Int J Theor Phys (2012) 51:999–1006
DOI 10.1007/s10773-011-0972-y

Painlevé Integrability and Complexiton-Like Solutions of a Coupled Higgs Model

Zhenyun Qin · Gui Mu · Wenxiu Ma

Received: 6 July 2011 / Accepted: 29 September 2011 / Published online: 15 October 2011

Abstract We perform the Painlevé test for a coupled Higgs system to determine its Painlevé integrability. Moreover, a class of exact complexiton-like solutions, including breather solutions and dark and bright solitary solutions, is explicitly constructed for the coupled Higgs model by using a generalized Hirota's bilinear form.

Keywords Coupled Higgs system · Painlevé analysis · Hirota's bilinear form · Solitary wave solution · Complexiton solution

1 Introduction

Since the soliton concept was introduced by Zabusky and Kruskal in 1965 [1], a great number of integrable systems have been discovered in the natural and applied sciences [2–19]. Integrable systems exhibit richness and variety of exact solutions such as soliton solutions, periodic solutions, rational solutions and complexiton solutions (see, e.g., [20, 21]). Hirota bilinear equations can even possess linear subspaces of their solution spaces [19]. This also shows that the multiple exp-function can explore diverse exact solutions to nonlinear equations [18]. Moreover, recently Kummer functions and methods are introduced to explore diverse exact solutions for nonlinear models by Prof. Dai et al. [22–25].

Z. Qin (✉) · W. Ma
School of Mathematics and LMNS, Fudan University, Shanghai 200433, P.R. China
e-mail: zyqin@fudan.edu.cn

G. Mu
College of Mathematics and Information Science, Qujing Normal University, Qujing 655011, P.R. China

W. Ma
Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620-5700, USA

In this letter, we study a coupled Higgs model which describes a system of conserved scalar nucleons interacting with neutral scalar mesons:

$$\begin{aligned} &u_{tt} - u_{xx} - \alpha u + \beta u|u|^2 - 2uv = 0, \\ &v_{tt} + v_{xx} - \beta(|u|^2)_{xx} = 0, \end{aligned} \tag{1}$$

where $u = u(x,t)$, $v = v(x,t)$. This model reduces to the so-called coupled nonlinear Klein-Gordonmodelin the case of $\alpha < 0$ and $\beta < 0$, and the so-called coupled Higgs field system in the case of $\alpha > 0$ and $\beta > 0$. For the coupled Higgs system (1), Tajiri obtained an N-soliton solution [26] and subsequently Hu got a homoclinic orbit solution using Hirota's bilinear method [27]. Here we concentrate on the Painlevé integrability and exact solutions of breather type and dark and bright solitary wave type to the coupled Higgs model (1).

The primary purpose of the letter is to explore integrability of the coupled Higgs model (1) by the Painlevé test and to construct its solitary wave solutions under the help of a generalized Hirota's bilinear form. In Sect. 2, we carry out the Painlevé analysis to determine when the coupled Higgs model (1) is integrable. In Sect. 3, we construct a class of complexiton-like solutions by Hirota's direct method and plot some of the presented solutions for the coupled Higgs model (1). The resulting solutions contain breathers and dark and bright solitary solutions. A few remarks are given in the final section.

2 Painlevé Integrability

In this section, we explore the Painlevé integrability of the coupled Higgs model (1). In order to make a Painlevé analysis, we define $p = \bar{u}$, where the bar represents the complex conjugate, and rewrite the above coupled Higgs model as

$$\begin{aligned} &u_{tt} - u_{xx} - \alpha u + \beta u^2 p - 2uv = 0, \\ &p_{tt} - p_{xx} - \alpha p + \beta u p^2 - 2pv = 0, \\ &v_{tt} + v_{xx} - \beta(up)_{xx} = 0, \end{aligned} \tag{2}$$

We begin with the following Laurent series for u, v, p:

$$u = \sum_{j=0}^{\infty} u_j \phi^{(j+\alpha_u)}, \qquad p = \sum_{j=0}^{\infty} p_j \phi^{(j+\alpha_p)}, \qquad v = \sum_{j=0}^{\infty} v_j \phi^{(j+\alpha_v)} \tag{3}$$

with a sufficient number of arbitrary functions among u_j, p_j, v_j in addition to ϕ. Moreover, the leading orders of $\alpha_u, \alpha_p, \alpha_v$ should be negative integers.

If we replace

$$u = u_0 \phi^{\alpha_u}, \qquad p = p_0 \phi^{\alpha_p}, \qquad v = v_0 \phi^{\alpha_v} \tag{4}$$

in (3), a balance of the dominant terms determines that

$$\alpha_u = \alpha_p = -1, \quad \alpha_v = -2.$$

Substituting

$$u = u_0 \phi^{-1}, \qquad p = p_0 \phi^{-1}, \qquad v = v_0 \phi^{-2}$$

into (3), and collecting the coefficients of $(\phi^{-3}, \phi^{-3}, \phi^{-4})$, we get

$$\begin{aligned} &u_0(2\phi_t^2 - 2\phi_x^2 + \beta u_0 p_0 - 2v_0) = 0, \\ &p_0(2\phi_t^2 - 2\phi_x^2 + \beta u_0 p_0 - 2q_0) = 0, \\ &6v_0\phi_t^2 + v_0\phi_x^2 - \beta v_0 u_0 \phi_x^2 = 0, \end{aligned} \tag{5}$$

which yields

$$v_0 = -2\phi_x^2,$$
$$u_0 = \frac{-2(\phi_t^2 + \phi_x^2)}{\beta p_0}.$$

Afterwards, inserting

$$u = \frac{u_0}{\phi} + u_j\phi^{j-1}, \qquad p = \frac{p_0}{\phi} + p_j\phi^{j-1}, \qquad v = \frac{w_0}{\phi^2} + v_j\phi^{j-2}$$

into (3), we find that the resonances appear at $j = 0, 2, 3, 3, 4$.

Let us check the resonance conditions at non-negative resonant points $j = 0, 2, 3, 3, 4$. The series (3) are truncated at $j = 4$. In order to make computation simpler, we adopt Kruskal's ansatz $\phi(x, t) = t + \psi(x)$.

At the level $j = 1$, the values of u_1, p_1, w_1 can be obtained explicitly by collecting the coefficients of $(\phi^{-2}, \phi^{-2}, \phi^{-3})$

$$\begin{aligned} u_1 = {} & \frac{1}{3\beta p_0^2(1-\phi_x^2)}(-2\beta p_0^2 u_{0t} - 2\beta p_0^2 u_{0x}\psi_x - \beta p_0^2 u_0 \psi_{xx} - 2p_{0t} - 2p_{0t}\psi_x^2 \\ & + 2\psi_x p_{0x} + 2p_{0x}\psi_x^3 + p_0\psi_{xx} + p_0\psi_{xx}\psi_x^2 + 4p_0 v_{0t} - 4\beta p_0 u_0 \psi_x p_{0x} + 4p_0 w_{0t}\psi_x \\ & + 2p_0 v_0 \psi_{xx}), \\ p_1 = {} & \frac{1}{12(1-\psi_x^4)}(-2\beta p_0^2 u_{0t} + 10\beta p_0^2 u_{0x}\psi_x + 5\beta p_0^2 u_0\psi_{xx} - 8p_{0t} - 8p_{0t}\psi_x^2 s \\ & + 8\psi_x p_{0x} + 8p_{0x}\psi_x^3 + 4p_0\psi_{xx} + 4p_0\psi_{xx}\psi_x^2 - 8p_0 v_{0t} + 8\beta p_0 u_0 \phi_x p_{0x} \\ & - 8p_0 v_{0x}\psi_x - 4p_0 v_0 \psi_{xx}), \\ w_1 = {} & -\frac{1}{6p_0(1-\psi_x^4)}(4\psi_x^3 p_{0x} + 2p_0\psi_{xx}\psi_x^2 + \beta p_0^2 u_0 \psi_x^2 \psi_{xx} + 4\psi_x^5 p_{0x} - 4\psi_x^2 p_{0t} \\ & - 4\psi_x^4 p_{0t} - 12p_0 v_{0t} - 4p_0\psi_x^3 v_{0x} + 2\beta p_0^2 \psi_x^3 u_{0x} - 12p_0 v_{0x}\psi_x - 6p_0 v_0 \psi_{xx} \\ & - 4p_0\psi_x^2 v_{0t} - 2p_0 v_0 \psi_x^2 \psi_{xx} + 2p_0^2\beta\psi_x^2 u_{0t} + 12\beta p_0^2 u_{0x}\psi_x + 6\beta p_0^2 u_0 \psi_{xx} \\ & + 12\beta p_0 p_{0x} u_0 \psi_x + 4\beta u_0 p_0 p_{0x}\psi_x^3 + 2p_0\psi_x^4\psi_{xx}). \end{aligned} \tag{6}$$

Similarly, at the level $j=2$, the values of u_2, p_2 can be obtained explicitly by collecting the coefficients of $(\phi^{-1},\phi^{-1},\phi^{-2})$

$$\begin{aligned} u_2 &= \frac{1}{4v_0^2-8v_0\beta u_0 p_0+3\beta^2 p_0^2 u_0^2}(\beta^2 p_1^2 u_0^3 - 2\beta^2 u_1 p_1 p_0 u_0^2 - 2\beta^2 u_1^2 p_0^2 u_0 + 2\beta u_0^2 p_0 v_2 \\ &\quad + \beta u_0^2 \alpha p_0 + \beta u_0^2 p_{0tt} - \beta u_0^2 p_{0xx} - 2\beta u_0^2 p_1 v_1 + 2\beta u_0 p_0 u_{0xx} + 4\beta u_0 p_0 u_1 v_1 \\ &\quad - 2\beta u_0 p_0 u_{0tt} + 4\beta u_0 u_1 p_1 v_0 + 2\beta u_1^2 v_0 p_0 - 4u_0 v_2 v_0 - 2u_0 \alpha v_0 - 2u_{0xx} v_0 \\ &\quad + 2u_{0tt} v_0 - 4u_1 v_1 v_0), \\ p_2 &= -\frac{1}{4v_0^2-8v_0\beta u_0 p_0+3\beta^2 p_0^2 u_0^2}(2\alpha p_0 v_0 - \beta p_0^2 u_{0tt} + 4p_1 v_1 v_0 + \beta p_0^2 u_{0xx} \\ &\quad + 2p_{0xx} v_0 - 2p_{0xx}\beta u_0 p_0 + 2p_{0tt}\beta u_0 p_0 - \alpha p_0^2 \beta u_0 - 2p_0^2 v_2 \beta u_0 - 2p_{0tt} v_0 \\ &\quad - \beta^2 p_0^3 u_1^2 + 4p_0 v_2 v_0 - 4p_1 v_1 \beta u_0 p_0 + 2\beta p_0^2 u_1 v_1 - 2\beta u_0 p_1^2 v_0 + 2\beta^2 u_0^2 p_1^2 p_0 \\ &\quad + 2\beta^2 p_0^2 u_0 u_1 p_1 - 4\beta u_1 p_0 p_1 v_0), \end{aligned} \tag{7}$$

with v_2 being arbitrary. This corresponds to the resonance at $j=2$.

The resonance condition at $j=3$ equivalently requires that

$$\begin{aligned} u_3 &= -\frac{1}{2(-v_0-\psi_x^2+1+\beta u_0 p_0)}(-2u_{2x}\psi_x + 2\beta u_0 u_1 p_2 - 2u_2 v_1 - \alpha u_1 + 2u_{2t} + u_{1tt} \\ &\quad + 2\beta u_0 u_2 p_1 + \beta u_1^2 p_1 + \beta u_0^2 p_3 - u_{1xx} + 2\beta u_1 u_2 p_0 - 2u_0 v_3 - 2u_1 v_2 \\ &\quad - u_2 \psi_{xx}), \end{aligned} \tag{8}$$

but p_3 and v_3 are two arbitrary functions. This corresponds to the resonance at $j=3,3$.

Under the conditions (6)–(8), the Painlevé test passes at the resonant point $j=4$. Therefore, this case does not present new resonance condition.

Now, to sum up, under the conditions (6)–(8), the coupled Higgs model (1) is integrable in Painlevé sense.

3 Complexiton-Like Solutions

In this section, we devote our efforts to constructing new complexiton-like solutions to the coupled Higgs model (1). To do this, we will introduce a suitable ansätz [15] by the aid of a generalized bilinear form.

Through the Painlevé series truncation, we obtain the dependent variable transformation

$$u = \frac{G}{F}, \qquad v = 2(\ln F)_{xx}, \quad F \textit{ real}. \tag{9}$$

In this case, the coupled Higgs model (1) is transformed into the bilinear form

$$(D_t^2 - D_x^2 + A - \alpha)G\cdot F = 0, \tag{10}$$

$$(D_t^2 + D_x^2 + A)F\cdot F - \beta GG^* = 0. \tag{11}$$

where G^* is the conjugate function of $G(x,t)$, A is an integration constant and the D-operator is defined by

$$D_x^m D_t^n f(x,t)\cdot g(x,t)=\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^m\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^n[f(x,t)g(x',t')]|_{x'=x,t'=t}.$$

When $A=0$, the above bilinear form can be reduced to Tajiri's form [26]. A class of complexiton-like solutions of (1) is sought as follows

$$G(x,t)=e^{-i(bx+at)}(\mathrm{e}^{-k_1(x-\omega t)}+b_1\cos[k(\omega x+t)]+b_2\mathrm{e}^{k_1(x-\omega t)}), \tag{12}$$

$$F(x,t)=\mathrm{e}^{-k_1(x-\omega t)}+b_3\cos[k(\omega x+t)]+b_4 e^{k_1(x-\omega t)}, \tag{13}$$

where $a,b,k,k_1,\omega,c_2,b_3,b_4$ are real and b_1,b_2 are complex.

Substituting (12)–(13) into (10)–(11) yields algebraic equations of

$$e^{jk_1(x-\omega t)},\ e^{jk_1(x-\omega t)}\cos[k(\omega x+t)],\ e^{jk_1(x-\omega t)}\sin[k(\omega x+t)]$$

for $j=-2,-1,0,1,2$. Collecting and equating all coefficients in front of these basic functions to zero, we obtain the following relations among the parameters:

$$\begin{aligned}&A=\beta,\quad a^2=\beta-\alpha+b^2,\\&b_1P^*=b_3P,\quad b_2P^{*2}=b_4P^2,\\&\Delta_1=k_1^2\Delta_2,\quad \Delta_3=\Delta_4k^2,\quad \Delta_5=b_4\Delta_6,\end{aligned} \tag{14}$$

where

$$\begin{aligned}&P=a-b\omega+2ik_1\omega,\quad \Delta_1=(\omega b-a)\Delta,\quad \Delta_2=4\omega^2(b\omega+a)(\omega^2+1)^2,\\&\Delta_3=\Delta\left(b\omega^3+3a\omega^2+3b\omega+a\right),\quad \Delta_4=4\omega^2(a+b\omega)(1-\omega^2)\left(\omega^2+1\right)^2,\\&\Delta_5={b_3}^2[(\omega b-a)^2+4{k_1}^2\omega^2]\left(b\omega^3+3a\omega^2+3b\omega+a\right),\\&\Delta_6=4[(-b^2+4{k_1}^2)\omega^4-2\omega^3ab+(-3b^2+3a^2-4{k_1}^2)\omega^2+2ab\omega\\&\qquad+a^2](a-b\omega),\\&\Delta=[-\omega^6b^2+\left(a^2-4\beta-2b^2\right)\omega^4+\left(2a^2+4\beta-b^2\right)\omega^2+a^2].\end{aligned}$$

As a result, inserting (13)–(14) into (10) and employing the above relations (15) among the parameters, explicit exact solutions of the coupled Higgs model (1) are obtained from (10)–(14). More precisely, the resulting solutions can be classified into the following three cases according to the sign of b_4.

Case I—$b_4>0$: The complexiton-like solution is given by

$$\begin{aligned}&u=\frac{e^{-i(bx+at)}(\mathrm{e}^{-k_1(x-\omega t)}+b_1\cos[k(\omega x+t)]+b_2\mathrm{e}^{k_1(x-\omega t)})}{\mathrm{e}^{-k_1(x-\omega t)}+b_3\cos[k(\omega x+t)]+b_4\mathrm{e}^{k_1(x-\omega t)}},\\&v=\frac{2\delta_1\sqrt{b_4}\cos[k(\omega x+t]\cosh(\xi)+4k_1b_3k\omega\sqrt{b_4}\sin[k(\omega x+t)]\sinh(\xi)+\delta_2}{\left(2\sqrt{b_4}\cosh(\xi)+b_3\cos[k(\omega x+t)]\right)^2},\end{aligned} \tag{15}$$

where

$$\xi = -k_1 x + k_1 \omega t - \frac{\ln(b_4)}{2}, \delta_1 = -k_1{}^2 b_3 + b_3 k^2 \omega^2, \delta_2 = -4k_1{}^2 b_4 + b_3{}^2 k^2 \omega^2.$$

Case II—$b_4 < 0$: The complexiton-like solution appears

$$u = \frac{e^{-i(bx+at)}(\mathrm{e}^{-k_1(x-\omega t)} + b_1 \cos[k(\omega x + t)] + b_2 \mathrm{e}^{k_1(x\omega t)})}{\mathrm{e}^{-k_1(x-\omega t)}} + b_3 \cos[k(\omega x + t)] + b_4 \mathrm{e}^{k_1(x-\omega t)},$$

$$v = \frac{2\delta_1 \sqrt{-b_4} \cos[k(\omega x + t)] \sinh(\xi) + 4k_1 b_3 k\omega \sqrt{-b_4} \sin[k(\omega x + t)] \cosh(\xi) + \delta_2}{\left(2\sqrt{-b_4} \sinh(\xi) + b_3 \cos[k(\omega x + t)]\right)^2}, \tag{16}$$

where

$$\xi = -k_1 x + k_1 \omega t + \frac{\ln(-b_4)}{2}, \quad \delta_1 = -k_1{}^2 b_3 + b_3 k^2 \omega^2, \quad \delta_2 = -4k_1{}^2 b_4 + b_3{}^2 k^2 \omega^2.$$

Case III—$b_4 = 0$: In this case, we have $b_2 = 0$, and hence the derived complexiton-like solution is

$$u = \frac{e^{-i(bx+at)}(\mathrm{e}^{-k_1(x-\omega t)} + b_1 \cos[k(\omega x + t)])}{\mathrm{e}^{-k_1(x-\omega t)} + b_3 \cos[k(\omega x + t)]},$$

$$v = \frac{b_3[\delta e^{-k_1(x-\omega t)}(\cos[k(\omega x + t)] + 2k_1 k\omega \sin[k(\omega x + t)]) + b_3 k^2 \omega^2]}{\left(e^{-k_1(x-\omega t)} + b_3 \cos[k(\omega x + t)]\right)^2}, \tag{17}$$

where $\delta = -k_1{}^2 + k^2\omega^2$. In what follows, the newly derived complexiton-like solutions are illustrated in figures. Figure 1 depicts that solitary waves produce breather behavior, Fig. 2 shows that complexiton-like solutions contain rich soliton structure, for example, dark solitons and bright solitons.

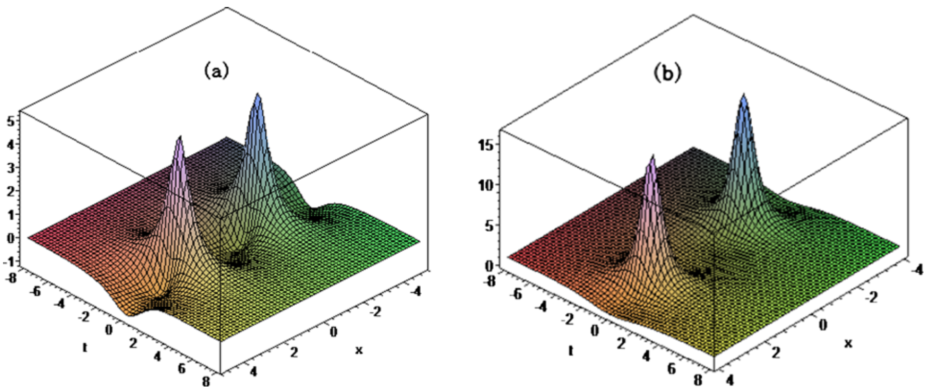

Fig. 1 The structure of breather solitary wave given by (16) with the parameters $\omega = 10$, $\beta = 1$, $a = 1$, $\alpha = 0.0001$, $b = 0.01$, $b_3 = 1$. (**a**) For v (**b**) For $|u|^2$

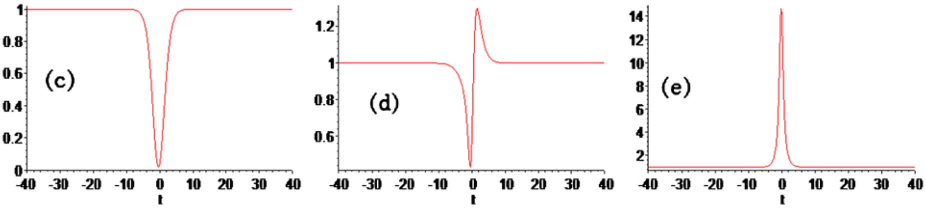

Fig. 2 (**c**)–(**e**) show the different interesting profiles in t-direction corresponding to Fig. 1(b), (**c**) $x = 0$ (**d**) $x = 1.2$ (**e**) $x = 2$

4 Discussions

Based on the presented figures, the derived complexiton-like solutions is likely to possess good stability. The stability problem of the coupled Higgs model will be analyzed in a forthcoming publication.

Complexiton-like solutions contain exponential functions and periodic functions, and they bring solitons, positions and breathers. However, it is still unclear if the newly derived complexiton-like solutions possess the elastic interaction property in the time dependence as solitons.

Also, some asymptotic behaviors of the obtained solutions can be found. Without loss of generality, we assume that $b_4 > 0$ and $k_1\omega < 0$, and obtain from (16)

$$(u, v) \to (e^{-i(bx+at)}, 0), \quad t \to \infty.$$

Noting that $P^2 = P^{*2}e^{i\theta}$, we obtain

$$(u, v) \to (e^{-i(bx+at-\theta)}, 0), \quad t \to -\infty.$$

Acknowledgements The work was supported in part by the State Administration of Foreign Experts Affairs of China, the National Natural Science Foundation of China (No. 10801037, No. 11061028, 61072147 and 11071159), the New Teacher Grant of Ministry of Education of China (No. 20080246), the Young Teachers Foundation (No. 1411018) of Fudan university, the Natural Science Foundation of Shanghai and the Shanghai Leading Academic Discipline Project (No. J50101) and Qujing Normal University Young Teacher Fundation No. 2010QN018.

References

1. Zabusky, N.Z., Kruskal, M.D.: Phys. Rev. Lett. B **15**, 240 (1965)
2. Ablowitz, M.J., Clarkson, P.A.: Soliton, Nonlinear Evolution Equations and Inverse Scattering. Cambridge University Press, Cambridge (1999)
3. Weiss, J., Tabor, M., Carnevale, G.: J. Math. Phys. B **24**, 522 (1983)
4. Weiss, J.: J. Math. Phys. B **24**, 14053 (1983)
5. Musette, M., Conter, R.: J. Math. Phys. B **32**, 1450 (1991)
6. Ma, W.X., Fudan Univ, J.: Natural Science E **33**, 319 (1994)
7. Lou, S.Y.: Phys. Rev. Lett. B **80**, 5027 (1998)
8. Lou, S.Y.: J. Math. Phys. B **39**, 5364 (1998)
9. Conte, R. (ed.): The Painlevé Property, One Century Later. Springer, New York (1999)
10. Zhang, S.L., Wu, B., Lou, S.Y.: Phys Lett A B **300**, 40 (2002)
11. Bekir, A.: Chaos, Soliton. Fract. B **32**, 449 (2007)
12. Ramani, A., Grammaticos, B., Bountis, T.: Phys. Rep. B **180**, 159 (1989)
13. Brugarino, T., Sciacca, M.: Opt. Commun. B **262**, 250 (2006)
14. Qin, Z.Y.: J. Phys. Soc. Jpn. B **76**, 124004 (2007)

15. Dai, Z.D., Li, S.L., Dai, Q.Y.: Chaos, Soliton. Fract. B **4**, 1148 (2007)
16. Ma, W.X., Lee, J.-H.: Chaos, Soliton. Fract. B **42**, 1356 (2009)
17. Ma, W.X., Chen, M.: Appl. Math. Comput. B **215**, 2835 (2009)
18. Ma, W.X., Huang, T.W., Zhang, Y.: Phys. Scripta B **82**, 065003 (2010)
19. Ma, W.X., Fan, E.G.: Comput. Math. Appl. B **61**, 950 (2011)
20. Ma, W.X.: Phys. Lett. A B **301**, 35 (2002)
21. Ma, W.X., You, Y.: Trans. Amer. Math. Soc. B **357**, 1753 (2005)
22. Wang, Y.Y., Dai, C.Q., Zhang, J.F.: Commun. Theor. Phys. B **51**, 81 (2009)
23. Dai, C.Q., Yang, Q., He, J.D., Wang, Y.Y.: Eur. Phys. J. D B **63**, 141 (2011)
24. Wu, L., Li, L., Zhang, J.F., Dumitru, M., Malomed, B.A., Liu, W.M.: Phys. Rev. A B **81**, 061805 (2010)
25. Dai, C.Q., Chen, R.P., Zhou, G.Q.: J. Phys. B : At. Mol. Opt. Phys. B **44**, 145401 (2011)
26. Tajiri, M.: J. Phys. Soc. Jpn. B **52**, 2277 (1983)
27. Hu, X.B., Guo, B.L., Tam, H.W.: J. Phys. Soc. Jpn. B **72**, 189 (2003)