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An Extended (G'/G)-expansion Method with Applications to Nonlinear Evolution Equations*

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Abstract

In this paper, an extended (G'/G)-expansion method with symbolic computation is generalized to construct a series of exact solutions of nonlinear evolution equation. Compared with classical (G'/G)-expansion method, the proposed method not only gives new a more general solutions, but also can be implemented on a computer with help of symbolic computation software. The efficiency of the method can be demonstrated on a large variety of nonlinear equations such as those considered in this paper, Ramani equation and coupled Ramani equation.

Keywords: NLEE equation; Extended (G'/G)-expansion method; The exact wave solutions.

Introduction

Seeking exact solutions of nonlinear evolution equations (NLEEs) is of important significance in soliton theory and has become one of the most extremely active areas of research investigation. In particular, the traveling wave solutions can provide much physical information to the physical problems and help one to understand the mechanism that governs these physical models. In the past several decades, many effective methods for obtaining exact solutions of NLEEs have been developed, such as inverse scattering method [1]-[2], bilinear transformation [3]-[6], the tanh-sech method [7]-[9], extended tanh method [10]-[12], improved tanh-function method [13], sine-cosine method [14]-[15], homogeneous balance method [16], Exp-function method [17]-[18] and (G'/G)-expansion method [19]. Among those, the (G'/G)-expansion method introduced by Wang provides a straightforward and effective algorithm to obtain such particular solutions for a large nonlinear equation [19]. In recent years, the (G'/G)-expansion method has been widely used by many authors [20]-[21].

The purpose of this paper is to establish exact wave solutions of two NLEE equations involving parameters by using the extended (G'/G)-expansion method. These solutions are reported here for the first time. Our paper is organized as follows. In Sec. 2, a methodology of the generalized (G'/G)-expansion method is presented. In Sec. 3, as applications of the proposed method, Ramani equation and coupled Ramini equation are presented. A conclusion is given in section 4.

2. The extended (G'/G)-expansion method

Let we outline the method. For a given nonlinear evolution equation, say in two independent variables x and t

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$$P(u,u_t,u_x,u_{tt},u_{xx},u_{xt},...)=0$$

where $u=u(x,t)$ is an unknown function, P is a polynomial interms of u and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. The main steps of the extended (G'/G)-expansion method is as follows:

Step 1:
Make the traveling wave transformation

$$u(x,t)=u(\xi),\xi=x-ct,$$

where V is a constant to be determined latter. Then, the NLEE (1) is reduced to the following nonlinear ordinary differential equation (NLODE) for $u(\xi)$

$$P(u,-cu',u',c^2u'',u'',-cu'',...)=0 \quad (1)$$

Step 2:
Assume that the NLODE (1) has the following solution

$$u(\xi)=a_0+\sum_{i=1}^{n}\left\{a_i\left(\frac{G'}{G}\right)^i+ +b_i\left(\frac{G'}{G}\right)^{i-1}\sqrt{\sigma[1+\frac{1}{\mu}\left(\frac{G'}{G}\right)^2]}\right\} \quad (2)$$

where $a_0,a_i,b_i(i=1,...,n)$ are constants to be determined latter, $\sigma=\pm1$, n is a positive integer, $G=G(\xi)$ satisfies the following second order LODE:

$$G''+\mu G=0\ , \quad (3)$$

where μ is a constant and $\mu\neq0$.

Step 3:
Determine the positive integer n by balancing the highest order derivatives and nonlinear terms in Eq. (2).

Step 4:
Substituting (2) along with Eq. (3) into Eq.(1) and collecting all the terms with the same order of $(G'/G)^k$ and $(G'/G)^{k-1}\sqrt{\sigma[1+(G'/G)^2/\mu]}$. Vanishing all the coefficients of this polynomial, yields a set of over-determined algebraic equations for $c,a_0,a_i,b_i(i=1,...,n)$.

Step 5:
Assuming that the constants $c,a_0,a_i,b_i(i=1,...,n)$ can be obtained by solving the algebraic equations in Step 4, then substituting these constants and the known general solutions of Eq. (3) into (2), we can obtain the explicit solutions of Eq. (1) immediately.

3. Applications

Example 1:
Consider the nonlinear Ramani equation [22]-[23]

$$u_{tt}+3u_{xt}u_{xx}+u_{xxxt}=0, \quad (4)$$

First using the wave variable

$$u(x,t)=U(\xi),\xi=x-ct, \quad (5)$$

Eq. (5) is carried to an ODE

$$c^2U''-3c(U'')^2-cU^{(4)}=0, \quad (6)$$

Further setting $v=U''$ results in

$$c^2v-3cv^2-cv''=0, \quad (7)$$

Balancing the highest order derivatives and the highest order nonlinear term gives n=2.
Assume Eq. (5) has the following solutions

$$v(\xi)=a_0+a_2\left(\frac{G'}{G}\right)^2+ +b_2\left(\frac{G'}{G}\right)^1\sqrt{\sigma[1+\frac{1}{\mu}\left(\frac{G'}{G}\right)^2]} \quad (8)$$

where G satisfies (4), $\sigma=\pm1$, c,a_0,a_2,b_2 are constants to be determined later.

Using (4), substituting Eq.(8) into Eq. (7)

and equating the coefficients of $(G'/G)^{k}$ and $(G'/G)^{k-1}\sqrt{\sigma[1+(G'/G)^2/\mu]}$ to zero, we obtain a set of over-determined algebraic equations for c, a_0, a_2, b_2

$$-6cb_2(a_2+1)=0,$$
$$-cb_2(6a_0-c+5\mu)=0,$$
$$\frac{-3c(a_2^2\mu+b_2^2\sigma+2a_2\mu)}{\mu}=0,$$
$$-c(-ca_2+6a_0a_2+8a_2\mu+3b_2^2\sigma)=0,$$
$$-c(-ca_0+2a_2\mu^2+3a_0^2)=0.$$

To solve the above equations by using Maple, we get the following results including the following different cases

Case 1.

$$a_0=-\frac{2\mu}{3}, a_2=-1, c=\mu, b_2=\pm\sqrt{\frac{\mu}{\sigma}}, \tag{9}$$

Case 2.

$$a_0=-\mu, a_2=-1, c=-\mu, b_2=\pm\sqrt{\frac{\mu}{\sigma}}. \tag{10}$$

Inserting (9) into (8) gives rise to

$$v_1(\xi)=-\frac{2\mu}{3}-\left(\frac{G'}{G}\right)^2\pm\sqrt{\frac{\mu}{\sigma}}\left(\frac{G'}{G}\right)\sqrt{\sigma[1+\frac{1}{\mu}\left(\frac{G'}{G}\right)^2} \tag{11}$$

where $\xi=x-\mu t$.

Similarly, Inserting (10) into (9) recasts as

$$v_2(\xi)=-\mu-\left(\frac{G'}{G}\right)^2\pm\sqrt{\frac{\mu}{\sigma}}\left(\frac{G'}{G}\right)\sqrt{\sigma[1+\frac{1}{\mu}\left(\frac{G'}{G}\right)^2}],$$

where $\xi=x+\mu t$.

Because of $v=U''$, through integrating twice, we have

$$U_1(\xi)=-\frac{\mu}{6}\xi^2+A_1\xi+A_2+\frac{1}{2}\ln[\mu+\left(\frac{G'}{G}\right)^2]$$
$$\pm\ln|\left(\frac{G'}{G}\right)+\sqrt{\left(\frac{G'}{G}\right)^2+\mu}\,|,$$

where $\xi=x-\mu t$, A_1, A_2 are arbitrary constants.

and

$$U_2(\xi)=A_1\xi+A_2+\frac{1}{2}\ln[\mu+\left(\frac{G'}{G}\right)^2]$$
$$-\ln|\left(\frac{G'}{G}\right)+\sqrt{\left(\frac{G'}{G}\right)^2+\mu}\,|,$$

where $\xi=x+\mu t$, A_1, A_2 are arbitrary constants.

By using the ODE (3), the exact solutions are listed as follows:

(1) when $\mu>0$

$$U_1(\xi)=-\frac{\mu}{6}\xi^2+A_1\xi+A_2+\frac{1}{2}\ln[\mu$$
$$+\mu(\frac{C_1\cos\sqrt{\mu}\xi-C_2\sin\sqrt{\mu}\xi}{C_1\sin\sqrt{\mu}\xi+C_2\cos\sqrt{\mu}\xi})^2]$$
$$\pm\ln|(\sqrt{\mu}\frac{C_1\cos\sqrt{\mu}\xi-C_2\sin\sqrt{\mu}\xi}{C_1\sin\sqrt{\mu}\xi+C_2\cos\sqrt{\mu}\xi})$$
$$+\sqrt{\mu(\frac{C_1\cos\sqrt{\mu}\xi-C_2\sin\sqrt{\mu}\xi}{C_1\sin\sqrt{\mu}\xi+C_2\cos\sqrt{\mu}\xi})^2+\mu}\,|,$$

Where $\xi=x-\mu t$, A_1, A_2 are arbitrary constants.

$$U_2(\xi)=A_1\xi+A_2+\frac{1}{2}\ln[\mu$$
$$+\mu(\frac{C_1\cos\sqrt{\mu}\xi-C_2\sin\sqrt{\mu}\xi}{C_1\sin\sqrt{\mu}\xi+C_2\cos\sqrt{\mu}\xi})^2]$$
$$\pm\ln|(\sqrt{\mu}\frac{C_1\cos\sqrt{\mu}\xi-C_2\sin\sqrt{\mu}\xi}{C_1\sin\sqrt{\mu}\xi+C_2\cos\sqrt{\mu}\xi})$$

$$+\sqrt{\mu(\frac{C_1\cos\sqrt{\mu}\xi - C_2\sin\sqrt{\mu}\xi}{C_1\sin\sqrt{\mu}\xi + C_2\cos\sqrt{\mu}\xi})^2 + \mu}\,|,$$

where $\xi = x + \mu t$, A_1, A_2 are arbitrary constants.

(2) when $\mu < 0$

$$U_1(\xi) = -\frac{\mu}{6}\xi^2 + A_1\xi + A_2 + \frac{1}{2}\ln[\mu - \mu(\frac{C_1\sinh\sqrt{-\mu}\xi + C_2\cosh\sqrt{-\mu}\xi}{C_1\cosh\sqrt{-\mu}\xi + C_2\sinh\sqrt{-\mu}\xi})^2]$$

$$\pm\ln|(-\sqrt{-\mu}(\frac{C_1\sinh\sqrt{-\mu}\xi + C_2\cosh\sqrt{-\mu}\xi}{C_1\cosh\sqrt{-\mu}\xi + C_2\sinh\sqrt{-\mu}\xi})$$

$$+\sqrt{-\mu(\frac{C_1\sinh\sqrt{-\mu}\xi + C_2\cosh\sqrt{-\mu}\xi}{C_1\cosh\sqrt{-\mu}\xi + C_2\sinh\sqrt{-\mu}\xi})^2 + \mu}\,|,$$

where $\xi = x - \mu t$, A_1, A_2 are arbitrary constants.

$$U_2(\xi) = A_1\xi + A_2 + \frac{1}{2}\ln[\mu - \mu(\frac{C_1\sinh\sqrt{-\mu}\xi + C_2\cosh\sqrt{-\mu}\xi}{C_1\cosh\sqrt{-\mu}\xi + C_2\sinh\sqrt{-\mu}\xi})^2]$$

$$-\ln|(-\sqrt{-\mu}(\frac{C_1\sinh\sqrt{-\mu}\xi + C_2\cosh\sqrt{-\mu}\xi}{C_1\cosh\sqrt{-\mu}\xi + C_2\sinh\sqrt{-\mu}\xi})$$

$$+\sqrt{-\mu(\frac{C_1\sinh\sqrt{-\mu}\xi + C_2\cosh\sqrt{-\mu}\xi}{C_1\cosh\sqrt{-\mu}\xi + C_2\sinh\sqrt{-\mu}\xi})^2 + \mu}\,|,$$

where $\xi = x + \mu t,$ A_1, A_2 are arbitrary constants.

Example 2:

Consider the coupled Ramani equation

$$u_{xt} + v_x v_t = 0,$$

$$v_t + v_{xxx} + v_x^3 + 3v_x u_{xx} = 0, \quad (11)$$

First using the wave variable

$$u(x,t) = U(\xi), v(x,t) = V(\xi),$$

(12)

Where $\xi = x - ct.$

Substituting (12) into (11) gives rise to

$$-cU'' - cV'^2 = 0,$$

(13)

$$-cV' + V''' + V'^3 + 3V'U'' = 0. \quad (14)$$

From (13) we have $U'' = -V'^2$, inserting it into (14) is

$$-cV' + V''' - 2V'^3 = 0, \quad (15)$$

introducing a new variable H and setting $H = V'$ is carried to

$$-cH - 2H^3 + H'' = 0. \quad (16)$$

Balancing the highest order derivatives and the highest order nonlinear result in n=1.

Setting the solutions have the form

$$H(\xi) = a_0 + a_1\left(\frac{G'}{G}\right)^2 + b_1\sqrt{\sigma[1 + \frac{1}{\mu}\left(\frac{G'}{G}\right)^2]}$$

,(17)

where G satisfies (3) , $\sigma = \pm 1$, c, a_0, a_1, b_1 are constants to be determined later.

Using (3), substituting Eq.(17) into Eq.(16) and equating the coefficients of $\left(\frac{G'}{G}\right)^k$ and $\left(\frac{G'}{G}\right)^{k-1}\sqrt{\sigma[1 + \frac{1}{\mu}\left(\frac{G'}{G}\right)^2]}$ to zero, we obtain a set of over-determined algebraic equations for c, a_0, a_1, b_1

$$-b_1(c + 6a_0^2 - \mu + 2\sigma b_1^2) = 0,$$

$$-12b_1a_0a_1=0,$$
$$\frac{-2b_1(-\mu+3\mu a_1^2+\sigma b_1^2)}{\mu}=0,$$
$$-\frac{2a_1(-\mu+\mu a_1^2+3\sigma b_1^2)}{\mu}=0,$$
$$-\frac{6a_0(\sigma b_1^2+\mu a_1^2)}{\mu}=0,$$
$$-a_1(c+6a_0^2-2\mu+6\sigma b_1^2)=0,$$
$$-a_0(c+6\sigma b_1^2+2a_0^2)=0.$$

To solve the above equations by using Maple, we get the following results including the following different cases

$$a_0=0, a_1=\pm\frac{1}{2}, b_1=\pm\frac{1}{2}\sqrt{\frac{\mu}{\sigma}}, c=\frac{1}{2}\mu, \quad (18)$$

Inserting (18) into (17) gives rise to

$$H(\xi)=\pm\frac{1}{2}\left(\frac{G'}{G}\right)\pm\frac{1}{2}\sqrt{\sigma[1+\frac{1}{\mu}\left(\frac{G'}{G}\right)^2]},$$

where $\xi=x-\frac{1}{2}\mu t$.

By integrating H once, V is

$$V(\xi)=\pm\frac{1}{4}\ln[\mu+\left(\frac{G'}{G}\right)^2]\pm\frac{1}{2}\ln|\left(\frac{G'}{G}\right)+\sqrt{\left(\frac{G'}{G}\right)^2+\mu}|,$$

$$\xi=x-\frac{1}{2}\mu t.$$

Then because of $-H^2=U''$, $U(\xi)$ is

$$U(\xi)=A_1\xi+A_2+\frac{1}{4}\ln[\mu+\left(\frac{G'}{G}\right)^2]-\frac{1}{4}\mu\xi^2\pm\frac{1}{2}\ln|\left(\frac{G'}{G}\right)+\sqrt{\left(\frac{G'}{G}\right)^2+\mu}|,$$

By using the ODE (3), the exact solutions are listed as follows

(1) when $\mu>0$, we get

$$U(\xi)=A_1\xi+A_2+\frac{1}{4}\ln(\mu+\mu(\frac{C_1\cos\sqrt{\mu}\xi-C_2\sin\sqrt{\mu}\xi}{C_1\sin\sqrt{\mu}\xi+C_2\cos\sqrt{\mu}\xi})^2)$$
$$-\frac{1}{4}\mu\xi^2\mp\frac{1}{2}\ln|\sqrt{\mu}(\frac{C_1\cos\sqrt{\mu}\xi-C_2\sin\sqrt{\mu}\xi}{C_1\sin\sqrt{\mu}\xi+C_2\cos\sqrt{\mu}\xi})$$
$$+\sqrt{\mu(\frac{C_1\cos\sqrt{\mu}\xi-C_2\sin\sqrt{\mu}\xi}{C_1\sin\sqrt{\mu}\xi+C_2\cos\sqrt{\mu}\xi})^2+\mu}|,$$
$$V(\xi)=\pm\frac{1}{4}\ln(\mu+\mu(\frac{C_1\cos\sqrt{\mu}\xi-C_2\sin\sqrt{\mu}\xi}{C_1\sin\sqrt{\mu}\xi+C_2\cos\sqrt{\mu}\xi})^2)$$
$$\pm\frac{1}{2}\ln|\sqrt{\mu}(\frac{C_1\cos\sqrt{\mu}\xi-C_2\sin\sqrt{\mu}\xi}{C_1\sin\sqrt{\mu}\xi+C_2\cos\sqrt{\mu}\xi})+\sqrt{\mu(\frac{C_1\cos\sqrt{\mu}\xi-C_2\sin\sqrt{\mu}\xi}{C_1\sin\sqrt{\mu}\xi+C_2\cos\sqrt{\mu}\xi})^2+\mu}|,$$

Where $\xi = x - \frac{1}{2}\mu t$, A_1, A_2, C_1, C_2 are arbitrary constants

(2) When $\mu < 0$, we have

$$U(\xi) = A_1\xi + A_2 + \frac{1}{4}\ln(\mu + \mu(\frac{C_1\cos\sqrt{\mu}\xi - C_2\sin\sqrt{\mu}\xi}{C_1\sin\sqrt{\mu}\xi + C_2\cos\sqrt{\mu}\xi})^2)$$

$$-\frac{1}{4}\mu\xi^2 \mp \frac{1}{2}\ln|\sqrt{\mu}(\frac{C_1\cos\sqrt{\mu}\xi - C_2\sin\sqrt{\mu}\xi}{C_1\sin\sqrt{\mu}\xi + C_2\cos\sqrt{\mu}\xi}) + \sqrt{-\mu(\frac{C_1\sinh\sqrt{-\mu}\xi + C_2\cosh\sqrt{-\mu}\xi}{C_1\cosh\sqrt{-\mu}\xi + C_2\sinh\sqrt{-\mu}\xi})^2 + \mu}\,|,$$

$$\mp\frac{1}{2}\ln|-\sqrt{-\mu}(\frac{C_1\sinh\sqrt{-\mu}\xi + C_2\cosh\sqrt{-\mu}\xi}{C_1\cosh\sqrt{-\mu}\xi + C_2\sinh\sqrt{-\mu}\xi})$$

$$V(\xi) = \pm\frac{1}{4}\ln(\mu - \mu(\frac{C_1\sinh\sqrt{-\mu}\xi + C_2\cosh\sqrt{-\mu}\xi}{C_1\cosh\sqrt{-\mu}\xi + C_2\sinh\sqrt{-\mu}\xi})^2)$$

$$\pm\frac{1}{2}\ln|-\sqrt{-\mu}(\frac{C_1\sinh\sqrt{-\mu}\xi + C_2\cosh\sqrt{-\mu}\xi}{C_1\cosh\sqrt{-\mu}\xi + C_2\sinh\sqrt{-\mu}\xi})$$

$$+\sqrt{-\mu(\frac{C_1\sinh\sqrt{-\mu}\xi + C_2\cosh\sqrt{-\mu}\xi}{C_1\cosh\sqrt{-\mu}\xi + C_2\sinh\sqrt{-\mu}\xi})^2 + \mu}\,|,$$

Where $\xi = x - \frac{1}{2}\mu t$, A_1, A_2 are arbitrary constants.

Remark 1
All solutions in this paper have been checked with Maple by putting them back into the original NLEE equation and the number of arbitrary constants has been reduced to a minimum.

Remark 2
To the best of our knowledge, exact wave solutions of Ramani equation (4) and the coupled Ramani equation (11) obtained in this paper have not been reported in previous literature.

4. Conclusion

In this work the extended (G'/G)-expansion method is proposed to seek a series of exact solution of NLEE equation. Being concise and straightforward, this approach is to establish the periodic wave solutions of NLEE equation involving parameters. The proposed method is effectively employed and more powerful than the classical (G'/G)-method. What is more, the method is also a direct and computerizable method, which allows us to solve complicated and tedious algebraic calculation. The method can also be extended to solve some nonlinear evolution equations with variable coefficients in physics and it can be generalized further by introducing new generalized ansäze equation. This is our task in the future.

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