pmatrix}, \tag{5.5}$$

the required result can be obtained as follows

$$X_0=\left(\frac{2B_0^{(1)}-B_1^{(2)}}{8\pi^2 k}, B_0^{(1)}\right)^T, \quad X_1=\left(\frac{2B_1^{(1)}-(B_2-A_2X_0)^{(2)}}{8\pi^2 k}, B_1^{(1)}\right)^T, \cdots, \tag{5.6a}$$

$$X_n=\left(\frac{2(B_{n+1}-\sum_{i=2}^{n} A_i X_{n-i})^{(1)}-\left(B_{n+1}-\sum_{i=2}^{n+1} A_i X_{n+1-i}\right)^{(2)}}{8\pi^2 k}, \quad (B_{n+1}-\textstyle\sum_{i=2}^{n} A_i X_{n-i})^{(1)} \right)^T, \quad n\geq 2, \quad n\in\mathbb{N}, \tag{5.6b}$$

in which $\alpha^{(1)}$ and $\alpha^{(2)}$ denote the first and second component of a two-dimensional vector α, respectively. In the following, we will present the relationship between the one-periodic wave solution (4.5) and the one-soliton solution (3.1).

Theorem 5.1. *If the vector* $(\omega,c)^T$ *is a solution of the system* (4.13) *and for the one-periodic wave solution* (4.5)*, we set*

$$u_0=0, \quad k=\frac{\mu}{2\pi i}, \quad \rho=\frac{\nu}{2\pi i}, \quad l=\frac{\sigma}{2\pi i}, \quad \varepsilon=\frac{\delta+\pi\tau}{2\pi i}, \tag{5.7}$$

where μ, ν, σ *and* δ *are determined by Eq.* (4.5)*, then we have the asymptotic properties as follows*

$$c\to 0, \quad \xi\to\frac{\eta+\pi\tau}{2\pi i}, \quad \vartheta(\xi,\tau)\to 1+e^{\eta}, \quad \text{when} \;\; \wp\to 0, \tag{5.8}$$

which implies that the one-periodic wave solution (4.5) *tends to the one-soliton solution* (3.1) *via a small amplitude limit, that is* $(u,\wp)\to(u_1,0)$.

Proof. Based on the system (4.7), the functions a_{ij}, b_i, $i,j=1,2$ can be rewritten as the series about $\wp$

$$a_{11}=-32\pi^2 k(\wp^2+4\wp^8+\cdots+n^2\wp^{2n^2}+\cdots), \tag{5.9a}$$

$$a_{12}=1+2(\wp^2+\wp^8+\cdots+\wp^{2n^2}+\cdots), \tag{5.9b}$$

$$a_{21}=-8\pi^2 k(\wp+9\wp^5+\cdots+(2n-1)^2\wp^{2n^2-2n+1}+\cdots), \tag{5.9c}$$

$$a_{22}=2(\wp+\wp^5+\cdots+\wp^{2n^2-2n+1}+\cdots), \tag{5.9d}$$

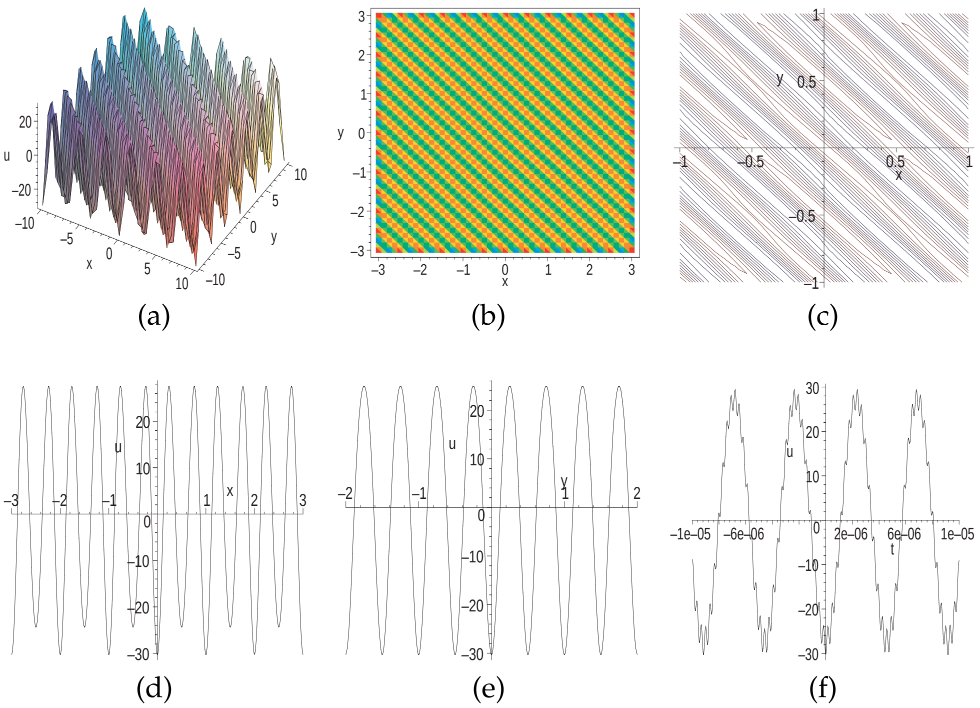

Figure 6: (Color online) A asymmetric two-periodic wave of the gCBS (1.1) via expression (4.15) with suitable parameters: $\frac{k_1}{k_2}\neq\frac{\rho_1}{\rho_2}\neq\frac{l_1}{l_2}$ and $u_0=0$, $k_1=2$, $k_2=3$, $\rho_1=2$, $\rho_2=4$, $l_1=2$, $l_2=5$, $\tau_{11}=\mathrm{i}$, $\tau_{12}=0.5\mathrm{i}$, $\tau_{22}=2\mathrm{i}$, $h_3=-1$, $h_4=2$, $h_5=4$, $h_6=6$, $h_7=8$, $\varepsilon_1=0$, $\varepsilon_2=0$ with $z=1$. (a) Perspective view of the real part of the periodic wave $Re(u)$. (b) The overhead view of the wave. (c) The corresponding contour plot. (d) The wave propagation pattern of the wave along the x-axis. (e) The wave propagation pattern of the wave along the y-axis. (f) The wave propagation pattern of the wave along the t-axis.

$$\begin{aligned}
b_1=32\pi^2\Big\{&[(16h_3\pi^2k^3\rho+16h_4\pi^2k^3l)(1+u_0)+h_5k^2+h_6\rho^2+h_7l^2]\wp^2\\
&+[(256h_3\pi^2k^3\rho+256h_4\pi^2k^3l)(1+u_0)+4h_5k^2+4h_6\rho^2+4h_7l^2]\wp^8+\cdots\\
&+[(16h_3\pi^2n^4k^3\rho+16h_4\pi^2n^4k^3l)(1+u_0)+h_5n^2k^2+h_6n^2\rho^2+h_7n^2l^2]\wp^{2n^2}+\cdots\Big\},
\end{aligned} \tag{5.9e}$$

$$\begin{aligned}
b_2=8\pi^2\Big\{&[(4h_3\pi^2k^3\rho+4h_4\pi^2k^3l)(1+u_0)+h_5k^2+h_6\rho^2+h_7l^2]\wp\\
&+[(324h_3\pi^2k^3\rho+324h_4\pi^2k^3l)(1+u_0)+9h_5k^2+9h_6\rho^2+9h_7l^2]\wp^5+\cdots\\
&+[(4h_3(2n-1)^4\pi^2k^3\rho+4h_4(2n-1)^4\pi^2k^3l)(1+u_0)+h_5(2n-1)^2k^2\\
&+h_6(2n-1)^2\rho^2+h_7(2n-1)^2l^2]\wp^{2n^2-2n+1}+\cdots\Big\}.
\end{aligned} \tag{5.9f}$$

According to Eqs. (5.2a) and (5.2c), one can get

$$A_0=\begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix},\quad A_1=\begin{pmatrix} 0 & 0\\ -8\pi^2k & 2\end{pmatrix}, \tag{5.10a}$$

$$A_2=\begin{pmatrix} -32\pi^2k & 2 \\ 0 & 0 \end{pmatrix},\quad A_5=\begin{pmatrix} 0 & 0 \\ -72\pi^2k & 2 \end{pmatrix}, \tag{5.10b}$$

$$B_1=\begin{pmatrix} 0 \\ (32h_3\pi^4k^3\rho+32h_4\pi^4k^3l)(1+u_0)+8h_5\pi^2k^2+8h_6\pi^2\rho^2+8h_7\pi^2l^2 \end{pmatrix}, \tag{5.10c}$$

$$B_2=\begin{pmatrix} (512h_3\pi^4k^3\rho+512h_4\pi^4k^3l)(1+u_0)+32h_5\pi^2k^2+32h_6\pi^2\rho^2+32h_7\pi^2l^2 \\ 0 \end{pmatrix}, \tag{5.10d}$$

$$B_5=\begin{pmatrix} 0 \\ (2592h_3\pi^4k^3\rho+2592h_4\pi^4k^3l)(1+u_0)+72h_5\pi^2k^2+72h_6\pi^2\rho^2+72h_7\pi^2l^2 \end{pmatrix}, \tag{5.10e}$$

$$A_3=A_4=0,\cdots,\quad B_0=B_3=B_4=0,\cdots. \tag{5.10f}$$

Combining Eq. (5.10) and the formulas (5.6), we obtain

$$X_0=\begin{pmatrix} \frac{(-4h_3\pi^2k^3\rho-4h_4\pi^2k^3l)(1+u_0)-h_5k^2-h_6\rho^2-h_7l^2}{k} \\ 0 \end{pmatrix}, \tag{5.11a}$$

$$X_2=\begin{pmatrix} 8kX_0^{(1)} \\ 32\pi^2kX_0^{(1)} \end{pmatrix}, \tag{5.11b}$$

$$X_4=\begin{pmatrix} \frac{(-324h_3\pi^2k^3\rho-324h_4\pi^2k^3l)(1+u_0)+(64k^2-25k)X_0^{(1)}-9h_5k^2-9h_6\rho^2-9h_7l^2}{k} \\ 256\pi^2k^2X_0^{(1)}-64\pi^2kX_0^{(1)} \end{pmatrix}, \tag{5.11c}$$

$$X_1=X_3=0,\cdots. \tag{5.11d}$$

With the aid of the formula (5.2b), we have

$$\begin{aligned}\omega=&\frac{(-4h_3\pi^2k^3\rho-4h_4\pi^2k^3l)(1+u_0)-h_5k^2-h_6\rho^2-h_7l^2}{k}+8kX_0^{(1)}\wp^2\\&+\left(\frac{(-324h_3\pi^2k^3\rho-324h_4\pi^2k^3l)(1+u_0)+(64k^2-25k)X_0^{(1)}-9h_5k^2-9h_6\rho^2-9h_7l^2}{k}\right)\wp^4\\&+o(\wp^4),\end{aligned} \tag{5.12a}$$

$$c=32\pi^2kX_0^{(1)}\wp^2+(256\pi^2k^2X_0^{(1)}-64\pi^2kX_0^{(1)})\wp^4+o(\wp^4). \tag{5.12b}$$

With the relation (5.7), we can get

$$c\to 0,\quad 2\pi\mathrm{i}\omega\to h_3\mu^2\nu+h_4\mu^2\sigma-h_5\mu-h_6\mu^{-1}\nu^2-h_7\mu^{-1}\sigma^2,\quad \text{when}\ \ \wp\to 0. \tag{5.13}$$

In order to show that the one-periodic wave solution (4.5) degenerates to the one-soliton solution (3.1) under the limit $\wp\to 0$, we first write the periodic function $\vartheta(\xi)$ in the form

$$\vartheta(\xi,\tau)=1+\left(e^{2\pi\mathrm{i}\xi}+e^{-2\pi\mathrm{i}\xi}\right)\wp+\left(e^{4\pi\mathrm{i}\xi}+e^{-4\pi\mathrm{i}\xi}\right)\wp^4+\cdots. \tag{5.14}$$

With the transformation (5.7), we can obtain

$$\vartheta(\xi,\tau)=1+e^{\tilde{\xi}}+\left(e^{-\tilde{\xi}}+e^{2\tilde{\xi}}\right)\wp^2+\left(e^{-2\tilde{\xi}}+e^{3\tilde{\xi}}\right)\wp^6+\cdots\to 1+e^{\tilde{\xi}},\quad \text{when}\ \ \wp\to 0, \tag{5.15a}$$

$$\tilde{\xi}=2\pi\mathrm{i}\xi-\pi\tau=\mu x+\nu y+\sigma z+2\pi\mathrm{i}\omega t+\delta. \tag{5.15b}$$

From Eqs. (5.13) and (5.15), we can deduce that

$$\tilde{\xi} \to \mu x+\nu y+\sigma z+(h_3\mu^2\nu+h_4\mu^2\sigma-h_5\mu-h_6\mu^{-1}\nu^2-h_7\mu^{-1}\sigma^2)t+\delta=\eta, \quad \text{when } \wp\to 0, \tag{5.16a}$$

$$\xi \to \frac{\eta+\pi\tau}{2\pi \mathrm{i}}, \quad \text{when } \wp\to 0. \tag{5.16b}$$

Combining Eqs. (5.15) and (5.16), we further have

$$\vartheta(\xi)\to 1+e^{\eta}, \quad \text{when } \wp\to 0. \tag{5.17}$$

From above analysis, we conclude that when the amplitude $\wp\to 0$, the one-periodic wave solution (4.5) just tends to the one-soliton solution (3.1). □

5.2 Feature and asymptotic properties of two-periodic waves

The two-periodic wave solution (4.15) has a similar simple characterization.

(i) Its surface pattern is two-dimensional, i.e., there are two phase variables ξ_1 and ξ_2, which implies that the two-periodic wave admits two independent spatial periods in two independent horizontal directions.

(ii) It has $2N$ fundamental periods $\{\zeta_i, i=1,2,\cdots,N\}$ and $\{\tau_i, i=1,2,\cdots,N\}$ in (ξ_1,ξ_2) with $\zeta_1=(1,0,\cdots,0)^T,\cdots\zeta_N=(0,0,\cdots,1)^T$.

(iii) Assuming that k_i, ρ_i, l_i satisfy the following relationship

$$\frac{k_2}{k_1}=\frac{\rho_2}{\rho_1}=\frac{l_2}{l_1}=m \quad (m \text{ is a constant}), \tag{5.18}$$

we can get

$$\omega_2\sim m\omega_1, \quad \xi_2\sim m\xi_1, \quad \vartheta(\xi_1,\xi_2)\sim\vartheta(\xi_1,m\xi_1). \tag{5.19}$$

The two-periodic wave is actually one dimensional and it degenerates to the one-periodic wave (see Fig. 4).

(iv) If the parameters do not satisfy the relationship, i.e.,

$$\frac{k_2}{k_1}\neq\frac{\rho_2}{\rho_1}, \tag{5.20}$$

then for any time t, the phase variables $\xi_1=m_1$ and $\xi_2=m_2$ (m_1,m_2 are constants) intersect at a unique point. This point moves in the (x,y,z,t) plane with a constant speed as the time t changes. In Figs. 5 and 6, every two-periodic wave is spatially periodic in three directions, but it need not be periodic in either the x, y, z or t directions.

Finally, we study the asymptotic properties of the two-periodic wave solution (4.15). Similarly to Theorem 5.1, the relationship between the two-periodic wave solution (4.15) and the two-soliton solution (3.2) can be established as follows.

Theorem 5.2. *If* $(\omega_1,\omega_2,u_0,c)^T$ *is a solution of the system* (4.24) *and for the two-periodic wave solution* (4.15), *we take*

$$k_i=\frac{\mu_i}{2\pi i},\quad \rho_i=\frac{\nu_i}{2\pi i},\quad l_i=\frac{\sigma_i}{2\pi i},\quad \varepsilon_i=\frac{\delta_i-\pi i\tau_{ii}}{2\pi i},\quad \tau_{12}=\frac{A_{12}}{2\pi i},\quad i=1,2, \tag{5.21}$$

where μ_i, ν_i, σ_i, δ_i, $i=1,2$ *and* A_{12} *can be obtained from Eq.* (3.3), *then we have the following asymptotic relations*

$$u_0\to 0,\quad c\to 0,\quad \xi_i\to\frac{\eta_i-\pi i\tau_{ii}}{2\pi i},\quad i=1,2, \tag{5.22a}$$

$$\vartheta(\xi_1,\xi_2,\tau)\to 1+e^{\eta_1}+e^{\eta_2}+e^{\eta_1+\eta_1+A_{12}},\quad \textit{when}\ \ \wp_1,\wp_2\to 0. \tag{5.22b}$$

It means that the two-periodic wave solution (4.15) *tends to the two-soliton solution* (3.2) *under a small amplitude limit* $(u,\wp_1,\wp_2)\to(u_1,0,0)$.

Proof. The periodic wave function $\vartheta(\xi_1,\xi_2,\tau)$ is expanded in the form as follows

$$\begin{aligned}\vartheta(\xi_1,\xi_2,\tau)=&1+\left(e^{2\pi \mathrm{i}\xi_1}+e^{-2\pi \mathrm{i}\xi_1}\right)e^{\pi\tau_{11}}+\left(e^{2\pi \mathrm{i}\xi_2}+e^{-2\pi \mathrm{i}\xi_2}\right)e^{\pi\tau_{22}}+\left(e^{2\pi \mathrm{i}(\xi_1+\xi_2)}+e^{-2\pi \mathrm{i}(\xi_1+\xi_2)}\right)\\&\times e^{\pi(\tau_{11}+2\tau_{12}+\tau_{22})}+\cdots.\end{aligned} \tag{5.23}$$

From Eq. (5.21), we have

$$\begin{aligned}\vartheta(\xi_1,\xi_2,\tau)=&1+e^{\tilde{\xi}_1}+e^{\tilde{\xi}_2}+e^{\tilde{\xi}_1+\tilde{\xi}_2-2\pi\tau_{12}}+\wp_1^2e^{-\tilde{\xi}_1}+\wp_2^2e^{-\tilde{\xi}_2}+\wp_1^2\wp_2^2e^{-\tilde{\xi}_1^2-\tilde{\xi}_2-2\pi\tau_{12}}+\cdots\\&\to 1+e^{\tilde{\xi}_1}+e^{\tilde{\xi}_2}+e^{\tilde{\xi}_1+\tilde{\xi}_2+A_{12}}\quad \text{as}\ \ \wp_1,\wp_2\to 0,\end{aligned} \tag{5.24}$$

where $\tilde{\xi}_i=\mu_ix+\nu_iy+\sigma_iz+\tilde{\omega}_it+\delta_i$, $\tilde{\omega}_i=2\pi\mathrm{i}\omega_i$, $i=1,2$. It remains to prove that

$$c\to 0,\ \tilde{\omega}_i\to h_3\mu_i^2\nu_i+h_4\mu_i^2\sigma_i-h_5\mu_i-h_6\mu_i^{-1}\nu_i^2-h_7\mu_i^{-1}\sigma_i^2,\ \tilde{\xi}_i\to\eta_i,\ i=1,2,\ \text{as}\ \wp_1,\wp_2\to 0. \tag{5.25}$$

According to the way used for the one periodic wave, we can expand each function in $\{h_{ij},b_i,i=1,2,3,4\}$ into a series with $\wp_1$, $\wp_2$. The expansions for the matrix H, the vector b and the solution of the system (4.24) are given by

$$H=\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}+\begin{pmatrix}0&0&0&0\\-8\pi^2k_1&0&-32h_3\pi^4k_1^3\rho_1-32h_4\pi^4k_1^3l_1&2\\0&0&0&0\\0&0&0&0\end{pmatrix}\wp_1$$

$$
\begin{aligned}
&+\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & -8\pi^2 k_2 & -32h_3\pi^4k_2^3\rho_2-32h_4\pi^4k_2^3l_2 & 2\\ 0 & 0 & 0 & 0 \end{pmatrix}\wp_2\\
&+\begin{pmatrix} -32\pi^2 k_1 & 0 & -512\pi^4k_1^3(h_3\rho_1+h_4l_1) & 2\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}\wp_1^2\\
&+\begin{pmatrix} 0 & -32\pi^2 k_2 & -512\pi^4k_2^3(h_3\rho_2+h_4l_2) & 2\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}\wp_2^2\\
&+\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ \Delta_1 & -\Delta_1 & \Delta_2 & 2 \end{pmatrix}\wp_1\wp_2\\
&+\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ \Delta_3 & \Delta_3 & \Delta_4 & 2 \end{pmatrix}\wp_1\wp_2\wp_3+o(\wp_1^i\wp_2^j\wp_3^k),\quad i+j+k\geq 3,
\end{aligned}
\tag{5.26a}
$$

$$
\begin{aligned}
b=&\begin{pmatrix} 0\\ \Upsilon_1\\ 0\\ 0 \end{pmatrix}\wp_1+\begin{pmatrix} 0\\ 0\\ \Upsilon_2\\ 0 \end{pmatrix}\wp_2+\begin{pmatrix} \Upsilon_3\\ 0\\ 0\\ 0 \end{pmatrix}\wp_1^2+\begin{pmatrix} \Upsilon_4\\ 0\\ 0\\ 0 \end{pmatrix}\wp_2^2+\begin{pmatrix} 0\\ 0\\ 0\\ \Upsilon_5 \end{pmatrix}\wp_1\wp_2\\
&+\begin{pmatrix} 0\\ 0\\ 0\\ \Upsilon_6 \end{pmatrix}\wp_1\wp_2\wp_3+o(\wp_1^i\wp_2^j\wp_3^k),\quad i+j+k\geq 3,
\end{aligned}
\tag{5.26b}
$$

$$
\begin{aligned}
\begin{pmatrix} \omega_1\\ \omega_2\\ u_0\\ c \end{pmatrix}=&\begin{pmatrix} \omega_1^{(00)}\\ \omega_2^{(00)}\\ u_0^{(00)}\\ c^{(00)} \end{pmatrix}+\begin{pmatrix} \omega_1^{(11)}\\ \omega_2^{(11)}\\ u_0^{(11)}\\ c^{(11)} \end{pmatrix}\wp_1+\begin{pmatrix} \omega_1^{(21)}\\ \omega_2^{(21)}\\ u_0^{(21)}\\ c^{(21)} \end{pmatrix}\wp_2+\begin{pmatrix} \omega_1^{(12)}\\ \omega_2^{(12)}\\ u_0^{(12)}\\ c^{(12)} \end{pmatrix}\wp_1^2+\begin{pmatrix} \omega_1^{(22)}\\ \omega_2^{(22)}\\ u_0^{(22)}\\ c^{(22)} \end{pmatrix}\wp_2^2\\
&+\begin{pmatrix} \omega_1^{(2)}\\ \omega_2^{(2)}\\ u_0^{(2)}\\ c^{(2)} \end{pmatrix}\wp_1\wp_2+\begin{pmatrix} \omega_1^{(3)}\\ \omega_2^{(3)}\\ u_0^{(3)}\\ c^{(3)} \end{pmatrix}\wp_1\wp_2\wp_3+o(\wp_1^i\wp_2^j\wp_3^k),\quad i+j+k\geq 3,
\end{aligned}
\tag{5.26c}
$$

where Δ_i and Υ_j are, respectively, given by

$$\Delta_1=-8\pi^2(k_1-k_2),\quad \Delta_2=-32h_3\pi^4(k_1-k_2)^3(\rho_1-\rho_2)-32h_4\pi^4(k_1-k_2)^3(l_1-l_2), \tag{5.27a}$$
$$\Delta_3=-8\pi^2(k_1+k_2),\quad \Delta_4=-32h_3\pi^4(k_1+k_2)^3(\rho_1+\rho_2)-32h_4\pi^4(k_1+k_2)^3(l_1+l_2), \tag{5.27b}$$
$$\Upsilon_1=32h_3\pi^4k_1^3\rho_1+32h_4\pi^4k_1^3l_1+8h_5\pi^2k_1^2+8h_6\pi^2\rho_1^2+8h_7\pi^2l_1^2, \tag{5.27c}$$
$$\Upsilon_2=32h_3\pi^4k_2^3\rho_2+32h_4\pi^4k_2^3l_2+8h_5\pi^2k_2^2+8h_6\pi^2\rho_2^2+8h_7\pi^2l_2^2, \tag{5.27d}$$
$$\Upsilon_3=512h_3\pi^4k_1^3\rho_1+512h_4\pi^4k_1^3l_1+32h_5\pi^2k_1^2+32h_6\pi^2\rho_1^2+32h_7\pi^2l_1^2, \tag{5.27e}$$
$$\Upsilon_4=512h_3\pi^4k_2^3\rho_2+512h_4\pi^4k_2^3l_2+32h_5\pi^2k_2^2+32h_6\pi^2\rho_2^2+32h_7\pi^2l_2^2, \tag{5.27f}$$
$$\begin{aligned}\Upsilon_5=&32h_3\pi^4(k_1-k_2)^3(\rho_1-\rho_2)+32h_4\pi^4(k_1-k_2)^3(l_1-l_2)+8h_5\pi^2(k_1-k_2)^2\\&+8h_6\pi^2(\rho_1-\rho_2)^2+8h_7\pi^2(l_1-l_2)^2,\end{aligned} \tag{5.27g}$$
$$\begin{aligned}\Upsilon_6=&32h_3\pi^4(k_1+k_2)^3(\rho_1+\rho_2)+32h_4\pi^4(k_1+k_2)^3(l_1+l_2)+8h_5\pi^2(k_1+k_2)^2\\&+8h_6\pi^2(\rho_1+\rho_2)^2+8h_7\pi^2(l_1+l_2)^2.\end{aligned} \tag{5.27h}$$

Substituting systems (5.26a)-(5.26c) into system (4.24) and comparing the same order of $\wp_1$, $\wp_2$ and $\wp_3$, we have some relationships as follows:

$$c^{(00)}=c^{(11)}=c^{(21)}=c^{(2)}=c^{(3)}=0, \tag{5.28a}$$
$$c^{(12)}-32\pi^2k_1\omega_1^{(00)}+(-512h_3\pi^4k_1^3\rho_1-512h_4\pi^4k_1^3l_1)u_0^{(00)}=\Upsilon_3, \tag{5.28b}$$
$$-8\pi^2k_1\omega_1^{(11)}+(-32h_3\pi^4k_1^3\rho_1-32h_4\pi^4k_1^3l_1)u_0^{(11)}=0, \tag{5.28c}$$
$$-8\pi^2k_2\omega_2^{(00)}+(-32h_3\pi^4k_2^3\rho_2-32h_4\pi^4k_2^3l_2)u_0^{(00)}=\Upsilon_2, \tag{5.28d}$$
$$8\pi^2k_2\omega_2^{(21)}+(-32h_3\pi^4k_2^3\rho_2-32h_4\pi^4k_2^3l_2)u_0^{(21)}=0, \tag{5.28e}$$
$$-8\pi^2k_1\omega_1^{(00)}+(-32h_3\pi^4k_1^3\rho_1-32h_4\pi^4k_1^3l_1)u_0^{(00)}=\Upsilon_1, \tag{5.28f}$$
$$c^{(22)}-32\pi^2k_2\omega_2^{(00)}+(-512h_3\pi^4k_2^3\rho_2-512h_4\pi^4k_2^3l_2)u_0^{(00)}=\Upsilon_4, \tag{5.28g}$$
$$-8\pi^2k_1\omega_1^{(21)}+(-32h_3\pi^4k_1^3\rho_1-32h_4\pi^4k_1^3l_1)u_0^{(21)}=0, \tag{5.28h}$$
$$-8\pi^2k_2\omega_2^{(11)}+(-32h_3\pi^4k_2^3\rho_2-32h_4\pi^4k_2^3l_2)u_0^{(11)}=0, \tag{5.28i}$$
$$\Delta_1\omega_1^{(00)}-\Delta_1\omega_2^{(00)}+\Delta_2u_0^{(00)}=\Upsilon_5,\quad \Delta_3\omega_1^{(00)}+\Delta_3\omega_2^{(00)}+\Delta_4u_0^{(00)}=\Upsilon_6. \tag{5.28j}$$

Taking $u_0^{(00)}=0$, we can show that

$$u_0=o(\wp_1,\wp_2)\to 0, \tag{5.29a}$$
$$\begin{aligned}c=&\left[16\pi^2k_1(\Upsilon_5\Delta_1^{-1}+\Upsilon_6\Delta_3)^{-1})+\Upsilon_3\right]\wp_1^2\\&+\left[16\pi^2k_2(\Upsilon_6\Delta_3)^{-1}-\Upsilon_5\Delta_1^{-1})+\Upsilon_4\right]\wp_2^2+o(\wp_1\wp_2)\to 0,\end{aligned} \tag{5.29b}$$

$$
\begin{aligned}
2\pi i\omega_1 = & -\frac{i}{k_1}\left(8h_3\pi^3k_1^2\rho_1+8h_4\pi^3k_1^2l_1+2h_5\pi k_1^2+2h_6\pi\rho_1^2+2h_7\pi l_1^2\right)+o(\wp_1\wp_2) \\
& \rightarrow h_3\mu_1^2\nu_1+h_4\mu_1^2\sigma_1-h_5\mu_1-h_6\mu_1^{-1}\nu_1^2-h_7\mu_1^{-1}\sigma_1^2, \qquad (5.29c) \\
2\pi i\omega_2 = & -\frac{i}{k_2}\left(8h_3\pi^3k_2^2\rho_2+8h_4\pi^3k_2^2l_2+2h_5\pi k_2^2+2h_6\pi\rho_2^2+2h_7\pi l_2^2\right)+o(\wp_1\wp_2) \\
& \rightarrow h_3\mu_2^2\nu_2+h_4\mu_2^2\sigma_2-h_5\mu_2-h_6\mu_2^{-1}\nu_2^2-h_7\mu_2^{-1}\sigma_2^2, \quad \text{as } (\wp_1,\wp_2)\rightarrow(0,0). \qquad (5.29d)
\end{aligned}
$$

From the above argument, we can draw the conclusion that the two-periodic wave solution (4.15) tends to the two-soliton solution (3.2) as $(\wp_1,\wp_2)\rightarrow(0,0)$. $\square$

6 Conclusions and remarks

In this paper, by virtue of binary Bell polynomials, Eq. (1.1) has been systematically investigated, which could be used to describe many nonlinear phenomena in plasma physics. We have obtained a Hirota bilinear form, soliton solutions and qusi-periodic wave solutions. Moreover, the relationships between the presented qusi-periodic wave solutions and soliton solutions were strictly established in detail. We have discussed the asymptotic properties of the one- and two- qusi-periodic wave solutions and verified that one- and two- qusi-periodic wave solutions tend to the one- and two-soliton solutions respectively as the amplitude $\wp\rightarrow 0$.

Based on the above results, we conclude that:

(i) With the help of binary Bell polynomials, a bilinear form (2.13) has been obtained for Eq. (1.1).

(ii) In virtue of the Hirota bilinear method and the multidimensional Riemann theta function, we have got the one- and two-soliton solutions and one- and two-periodic wave solutions [see Solutions (3.1), (3.2), (4.5) and (4.15)] of Eq. (1.1) and given the graphical analysis. The figures of soliton solutions were presented in Fig. 1 and Fig. 2. The analogues of periodic wave solutions were presented in Figs. 3-6. Furthermore, the asymptotic behaviors of one- and two- qusi-periodic wave solutions were investigated, respectively. It is of interest that we have provided the relationships between the qusi-periodic wave solutions and the soliton solutions by two theorems with the strict proofs in details.

(iii) The presented analysis is very helpful for us to do further studies on nonlinear problems in the fields of mathematical physics and engineering.

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