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Construction of lump-type and interaction solutions to an extended (3 + 1)-dimensional Jimbo–Miwa-like equation

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We present lump-type solutions and interaction solutions to an extended (3 + 1)-dimensional Jimbo–Miwa-like equation. Three classes of lump-type solutions are obtained by the Hirota bilinear method. Interaction solutions are among lump-type solutions, two kink waves and periodic waves, and between two kink waves and a periodic wave are computed. Dynamical characters of the obtained solutions are graphically exhibited. These wave solutions enrich the dynamical theory of higher-dimensional nonlinear dispersive wave equations.

Keywords: Lump-type solution; interaction solution; kink solution; periodic solution.

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1. Introduction

Nonlinear evolution equations model varieties of physical phenomena.[1–14] Recently, Wazwaz presented two extended $(3+1)$-dimensional Jimbo–Miwa equations[15–28]:

$$u_{xxxy} + 3u_x u_{xy} + 3u_y u_{xx} + 2u_{yt} - 3(u_{xz} + u_{yz} + u_{zz}) = 0\,, \tag{1}$$

and

$$u_{xxxy} + 3u_x u_{xy} + 3u_y u_{xx} - 3u_{yt} + 2(u_{xt} + u_{yt} + u_{zt}) = 0\,, \tag{2}$$

which describe certain $(3+1)$-dimensional waves in plasmas and optics.[15] Here, $u(x, y, z, t)$ is a function of the spatial variables x, y, z and the temporal variable t, and the subscripts denote the corresponding partial derivatives.[16] Exact solitary wave solutions,[29] multiple soliton solutions,[16,30] meromorphic exact solutions,[31] resonant multi-soliton solutions,[32] periodic solitary wave solutions,[33] lump solutions and the lump–kink solutions[34] of Eqs. (1) and (2) have been obtained by various methods.[35]

Under the transformation $u = 2(\ln F)_x$, a Hirota bilinear form for Eq. (1) reads

$$(D_x^3 D_y + 2D_t D_y - 3D_x D_z - 3D_y D_z - 3D_z^2)F \cdot F = 0\,. \tag{3}$$

Recently, a generalization of the bilinear differential operators has been presented as[36]

$$\begin{aligned} D_{p,x_1}^{n_1} \ldots D_{p,x_M}^{n_M}(f \cdot g) = \prod_{i=1}^{M} \left(\frac{\partial}{\partial x_i} + \alpha \frac{\partial}{\partial x_i'} \right)^{n_i} \\ \times f(x_1, \ldots, x_M) g(x_1', \ldots, x_M')|_{x_1' = x_1, \ldots, x_M' = x_M}\,, \end{aligned} \tag{4}$$

in which $n_1, \ldots, n_M$ are nonnegative integers, with an integer m, $\alpha^m = (-1)^{r(m)}$, if $m \equiv r(m)$ with $0 \le r(m) < p$. Taking $p = 3$, we have

$$\alpha_3 = -1, \quad \alpha_3^2 = \alpha_3^3 = 1, \quad \alpha_3^4 = -1, \quad \alpha_3^5 = \alpha_3^6 = 1\,,$$

and we can generalize the bilinear Eq. (3) into

$$\begin{aligned} &(D_{3,x}^3 D_{3,y} + 2D_{3,t} D_{3,y} - 3D_{3,x} D_{3,z} - 3D_{3,y} D_{3,z} - 3D_{3,z}^2)F \cdot F \\ &\quad = 2(3\,F_{xx} F_{xy} + 2F_{yt} F - 2F_y F_t - 3F_{xz} F + 3F_x F_z \\ &\qquad - 3F_{yz} F + 3F_y F_z - 3F_{zz} F + 3F_z^2) = 0\,, \end{aligned} \tag{5}$$

which is equivalently linked with an extended $(3+1)$-dimensional Jimbo–Miwa-like equation

$$\begin{aligned} &2u_{yt} + \frac{3}{8} u^3 u_y + \frac{9}{4} u u_y u_x + \frac{3}{4} u^2 u_{xy} + \frac{3}{2} u_x u_{xy} + \frac{3}{2} u_y u_{xx} \\ &\quad - 3\,(u_{xz} + u_{yz} + u_{zz}) + \frac{9}{8} v u^2 u_x + \frac{3}{4} v u_x^2 + \frac{3}{4} v u u_{xx} = 0\,, \end{aligned} \tag{6}$$

where $u_y = v_x$.

In this paper, we will consider Eq. (6). This paper will be structured as follows. In Sec. 2, we will present three classes of lump-type solutions of Eq. (6) through a generalized Hirota bilinear method. In Sec. 3, we will construct and analyze some interaction solutions. In Sec. 4, dynamical properties of the above presented solutions will be graphically discussed. Conclusions will be provided in Sec. 5.

2. Lump-Type Solutions

To search for lump-type solutions of Eq. (6) (see, for example, Refs. 37–45 for other examples), we begin by assuming

$$\begin{cases} F = G^2 + H^2 + a_{11}, \quad G = a_1x + a_2y + a_3z + a_4t + a_5\,, \\ H = a_6x + a_7y + a_8z + a_9t + a_{10}\,, \end{cases} \tag{7}$$

where $a_i (1 \leq i \leq 11)$ are real parameters to be determined. Substituting such a function F in Eqs. (7) into the bilinear Eq. (5), we can, with Maple symbolic computation, get

Case 1.

$$\begin{cases} a_4 = \dfrac{\begin{array}{l}3(a_3a_2^2 + a_3^2a_2 - a_8^2a_2 + a_1a_3a_2 - a_6a_8a_2 + a_3a_7^2 \\ \quad + a_3a_6a_7 + a_1a_7a_8 + 2a_3a_7a_8)\end{array}}{2(a_2^2 + a_7^2)}\,, \\ a_9 = \dfrac{\begin{array}{l}3(a_8a_2^2 + a_3a_6a_2 + a_1a_8a_2 + 2a_3a_8a_2 + a_7a_8^2 - a_3^2a_7 \\ \quad - a_1a_3a_7 + a_7^2a_8 + a_6a_7a_8)\end{array}}{2(a_2^2 + a_7^2)}\,, \\ a_{11} = \dfrac{(a_1^2 + a_6^2)(a_1a_2 + a_6a_7)(a_2^2 + a_7^2)}{(a_3a_7 - a_2a_8)[(a_1 + a_3)a_7 - a_2(a_6 + a_8)]}\,, \end{cases} \tag{8}$$

where the involved constants should satisfy $(a_2^2 + a_7^2)(a_3a_7 - a_2a_8)[(a_1 + a_3)a_7 - a_2(a_6 + a_8)] \neq 0$.

Case 2.

$$\begin{aligned} a_2 &= \frac{a_3a_7}{a_8}, \quad a_4 = \frac{3(a_3a_7 + a_1a_8 + a_3a_8)}{2a_7}\,, \\ a_8 &= -\frac{a_1a_3}{a_6}, \quad a_9 = \frac{3a_8(a_6 + a_7 + a_8)}{2a_7}\,, \end{aligned} \tag{9}$$

where $a_6a_7a_8 \neq 0$.

Case 3.

$$\begin{cases} a_2 = \dfrac{a_3^2 a_7 - a_7 a_8^2}{2a_3 a_8}\,, \\ a_4 = \dfrac{3a_3(2a_8^3 + 2a_6 a_8^2 + a_7 a_8^2 + 2a_3^2 a_8 + 2a_1 a_3 a_8 + a_3^2 a_7)}{2a_7(a_3^2 + a_8^2)}\,, \\ a_9 = \dfrac{3a_8(2a_6 a_3^2 + a_7 a_3^2 - 2a_1 a_8 a_3 + a_7 a_8^2)}{2a_7(a_3^2 + a_8^2)}\,, \\ a_{11} = \dfrac{(a_1^2 + a_6^2)a_7(a_3^2 + a_8^2)(a_1 a_3^2 + 2a_6 a_8 a_3 - a_1 a_8^2)}{2a_3 a_8^2(a_8^3 + a_6 a_8^2 + a_3^2 a_8 + 2a_1 a_3 a_8 - a_3^2 a_6)}\,, \end{cases} \tag{10}$$

where $a_3 a_7 a_8(a_8^3 + a_6 a_8^2 + a_3^2 a_8 + 2a_1 a_3 a_8 - a_3^2 a_6) \neq 0$.

All the above cases of parameter selections result in three classes of solutions to Eq. (7); and then through the transformation $u = 2(\ln F)_x$ we get three classes of lump-type solutions of Eq. (6), which can be presented as

$$u = \frac{4a_1(a_4 t + a_1 x + a_2 y + a_3 z + a_5) + 4a_6(a_9 t + a_6 x + a_7 y + a_8 z + a_{10})}{(a_4 t + a_1 x + a_2 y + a_3 z + a_5)^2 + (a_9 t + a_6 x + a_7 y + a_8 z + a_{10})^2 + a_{11}}, \tag{11}$$

in which it is sufficient for u to be analytic if $a_{11} > 0$. For example, choosing exact values of parameters for the three cases above, we can get three kinds of lump-type solutions separately.

(1) Taking $a_1 = 1, a_2 = 2, a_3 = 1, a_5 = 1, a_6 = 1, a_7 = 1, a_8 = 1$ and $a_{10} = 1$ for Case 1, we can get

$$u = \frac{2(4x + 6y + 4z + \frac{66t}{5} + 4)}{(x + y + z + \frac{39t}{10} + 1)^2 + (x + 2y + z + \frac{27t}{10} + 1)^2 + 15}\,; \tag{12}$$

(2) Taking $a_1 = 1, a_3 = 1, a_5 = 1, a_6 = 1, a_7 = 1, a_{10} = 1$ and $a_{11} = 1$ for Case 2, we can get

$$u = \frac{8x - 12t + 8}{(x + y - z - \frac{3t}{2} + 1)^2 + (x - y + z - \frac{3t}{2} + 1)^2 + 15}\,; \tag{13}$$

(3) Taking $a_1 = 1, a_3 = 1, a_5 = 1, a_6 = 1, a_7 = 1, a_8 = 1$ and $a_{10} = 1$ for Case 3, we can get

$$u = \frac{4(2x + y + 2z + 9t + 2)}{(x + y + z + \frac{3t}{2} + 1)^2 + (x + z + \frac{15t}{2} + 1)^2 + 1}\,. \tag{14}$$

3. Interaction Solutions

First, in order to construct interaction solutions (see, for example, Refs. 41, 46, 47 and Refs. 48–52 for other examples of nonlinear PDEs and linear PDEs, respectively), we assume that

$$\begin{cases} F = G^2 + H^2 + L + b_{11}, \quad G = b_1 x + b_2 y + b_3 z + b_4 t + b_5, \\ H = b_6 x + b_7 y + b_8 z + b_9 t + b_{10}\,, \\ L = v_1 e^{k_1 x + k_2 y + k_3 z + k_4 t + k_5} + v_2 e^{-k_1 x - k_2 y - k_3 z - k_4 t - k_5} \\ \qquad + v_3 \cos(k_6 x + k_7 y + k_8 z + k_9 t + k_{10})\,, \end{cases} \tag{15}$$

where $b_i (1 \leq i \leq 11)$, $k_j (1 \leq j \leq 10)$, v_1, v_2 and v_3 are all real parameters to be determined. Substituting such a function F by Eqs. (15) into the bilinear Eq. (5), we can, through Maple, obtain

Case 1.

$$\begin{cases} b_4 = \dfrac{3b_3(b_1 + b_2 + b_3)}{2b_2}, \quad b_6 = -\dfrac{b_1 b_2}{b_7}, \quad b_8 = \dfrac{b_3 b_7^2 + b_1(b_2^2 + b_7^2)}{b_2 b_7}\,, \\ k_1 = 0, \quad k_3 = \dfrac{(b_1 + b_3)k_2}{b_2}\,, \\ b_9 = \dfrac{3(b_2^2 + b_7^2)b_1^2 + 3(b_2^3 + b_3 b_2^2 + b_7^2 b_2 + 2b_3 b_7^2)b_1 + 3b_3(b_2 + b_3)b_7^2}{2b_2^2 b_7}\,, \\ k_8 = \dfrac{(b_1 + b_3)k_7}{b_2}, \quad k_4 = \dfrac{3(b_1 + b_3)(b_1 + b_2 + b_3)k_2}{2b_2^2}\,, \\ k_6 = 0, \quad k_9 = \dfrac{3(b_1 + b_3)(b_1 + b_2 + b_3)k_7}{2b_2^2}\,, \end{cases} \tag{16}$$

which needs to satisfy $b_2 b_7 \neq 0$, to make F to be analytic.

Case 2.

$$\begin{cases} b_2 = 0, \quad b_3 = -b_1, \quad b_4 = \dfrac{3}{2}k_1(k_6^2 - 1), \quad b_7 = 0, \quad b_8 = -b_6\,, \\ b_9 = -\dfrac{3b_6}{2}, \quad k_2 = 0\,, \\ k_3 = -k_1, \quad k_4 = \dfrac{3}{2}k_1(k_6^2 - 1), \quad k_7 = 0, \quad k_8 = -k_6, \quad b_9 = -\dfrac{3k_6}{2}\,. \end{cases} \tag{17}$$

With $u = 2(\ln F)_x$, a kind of interaction solutions of Eq. (6) can be presented as

$$\begin{aligned} u = {} & [4b_1(b_1 x + b_2 y + b_3 z + b_4 t + b_5) + 4b_6(b_6 x + b_7 y + b_8 z + b_9 t + b_{10}) \\ & + 2v_1 k_1 e^{k_1 x + k_2 y + k_3 z + k_4 t + k_5} - 2k_1 v_2 e^{-k_1 x - k_2 y - k_3 z - k_4 t - k_5} \\ & - k_6 v_3 \sin(k_6 x + k_7 y + k_8 z + k_9 t + k_{10})] \times [(b_1 x + b_2 y + b_3 z + b_4 t + b_5)^2 \\ & + (b_9 t + b_6 x + b_7 y + b_8 z + b_{10})^2 + b_{11} + v_1 e^{k_1 x + k_2 y + k_3 z + k_4 t + k_5} \\ & + v_2 e^{-k_1 x - k_2 y - k_3 z - k_4 t - k_5} + v_3 \cos(k_7 y + k_8 z + k_9 t + k_{10})]^{-1}\,. \end{aligned} \tag{18}$$

Taking Case 2 as an example, by selecting the parameters of Eq. (18) specially as: $b_3 = 1, b_5 = 1, b_6 = 1, b_{10} = 1, b_{11} = 1, k_1 = 1, k_5 = 1, k_6 = 1$ and $k_{10} = 1$, we can obtain the interaction solutions among the lump, kink wave and periodic wave

$$u = \frac{2[-v_3 \sin(x - z - \frac{3t}{2} + 1) - 3t + v_1 e^{x-z+1} - v_2 e^{-x+z-1} + 4x - 4z + 4]}{\begin{array}{l} v_3 \cos(x - z - \frac{3t}{2} + 1) + (x - z - \frac{3t}{2} + 1)^2 + v_1 e^{x-z+1} \\ \quad + v_2 e^{-x+z-1} + (x - z + 1)^2 + 1 \end{array}}, \tag{19}$$

and if further assuming

(1) $v_1 = 1, v_2 = 1$ and $v_3 = 0$, we can obtain the interaction solution between the lump and two kink waves of Eq. (6)

$$u = \frac{2(e^{x-z+1} - e^{-x+z-1} + 4x - 4z - 3t + 4)}{(x - z - \frac{3t}{2} + 1)^2 + (x - z + 1)^2 + e^{x-z+1} + e^{-x+z-1} + 1}; \tag{20}$$

(2) $v_1 = 1, v_2 = 0$ and $v_3 = 0$, we can obtain the mixed lump–kink wave solution of Eq. (6)

$$u = \frac{2(e^{x-z+1} + 4x - 4z - 3t + 4)}{(x - z - \frac{3t}{2} + 1)^2 + (x - z + 1)^2 + e^{x-z+1} + 1}; \tag{21}$$

(3) $v_1 = 0, v_2 = 0$ and $v_3 = 1$, we can obtain the interaction solution between a lump and a periodic wave of Eq. (6)

$$u = \frac{2[-\sin(x - z - \frac{3t}{2} + 1) + 4x - 4z - 3t + 4]}{(x - z - \frac{3t}{2} + 1)^2 + \cos(x - z - \frac{3t}{2} + 1) + (x - z + 1)^2 + 1}. \tag{22}$$

Second, for the purpose of constructing periodic wave solutions,[19,23,53,54] we assume

$$\begin{aligned} F = {} & m_1 e^{l_1 x + l_2 y + l_3 z + l_4 t + l_5} + m_2 e^{-l_1 x - l_2 y - l_3 z - l_4 t - l_5} \\ & + m_3 \cos(l_6 x + l_7 y + l_8 z + l_9 t + l_{10}), \end{aligned} \tag{23}$$

in which $l_j (1 \leq j \leq 10)$, m_1, m_2 and m_3 are all real parameters to be determined. Substituting such a function F by Eq. (23) into the bilinear Eq. (5), we can, through Maple, obtain

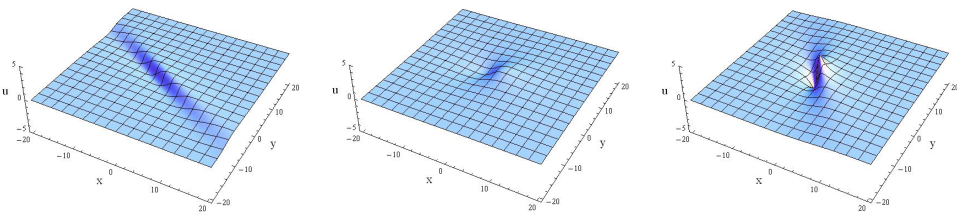
$$l_2 = 0, \quad l_3 = -l_1, \quad l_7 = 0, \quad l_8 = -l_6, \tag{24}$$

and then through the transformation $u = 2(\ln F)_x$ we get periodic wave solutions of Eq. (6)

$$u = \frac{2[l_1 m_1 e^{l_1 x - l_1 z + l_4 t + l_5} - l_1 m_2 e^{-l_1 x + l_1 z - l_4 t - l_5} - l_6 m_3 \sin(l_6 x - l_6 z + l_9 t + l_{10})]}{m_1 e^{l_1 x - l_1 z + l_4 t + l_5} + m_2 e^{-l_1 x + l_1 z - l_4 t - l_5} + m_3 \cos(l_6 x - l_6 z + l_9 t + l_{10})}. \tag{25}$$

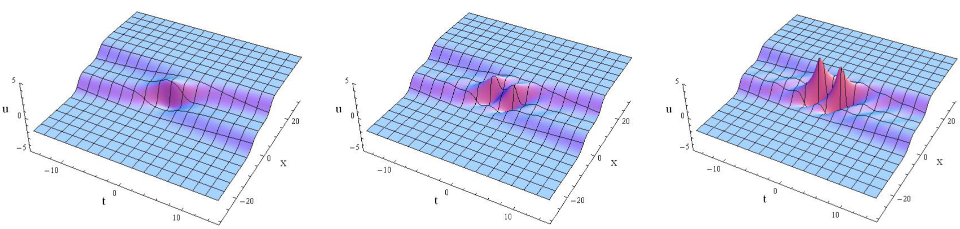
4. Dynamical Features

In the following, Figs. 1–4 depict dynamic features and energy distributions of the lump-type solutions, interaction solutions and periodic wave solutions. Figure 1 illustrates three-dimensional plots of three types of lump-type waves with Eqs. (12)–(14) separately. Figures 2 and 3 show the different interaction waves with different parameters. Characters of the interaction process among lump, kink wave and periodic wave are dependent on the rational quadratic function, exponential function (including hyperbolic function) and triangle function.[51] Figure 2 depicts interactions among the lump, kink wave and periodic wave. If we take $v_3 = 1$ (see Fig. 2(a)), the wave has one peak and one valley, when we increase the value of v_3



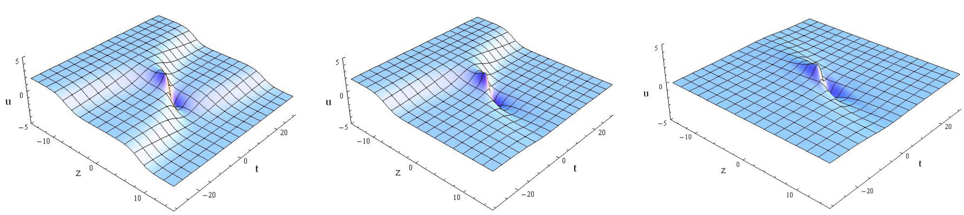
(a) (b) (c)

Fig. 1. (Color online) Lump-type solutions of (a) Eq. (12), (b) Eq. (13) and (c) Eq. (14).



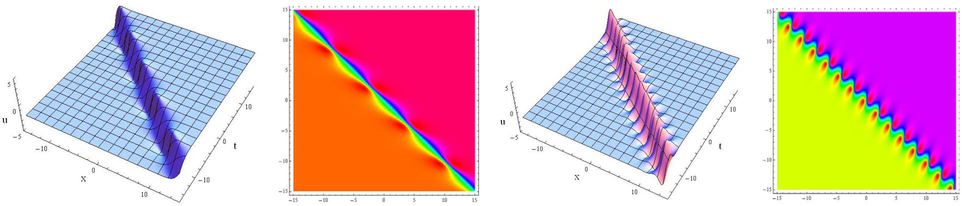
(a) (b) (c)

Fig. 2. (Color online) Interaction solutions of Eq. (19) with $v_1 = v_2 = 1$: (a) $v_3 = 1$, (b) $v_3 = 5$ and (c) $v_3 = 10$.



(a) (b) (c)

Fig. 3. (Color online) Interaction solutions between: (a) a lump and two kink waves of Eq. (20), (b) a mixed lump–kink wave of Eq. (21) and (c) a lump and a periodic wave of Eq. (22).



(a) (b) (c) (c)

Fig. 4. (Color online) Periodic wave solutions (25) with $l_1 = 1, l_4 = 1, l_5 = 1, l_9 = 1, m_1 = 1, m_2 = 1, m_3 = 1$: (a) $l_6 = 0$; (b) is the corresponding contour plot of (a); (c) $l_6 = 2$; (d) is the corresponding contour plot of (c).

(by taking $v_3 = 5$ and $v_3 = 10$), it splits into two peaks and two valleys gradually with the increase of the amplitudes of the peaks and valleys (see Figs. 2(b) and 2(c)). Figure 3(a) is an interaction between a lump and two kink wave of Eq. (20). Figure 3(b) is a lump–kink wave based on Eq. (21). Figure 3(c) is an interaction wave between a lump and a periodic wave of Eq. (22). Figure 4 shows periodic waves (25) with $l_1 = 1, l_4 = 1, l_5 = 1, l_9 = 1, m_1 = 1, m_2 = 1, m_3 = 1$ along with different value of l_6, Fig. 4(a) is with $l_6 = 0$ and Fig. 4(c) with $l_6 = 2$, Figs. 4(b) and 4(d) are corresponding contour plots of Figs. 4(a) and 4(c), respectively.

5. Conclusions

In this paper, we have considered the construction of lump-type and interaction solutions to an extended $(3+1)$-dimensional Jimbo–Miwa-like equation, which has potential applications in engineering and physical sciences. All the computations are based on a generalized Hirota bilinear form. Three classes of lump-type solutions have been obtained. Interaction solutions are among lump-type solutions, two kink waves and periodic waves, and between two kink waves and periodic waves. Dynamical features and energy distributions of the presented solutions have been depicted through various plots. Such wave solutions are expected to have applications in modeling nonlinear wave phenomena in nature.

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