The generalized Kupershmidt deformation for integrable bi-Hamiltonian systems

Yuqin Yao

Tsinghua University, Beijing, China

(with Yunbo Zeng)

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catalogue

Introduction

It is an effective approach to construct a new integrable system starting from a bi-Hamiltonian system.

- Fuchssteiner and Fokas (1981) showed that compatible symplectic structures lead to hereditary symmetries, which provides a method to construct a hierarchy of exactly solvable evolution equations.
- Olver and Rosenau(1996) demonstrated that most integrable bi-Hamiltonian systems are governed by a compatible trio of Hamiltonian structures, and their recombination leads to integrable hierarchies of nonlinear equations.
- Kupershmidt (2008) proposed the Kupershmidt deformation of the bi-Hamiltonian systems.

KdV6 equation

Recently, KdV6 equation attract more attentions. Karasu-Kalkani et al applied the Painleve analysis to the class of 6th-order nonlinear wave equation and they have found 4 cases that pass the Painleve test. Three of these were previously known, but the 4th one turned out to be new

$$(\partial_x^3 + 8u_x\partial_x + 4u_{xx})(u_t + u_{xxx} + 6u_x^2) = 0. \tag{1}$$

This equation, as it stands, does not belong to any recognizable theory. In the variables $v = u_x, \; w = u_t + u_{xxx} + 6u_x^2$, (1) is converted to

$$v_t + v_{xxx} + 12vv_x - w_x = 0, \tag{2a}$$

$$w_{xxx} + 8vw_x + 4wv_x = 0, \tag{2b}$$

which is referred as KdV6 equation.

KdV6 equation

- The authors found Lax pair and an auto-Bäcklund transformation for KdV6 equation, but they were unable to find higher symmetries and asked if higher conserved densities and a Hamiltonian formalism exist for KdV6 equation.
- Kundu A, Sahadevan R et al show that KdV6 equation possess infinitely many generalized symmetries, conserved quantities and a recursion operator.
- Kupershmidt described KdV6 equation as a nonholonomic perturbations of bi-Hamiltonian systems. By rescaling v and t in (2), one gets

$$u_t = 6uu_x + u_{xxx} - w_x, \tag{3a}$$

$$w_{xxx} + 4uw_x + 2wu_x = 0, \tag{3b}$$

which can be converted into

$$u_t = B_1(\frac{\delta H_3}{\delta u}) - B_1(\omega), \quad (4a)$$

$$B_2(\omega) = 0, \quad (4b)$$

where

$$B_1 = \partial = \partial_x, \;\; B_2 = \partial^3 + 2(u\partial + \partial u) \quad (5)$$

are the two standard Hamiltonian operators of the KdV hierarchy and $H_3 = u^3 - \frac{u_x^2}{2}$. (4) is called the Kupershmidt deformed system. In general, for a bi-Hamiltonian system

$$u_{t_n} = B_1(\frac{\delta H_{n+1}}{\delta u}) = B_2(\frac{\delta H_n}{\delta u}) \quad (6)$$

where B_1 and B_2 are the standard Hamiltonian operators.

KdV6 equation

The Kupershmidt deformation of the bi-Hamiltonian system (6) is constructed as follows

$$u_{t_n} = B_1(\frac{\delta H_{n+1}}{\delta u}) - B_1(\omega),$$

$$B_2(\omega) = 0. \tag{7}$$

This deformation is conjectured to preserve integrability and the conjecture is verified in a few representative cases(Kupershmidt, 2008)

- We show that the KdV6 equation is equivalent to the Rosochatius deformation of KdV equation with self-consistent sources. We also give the t-type bi-Hamiltonian formalism of KdV6 equation and some new solutions.

The generalized Kupershmidt deformed KdV hierarchy

The KdV hierarchy read

$$u_{t_n} = B_1(\frac{\delta H_{n+1}}{\delta u}) = B_2(\frac{\delta H_n}{\delta u}), \; n = 1, 2, \cdots \tag{8}$$

where

$$B_1 = \partial = \partial_x, \; B_2 = \partial^3 + 2(u\partial + \partial u)$$

$$H_{n+1} = -\frac{2}{2n+1} L^n u, \; L = -\frac{1}{4}\partial^2 - u + \frac{1}{2}\partial^{-1} u_x.$$

For N distinct real λ_j, consider the spectral problem

$$\varphi_{jxx} + (u - \lambda_j)\varphi_j = 0, \; j = 1, 2, \cdots, N.$$

It is easy to find that

$$\frac{\delta \lambda_j}{\delta u} = \varphi_j^2.$$

The generalized Kupershmidt deformed KdV hierarchy

We generalize Kupershmidt deformation of KdV hierarchy

$$u_{t_n} = B_1(\frac{\delta H_{n+1}}{\delta u}) - B_1(\sum_{j=1}^{N} \omega_j), \tag{9a}$$

$$(B_2 - \lambda_j B_1)(\omega_j) = 0,\ j = 1, 2, \cdots, N. \tag{9b}$$

Since ω_j is at the same position as $\frac{\delta H_{n+1}}{\delta u}$, it is reasonable to take $\omega_j = \frac{\delta \lambda_j}{\delta u}$.

The generalized Kupershmidt deformed KdV hierarchy

So the generalized Kupershmidt deformation for a bi-Hamiltonian systems is proposed as follows

$$u_{t_n} = B_1\left(\frac{\delta H_{n+1}}{\delta u} - \sum_{j=1}^{N} \frac{\delta \lambda_j}{\delta u}\right), \tag{10a}$$

$$(B_2 - \lambda_j B_1)\left(\frac{\delta \lambda_j}{\delta u}\right) = 0,\ j = 1, 2, \cdots, N. \tag{10b}$$

The generalized Kupershmidt deformed KdV hierarchy

From (10b), we can obtain

$$\varphi_{jxx} + (u - \lambda_j)\varphi_j = \frac{\mu_j}{\varphi_j^3},$$

where $\mu_j,\ j = 1, 2, \cdots, N$ are integrable constants.

When $n = 2$, (10) gives rise to the generalized Kupershmidt deformed KdV equation

$$u_t = \frac{1}{4}(u_{xxx} + 6uu_x) - \sum_{j=1}^{N}(\varphi_j^2)_x, \tag{11a}$$

$$\varphi_{jxx} + (u - \lambda_j)\varphi_j = \frac{\mu_j}{\varphi_j^3},\ j = 1, 2, \cdots, N \tag{11b}$$

which is just the Rosochatius deformation of KdV equation with self-consistent sources.

The generalized Kupershmidt deformed KdV hierarchy

The Lax pair is

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = U \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ \lambda - u & 0 \end{pmatrix}, \tag{12a}$$

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_t = V \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

$$V = \begin{pmatrix} -\frac{u_x}{4} & -\lambda + \frac{u}{2} \\ -\lambda^2 - \frac{u}{2}\lambda - \frac{u_{xx}}{4} - \frac{u^2}{2} + \frac{1}{2}\sum_{j=1}^{N} \varphi_j^2 & \frac{u_x}{4} \end{pmatrix}$$

$$- \frac{1}{2}\sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} \begin{pmatrix} \varphi_j \varphi_{jx} & -\varphi_j^2 \\ \varphi_{jx}^2 + \frac{\mu_j}{\varphi_j^2} & -\varphi_j \varphi_{jx} \end{pmatrix}. \tag{12b}$$

The generalized Kupershmidt deformed Camassa-Holm equation

The Camassa-Holm (CH) equation read

$$m_t = B_1 \frac{\delta H_1}{\delta u} = B_2 \frac{\delta H_0}{\delta u} = -2u_x m - u m_x, \quad m = u - u_{xx} + \omega \quad (13)$$

where

$$B_1 = -\partial + \partial^3, \; B_2 = m\partial + \partial m$$

$$H_0 = \frac{1}{2} \int (u^2 + u_x^2) dx, \; H_1 = \frac{1}{2} \int (u^3 + u u_x^2) dx.$$

The generalized Kupershmidt deformed Camassa-Holm equation

We have

$$\frac{\delta \lambda_j}{\delta m} = \lambda_j \varphi_j^2.$$

The generalized Kupershmidt deformed CH equation is constructed as follows

$$m_t = B_1\left(\frac{\delta H_1}{\delta m} - \sum_{j=1}^{N} \frac{1}{\lambda_j}\frac{\delta \lambda_j}{\delta m}\right) = -2u_x m - u m_x + \sum_{j=1}^{N}[(\varphi_j^2)_x - (\varphi_j^2)_{xxx}], \tag{14a}$$

$$(B_2 - \frac{1}{\lambda_j}B_1)\left(\frac{1}{\lambda_j}\frac{\delta \lambda_j}{\delta m}\right) = 0,\ j = 1, 2, \cdots, N. \tag{14b}$$

The generalized Kupershmidt deformed Camassa-Holm equation

(14b) gives $\varphi_{jxx} = \frac{1}{4}\varphi_j - \frac{1}{2}m\lambda_j\varphi_j + \frac{\mu_j}{\varphi_j^3}$.

So Eq.(14) gives the Kupershmidt deformed Camassa-Holm equation

$$m_t = -2u_x m - u m_x + \sum_{j=1}^{N}[(\varphi_j^2)_x - (\varphi_j^2)_{xxx}], \tag{15a}$$

$$\varphi_{jxx} = \frac{1}{4}\varphi_j - \frac{1}{2}m\lambda_j\varphi_j + \frac{\mu_j}{\varphi_j^3}, \; j = 1, 2, \cdots, N \tag{15b}$$

which is called as the RD-CHESCS. Eq.(15) has the lax pair

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = U \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ \frac{1}{4} - \frac{1}{2}\lambda m & 0 \end{pmatrix} \tag{16a}$$

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_t = V \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

$$V = \begin{pmatrix} \frac{u_x}{2} & -\frac{1}{\lambda} - u \\ \frac{u}{4} - \frac{1}{4\lambda} + \frac{mu\lambda}{2} & -\frac{u_x}{2} \end{pmatrix} - \sum_{j=1}^{N} \frac{\lambda\lambda_j}{\lambda - \lambda_j} \begin{pmatrix} \varphi_j\varphi_{jx} & -\varphi_j^2 \\ \varphi_{jx}^2 + \frac{\mu_j}{\varphi_j^2} & -\varphi_j\varphi_{jx} \end{pmatrix}$$

The generalized Kupershmidt deformed Boussinesq equation

The Boussinesq equation is

$$\begin{pmatrix} v \\ w \end{pmatrix}_t = B_1 \begin{pmatrix} \frac{\delta H_2}{\delta v} \\ \frac{\delta H_2}{\delta w} \end{pmatrix} = B_2 \begin{pmatrix} \frac{\delta H_1}{\delta v} \\ \frac{\delta H_1}{\delta w} \end{pmatrix} = \begin{pmatrix} 2w_x \\ -\frac{2}{3}vv_x - \frac{1}{6}w_{xxx} \end{pmatrix}, \tag{17}$$

where

$$B_1 = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix},$$

$$B_2 = \frac{1}{3}\begin{pmatrix} 2\partial^3 + 2v\partial + v_x & 3w\partial + 2w_x \\ 3w\partial + w_x & -\frac{1}{6}(\partial^5 + 5v\partial^3 + \frac{15}{2}v_x\partial^2 + \frac{9}{2}v_{xx}\partial + 4v^2\partial + v_{xxx} + 4vv_x) \end{pmatrix}$$

are the two standard Hamiltonian operators of the Boussinesq equation and

$$H_1 = \int w dx, \;\; H_2 = \int (\frac{1}{12}v_x^2 - \frac{1}{9}v^3 + w^2)dx.$$

The generalized Kupershmidt deformed Boussinesq equation

From the following spectral problem and its adjoint spectral problem

$$\varphi_{jxxx} + v\varphi_{jx} + (\frac{1}{2}v_x + w)\varphi_j = \lambda\varphi_j, \tag{18a}$$

$$\varphi^*_{jxxx} + v\varphi^*_{jx} + (\frac{1}{2}v_x - w)\varphi^*_j = -\lambda\varphi^*_j, \ j = 1, 2, \cdots, N. \tag{18b}$$

we have

$$\frac{\delta\lambda_j}{\delta v} = \frac{3}{2}(\varphi_{jx}\varphi^*_j - \varphi_j\varphi^*_{jx}), \ \frac{\delta\lambda_j}{\delta w} = 3\varphi_j\varphi^*_j.$$

The generalized Kupershmidt deformed Boussinesq equation

$$\begin{pmatrix} v \\ w \end{pmatrix}_t = B_1(\begin{pmatrix} \frac{\delta H_2}{\delta v} \\ \frac{\delta H_2}{\delta w} \end{pmatrix} - \sum_{j=1}^{N} \begin{pmatrix} \frac{\delta\lambda_j}{\delta v} \\ \frac{\delta\lambda_j}{\delta w} \end{pmatrix}), \tag{19a}$$

$$(B_2 - \lambda_j B_1)\begin{pmatrix} \frac{\delta\lambda_j}{\delta v} \\ \frac{\delta\lambda_j}{\delta w} \end{pmatrix} = 0, \ j = 1, 2, \cdots, N. \tag{19b}$$

The generalized Kupershmidt deformed Boussinesq equation

By the complicated computation, from (19) we obtain

the generalized Kupershmidt deformed Boussinesq equation

$$v_t = 2w_x - 3\sum_{j=1}^{N}(\varphi_j\varphi_j^*)_x, \tag{20a}$$

$$w_t = -\frac{1}{6}(4vv_x + v_{xxx}) - \frac{3}{2}(\varphi_{jxx}\varphi_j^* - \varphi_j\varphi_{jxx}^*), \tag{20b}$$

$$\varphi_{jxxx} + v\varphi_{jx} + (\frac{1}{2}v_x + w)\varphi_j = \lambda_j\varphi_j, \tag{20c}$$

$$\varphi_{jxxx}^* + v\varphi_{jx}^* + (\frac{1}{2}v_x - w)\varphi_j^* = -\lambda_j\varphi_j^*,\ j = 1, 2, \cdots, N \tag{20d}$$

which just is the Boussinesq equation with self-consistent sources.

The generalized Kupershmidt deformed Boussinesq equation

Lax representation

$$L_t = [\partial^2 + \frac{2}{3}v + \sum_{j=1}^{N} \varphi_j \partial^{-1} \varphi_j^*, L] \tag{21a}$$

$$L\psi = (\partial^3 + v\partial + \frac{1}{2}v_x + w)\psi = \lambda\psi, \tag{21b}$$

$$\psi_t = (\partial^2 + \frac{2}{3}v + \sum_{j=1}^{N} \varphi_j \partial^{-1} \varphi_j^*)\psi. \tag{21c}$$

The generalized Kupershmidt deformed Jaulent-Miodek equation

The JM hierarchy is

$$\begin{pmatrix} q \\ r \end{pmatrix}_{t_n} = B_1 \begin{pmatrix} b_{n+2} \\ b_{n+1} \end{pmatrix} = B_1 \begin{pmatrix} \frac{\delta H_{n+1}}{\delta q} \\ \frac{\delta H_{n+1}}{\delta r} \end{pmatrix} = B_2 \begin{pmatrix} \frac{\delta H_n}{\delta q} \\ \frac{\delta H_n}{\delta r} \end{pmatrix}$$

where

$$B_1 = \begin{pmatrix} 0 & 2\partial \\ 2\partial & -q_x - 2q\partial \end{pmatrix}, \quad B_2 = \begin{pmatrix} 2\partial & 0 \\ 0 & r_x + 2r\partial - \frac{1}{2}\partial^3 \end{pmatrix},$$

$$\begin{pmatrix} b_{n+2} \\ b_{n+1} \end{pmatrix} = L \begin{pmatrix} b_{n+1} \\ b_n \end{pmatrix}, \quad n = 1, 2, \cdots$$

$$b_0 = b_1 = 0, \; b_2 = -1, \; H_n = \frac{1}{n-1}(2b_{n+2} - q b_{n+1}).$$

The generalized Kupershmidt deformed Jaulent-Miodek equation

Similarly, the generalized Kupershmidt deformed JM hierarchy is constructed as follows

$$\begin{pmatrix} q \\ r \end{pmatrix}_{t_n} = B_1\left(\begin{pmatrix} \frac{\delta H_{n+1}}{\delta q} \\ \frac{\delta H_{n+1}}{\delta r} \end{pmatrix} + \sum_{j=1}^{N} \begin{pmatrix} \frac{\delta \lambda_j}{\delta q} \\ \frac{\delta \lambda_j}{\delta r} \end{pmatrix}\right), \tag{22a}$$

$$(B_2 - \lambda_j B_1)\begin{pmatrix} \frac{\delta \lambda_j}{\delta q} \\ \frac{\delta \lambda_j}{\delta r} \end{pmatrix} = 0, \ j = 1, 2, \cdots, N. \tag{22b}$$

(22b) leads to

$$\varphi_{2jx} = (-\lambda_j^2 + \lambda_j q + r)\varphi_{1j} + \frac{\mu_j}{\varphi_{1j}^3}, \ j = 1, 2, \cdots, N.$$

The generalized Kupershmidt deformed Jaulent-Miodek equation

Then Eqs.(22) with $n = 3$ gives rise to the generalized Kupershmidt deformed JM equation

$$q_t = -r_x - \frac{3}{2}qq_x + 2\sum_{j=1}^{N}\varphi_{1j}\varphi_{2j}, \tag{23a}$$

$$r_t = \frac{1}{4}q_{xxx} - q_x r - \frac{1}{2}qr_x + \sum_{j=1}^{N}[2(\lambda_j - q)\varphi_{1j}\varphi_{2j} - \frac{1}{2}q_x\varphi_{1j}^2], \tag{23b}$$

$$\varphi_{1jx} = \varphi_{2j},\ \varphi_{2jx} = (-\lambda_j^2 + \lambda_j q + r)\varphi_{1j} + \frac{\mu_j}{\varphi_{1j}^3}\ j = 1, 2, \cdots, N \tag{23c}$$

which just is the RD-JMESCS

Eq.(23) has the Lax representation (12a) with

$$U = \begin{pmatrix} 0 & 1 \\ -\lambda^2 + \lambda q + r & 0 \end{pmatrix},$$

$$\begin{aligned} V = & \begin{pmatrix} \frac{1}{4}q_x & -\lambda - \frac{1}{2}q \\ \lambda^3 - \frac{1}{2}q\lambda^2 - (\frac{1}{2}q^2 + r)\lambda + \frac{1}{4}q_{xx} - \frac{1}{2}qr & -\frac{1}{4}q_x \end{pmatrix} \\ & + \frac{1}{2}\begin{pmatrix} 0 & 0 \\ \lambda\langle\Phi_1, \Phi_1\rangle - \langle\Lambda\Phi_1, \Phi_1\rangle - q\langle\Phi_1, \Phi_1\rangle & 0 \end{pmatrix} \\ & + \frac{1}{2}\sum_{j=1}^{N}\frac{1}{\lambda - \lambda_j}\begin{pmatrix} \phi_{1j}\phi_{2j} & -\phi_{1j}^2 \\ \phi_{2j}^2 + \frac{\mu_j}{\phi_{1j}^2} & -\phi_{1j}\phi_{2j} \end{pmatrix} \end{aligned}$$

The generalized Kupershmidt deformed Jaulent-Miodek equation

Denote the inner product in $\mathbb{R}^N$ by $\langle ., . \rangle$ and

$$\Phi_i = (\varphi_{i1}, \varphi_{i2}, \cdots, \varphi_{iN})^T, \;\; i = 1, 2, \;\; \mu = (\mu_1, \cdots, \mu_N)^T, \;\; \Lambda = diag(\lambda_1, \cdots, \lambda_N)$$

Eq.(23) can be written as

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = B_1 \begin{pmatrix} \frac{1}{8} q_{xx} - \frac{3}{4} qr - \frac{5}{16} q^3 + \frac{1}{2} \langle \Lambda\Phi_1, \Phi_1 \rangle \\ -\frac{1}{2} r - \frac{3}{8} q^2 + \frac{1}{2} \langle \Phi_1, \Phi_1 \rangle \end{pmatrix} \tag{24a}$$

$$\varphi_{1jx} = \varphi_{2j}, \;\; \varphi_{2jx} = -\lambda_j^2 \varphi_{1j} + q\lambda_j \varphi_{1j} + r\varphi_{1j} + \frac{\mu_j}{\varphi_{1j}^3}. \tag{24b}$$

Notices that Kernel of B_1 is $(c_1 + \frac{1}{2} q c_2, c_2)^T$, we may rewrite (24a) as

$$\frac{1}{8} q_{xx} - \frac{3}{4} qr - \frac{5}{16} q^3 + \frac{1}{2} \langle \Lambda\Phi_1, \Phi_1 \rangle = c_1 + \frac{1}{2} q c_2, \;\; -\frac{1}{2} r - \frac{3}{8} q^2 + \frac{1}{2} \langle \Phi_1, \Phi_1 \rangle = c_2 \tag{25a}$$

$$c_{1x} = \frac{1}{2} \partial_t (r + \frac{1}{4} q^2), \;\; c_{2x} = \frac{1}{2} \partial_t q. \tag{25b}$$

The generalized Kupershmidt deformed Jaulent-Miodek equation

By introducing $q_1 = q, \quad p_1 = -\frac{1}{8}q_x,$

Eqs. (24b) and (25b) give rise to the t-type Hamiltonian form

$$R_x = G_1 \frac{\delta F_1}{\delta R}, \tag{26a}$$

where

$$R = (\Phi_1^T, q_1, \Phi_2^T, p_1, c_1, c_2)^T,$$

$$F_1 = -4p_1^2 - \frac{1}{16}q_1^4 - \frac{1}{2}q_1^2 c_2 + q_1 c_1 - c_2^2 + \frac{3}{8}q_1^2\langle\Phi_1, \Phi_1\rangle - \frac{1}{2}q_1\langle\Lambda\Phi_1, \Phi_1\rangle$$
$$+ \frac{1}{2}\langle\Phi_2, \Phi_2\rangle + \frac{1}{2}\langle\Lambda^2\Phi_1, \Phi_1\rangle + c_2\langle\Phi_1, \Phi_1\rangle - \frac{1}{4}\sum_{j=1}^{N}\varphi_{1j}^4 + \frac{1}{2}\sum_{j=1}^{N}\frac{\mu_j}{\varphi_{1j}^2}, \tag{26b}$$

and the $t-$ type Hamiltonian operator G_1 is given by

$$G_1 = \begin{pmatrix} 0 & I_{(N+1)\times(N+1)} & 0 & 0 \\ -I_{(N+1)\times(N+1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}\partial_t \\ 0 & 0 & \frac{1}{2}\partial_t & 0 \end{pmatrix}. \tag{26c}$$

The generalized Kupershmidt deformed Jaulent-Miodek equation

The Rosochatius deformation of MJM equation with self-consistent sources (RD-MJMSCS) is defined as

$$\begin{pmatrix} \tilde{r} \\ \tilde{q} \end{pmatrix}_t = \tilde{B}_1\left(\frac{\delta \tilde{H}_2}{\delta \tilde{u}} + \frac{\delta \lambda}{\delta \tilde{u}}\right) = \tilde{B}_1 \begin{pmatrix} -\frac{1}{2}\tilde{q}_x - \tilde{q}\tilde{r} + \langle \tilde{\Phi}_1, \tilde{\Phi}_2 \rangle \\ -\frac{1}{2}\tilde{r}^2 - \frac{3}{8}\tilde{q}^2 + \frac{1}{2}\tilde{r}_x + \frac{1}{2}\langle \tilde{\Phi}_1, \tilde{\Phi}_1 \rangle \end{pmatrix} \tag{27a}$$

$$\tilde{\varphi}_{1jx} = -\tilde{r}\tilde{\varphi}_{1j} + \lambda_j \tilde{\varphi}_{2j}, \; \tilde{\varphi}_{2jx} = -\lambda_j \tilde{\varphi}_{1j} + \tilde{q}\tilde{\varphi}_{1j} + \tilde{r}\tilde{\varphi}_{2j} + \frac{\mu_j}{\lambda_j \tilde{\varphi}_{1j}^3}. \tag{27b}$$

where $\tilde{B}_1 = \begin{pmatrix} \frac{1}{2}\partial & 0 \\ 0 & 2\partial \end{pmatrix}, \; \tilde{H}_2 = -\frac{1}{2}\tilde{q}_x\tilde{r} - \frac{1}{2}\tilde{q}\tilde{r}^2 - \frac{1}{8}\tilde{q}^3.$

The generalized Kupershmidt deformed Jaulent-Miodek equation

Since the Kernel of $\tilde{B}_1$ is $(\tilde{c}_1, \tilde{c}_2)^T$, let

$$-\frac{1}{2}\tilde{q}_x - \tilde{q}\tilde{r} + \langle\tilde{\Phi}_1, \tilde{\Phi}_2\rangle = \tilde{c}_1, \quad -\frac{1}{2}\tilde{r}^2 - \frac{3}{8}\tilde{q}^2 + \frac{1}{2}\tilde{r}_x + \frac{1}{2}\langle\tilde{\Phi}_1, \tilde{\Phi}_1\rangle = \tilde{c}_2,$$

$$\tilde{q}_1 = \tilde{q}, \ \tilde{p}_1 = -\frac{1}{2}\tilde{r}, \ \tilde{R} = (\tilde{\Phi}_1^T, \tilde{q}_1, \tilde{\Phi}_2^T, \tilde{p}_1, \tilde{c}_1, \tilde{c}_2)^T,$$

then RD-MJMSCS (27) can be written as a t-type Hamiltonian system

$$\tilde{R}_x = \tilde{G}_1 \frac{\delta \tilde{F}_1}{\delta \tilde{R}} \tag{28a}$$

$$\tilde{F}_1 = -2\tilde{p}_1\tilde{c}_1 + \tilde{q}_1\tilde{c}_2 + 2\tilde{p}_1^2\tilde{q}_1 + \frac{1}{8}\tilde{q}_1^3 + 2\tilde{p}_1\langle\tilde{\Phi}_1, \tilde{\Phi}_2\rangle - \frac{1}{2}\tilde{q}_1\langle\tilde{\Phi}_1, \tilde{\Phi}_1\rangle + \frac{1}{2}\langle\Lambda\tilde{\Phi}_2, \tilde{\Phi}_2\rangle + \frac{1}{2}\langle\Lambda\tilde{\Phi}_1, \tilde{\Phi}_1\rangle + \sum_{j=1}^{N} \frac{\mu_j}{2\lambda_j\tilde{\varphi}_{1j}^2}, \tag{28b}$$

$$\tilde{G}_1 = \begin{pmatrix} 0 & I_{(N+1)\times(N+1)} & 0 & 0 \\ -I_{(N+1)\times(N+1)} & 0 & 0 & 0 \\ 0 & 0 & 2\partial_t & 0 \\ 0 & 0 & 0 & \frac{1}{2}\partial_t \end{pmatrix}. \tag{28c}$$

The generalized Kupershmidt deformed Jaulent-Miodek equation

The Miura map relating systems (26) and (28), i.e. $R = M(\tilde{R})$, is given by

$$\Phi_1 = \tilde{\Phi}_1, \ \Phi_2 = \Lambda\tilde{\Phi}_2 + 2\tilde{p}_1\tilde{\Phi}_1, \tag{29a}$$

$$q_1 = \tilde{q}_1, \ p_1 = -\frac{1}{2}\tilde{q}_1\tilde{p}_1 - \frac{1}{4}\langle\tilde{\Phi}_1, \tilde{\Phi}_2\rangle + \frac{1}{4}\tilde{c}_1, \tag{29b}$$

$$c_1 = \frac{1}{2}\tilde{F}_1 + \partial_t\tilde{p}_1, \quad c_2 = \tilde{c}_2. \tag{29c}$$

Denote

$$M' \equiv \frac{DR}{D\tilde{R}^T}$$

where $\frac{DR}{D\tilde{R}^T}$ is the Jacobi matrix consisting of Frechet derivative of M, $M^{'*}$ denotes adjoint of M'.

The generalized Kupershmidt deformed Jaulent-Miodek equation

The second Hamiltonian operator of Eq.(26)

$$G_2 = M\tilde{G}_1 M^* = \begin{pmatrix} 0 & 0 & \Lambda & -\frac{1}{4}\Phi_1 & \frac{1}{2}\Phi_2 & 0 \\ 0 & 0 & 2\Phi_1^T & -\frac{1}{2}q_1 & -4p_1 - \partial_t & 0 \\ -\Lambda & 2\Phi_1 & 0 & \frac{1}{4}\Phi_2 & g_{35} & 0 \\ \frac{1}{4}\Phi_1^T & \frac{1}{2}q_1 & -\frac{1}{4}\Phi_2^T & \frac{1}{8}\partial_t & g_{45} & 0 \\ -\frac{1}{2}\Phi_2^T & 4p_1 - \partial_t & -g_{35} & -g_{45} & g_{55} & \partial_t q_1 \\ 0 & 0 & 0 & 0 & q_1\partial_t & 2\partial_t \end{pmatrix} \tag{30}$$

where

$$g_{35} = \frac{1}{2}q_1\Lambda\Phi_1 - \frac{1}{2}\Lambda^2\Phi_1 - \frac{3}{8}q_1^2\Phi_1 - c_2\Phi_1 + \frac{1}{4}\Phi_1\langle\Phi_1, \Phi_1\rangle + (\frac{\mu_1}{\varphi_{1j}^3}, \cdots, \frac{\mu_N}{\varphi_{1N}^3})^T$$

$$g_{45} = -\frac{1}{2}c_1 + \frac{1}{4}\langle\Lambda\Phi_1, \Phi_1\rangle - \frac{3}{8}q_1\langle\Phi_1, \Phi_1\rangle + \frac{1}{2}q_1c_2 + \frac{1}{8}q_1^3$$

$$g_{55} = \partial_t(\frac{1}{4}\langle\Phi_1, \Phi_1\rangle - \frac{1}{2}c_2) + (\frac{1}{4}\langle\Phi_1, \Phi_1\rangle - \frac{1}{2}c_2)\partial_t - \frac{1}{4}q_1\partial_t q_1.$$

Thus we get the bi-Hamiltonian structure for Eq.(26a)-(26c)

$$R_x = G_1\frac{\delta F_1}{\delta R} = G_2\frac{\delta F_0}{\delta R}, \quad F_0 = 2c_1. \tag{31}$$

Thank you

for your attention!