Monodromy preserving deformation and 4-dimensional Painlevé type equations

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§1. Introduction

Two important aspects about the Painlevé equations:

1. Generalization of equations of elliptic functions (non-autonomous).

$$(y')^2 = 4y^3 - g_2 y - g_3 \qquad \leftrightarrow \qquad y'' = 6y^2 - g_2/2$$

$$\downarrow$$

$$y'' = 6y^2 + t$$

2. It is derived from a deformation theory of linear equations.

   (Monodromy preserving deformation)

Degeneration of the Painlevé equations

$$
\begin{array}{ccccccccc}
P_{\mathrm{VI}} & \rightarrow & P_{\mathrm{V}} & \rightarrow & P_{\mathrm{III}}(D_6) & \rightarrow & P_{\mathrm{III}}(D_7) & \rightarrow & P_{\mathrm{III}}(D_8) \\
 & & & \searrow & & \searrow & & \searrow & \\
 & & & & P_{\mathrm{IV}} & \rightarrow & P_{\mathrm{II}} & \rightarrow & P_{\mathrm{I}}
\end{array}
$$

Schlesinger system:

$$\begin{aligned}\frac{\partial}{\partial u_j} A_i &= \frac{[A_j, A_i]}{u_j - u_i}, \qquad (j \neq i),\\ \frac{\partial}{\partial u_i} A_i &= -\sum_{j \neq i} \frac{[A_j, A_i]}{u_j - u_i}\end{aligned}$$

is derived from a defomation theory of Fuchsian equation in the form

$$\frac{d}{dx} Y = \left(\sum_{1=1}^{n} \frac{A_i}{x - u_i} \right) Y,$$

called Schlesinger normal form.

§2. Classification theory of Fuchsian equations

2.1. Spectral types

$n+1$-tuples of partitions of m:

$$m_1^1 m_2^1 \ldots m_{l_1}^1, m_1^2 \ldots m_{l_2}^2, \ldots, m_1^n \ldots m_{l_n}^n, m_1^\infty \ldots m_{l_0}^\infty$$

$$\left(\sum_{j=1}^{l_i} m_j^j = m \quad \text{for} \quad 0 \leq \forall i \leq n \right).$$

It means that an information about the multiplicity of eigen values of A_i is given by the i-th partition; m_j^i ($1 \leq j \leq l_i$ same eigen values exist.

2.2. N. Katz's two operations:

1. addition

2. middle convolution

Addition is a transformation

$$A = (A_1, \ldots, A_n) \mapsto (A_1 + \alpha_1, \ldots, A_n + \alpha_n)$$

for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$.

Middle convolution

A convolution is defined as a transformation $A \mapsto (G_1, \ldots, G_n)$ for $\lambda \in \mathbb{C}$, where

$$G_i = \begin{pmatrix} & & & O & & & \\ A_1 & A_2 & \cdots & \cdots & A_i + \lambda 1_m & \cdots & A_n \\ & & & O & & & \end{pmatrix} < i,$$

$$G_i \in M_{n\times m}(\mathbb{C}).$$

Then, we consider two invariant subspaces with respect to $G = (G_1, \ldots, G_n)$:

$$\mathcal{K} = \begin{pmatrix} \mathrm{Ker} A_1 \\ \vdots \\ \mathrm{Ker} A_n \end{pmatrix} \subset \mathbb{C}^{n\times m},$$

$$\mathcal{L}_\lambda = \mathrm{Ker}(G_1 + \cdots + G_n),$$

and we express the action of $G = (G_1, \ldots, G_n)$ on the quotient space $\mathbb{C}^{n \times m}/\mathcal{K} + \mathcal{L}_\lambda$ as $\overline{G} = (\overline{G_1}, \ldots \overline{G_n})$. The transformation

$$A = (A_1, \ldots, A_n) \mapsto \overline{G} = (\overline{G_1}, \ldots \overline{G_n})$$

is called a middle convolution.

e.g. hypergeometric equation

$$\frac{dy}{dx} = \frac{a_0}{x}y + \frac{a_1}{x-1}y$$

$$\downarrow \qquad mc_\lambda$$

$$\frac{dY}{dx} = \left(\frac{1}{x}\begin{pmatrix} a_0+\lambda & a_1 \\ 0 & 0 \end{pmatrix} + \frac{1}{x-1}\begin{pmatrix} 0 & 0 \\ a_0 & a_1+\lambda \end{pmatrix}\right)Y$$

Theorem 1 (N. Katz). *Any regid, irreducible Fuchsian system is constructed by finite procedures of additions, and middle convolutions from a Fuchsian system of order one.*

Theorem 2 (T. Oshima). *Any irreducible Fuchsian system with 4 accessary parameters is constructed by finite procedures of addtions and middle convoltuions from a system of Fuchsian equations of the following 13 types:*

11,11,11,11,11

21,21,111,111 31,22,22,1111 22,22,22,211

211,1111,1111 221,221,11111 32,11111,11111
222,222,2211 33,2211,111111 44,2222,22211
44,332,11111111 55,3331,22222 66,444,2222211.

Theorem 3 (Haraoka-Filipuk). *Schlesinger systems are invariant under the Katz's two operations.*

§3. Space of accessory parameters

3.1. Dimension

The number of the accessory parameters is given by the following formula:

$$N = (n-1)m^2 - \sum_{i=0}^{n} \left(\sum_{j=1}^{l_i} (m_j^i)^2 \right) + 2.$$

3.2. Poisson structure

Kostant-Kirillov structure is introduced by the Poisson bracket

$$\{(A_p)_{i,j}, (A_q)_{k,l}\} = \delta_{p,q}(\delta_{i,l}(A_p)_{k,j} - \delta_{k,j}(A_p)_{i,l}).$$

Schlesinger system is rewritten as

$$\frac{\partial}{\partial u_k} A_l = \{A_l, H_k\},$$

by using the Hamiltonian

$$H_k = \sum_{l \neq k} \frac{\mathsf{Tr}(A_k A_l)}{u_k - u_l}.$$

3.3. Symplectic structure

If we put $A_i = B^i \cdot C^i$, $B^i \in M_{m,\mathsf{rk} A_i}(\mathbb{C})$, $\quad C^i \in M_{\mathsf{rk} A_i,m}(\mathbb{C})$, then the symplectic form is given as

$$\omega = \sum_{i=1}^{n} \mathsf{Tr}(dB^i \wedge dC^i).$$

§4.1. Hamiltonian of the type 21,21,111,111

Eigen values of each matrices are given as

$$0, 0, \theta^0; \quad 0, 0, \theta^1; \quad 0, \theta_1^t, \theta_2^t; \quad \kappa_1, \kappa_2, \kappa_3,$$

$$\theta^0 + \theta^1 + \theta_1^t + \theta_2^t + \kappa_1 + \kappa_2 + \kappa_3 = 0 \text{ (Fuchs)}$$

The hamltonian H can be written as

$$\begin{aligned} s(s-1)H &= s(s-1)H_{\mathrm{VI}}\left(\begin{matrix} -\theta^0, \ \theta_2^t - \theta_1^t \\ \kappa_2 - \kappa_1, -\kappa_2 - \theta_2^t \end{matrix}; s; q_1, p_1\right) \\ &+ s(s-1)H_{\mathrm{VI}}\left(\begin{matrix} -\theta^0, \ -\theta_1^t \\ \kappa_3 - \kappa_1, -\kappa_3 \end{matrix}; s; q_2, p_2\right) \\ &-(q_1 + q_2)(q_1 - 1)p_1(q_2 - s)p_2 \\ &+\kappa_3(q_1 - 1)p_1 q_2 + (\kappa_2 + \theta_2^t)(q_2 - s)p_2 q_1 + f(s), \end{aligned}$$

where $s = \frac{t-1}{t}$.

Here H_{VI} is the hamiltonian of the sixth Painlevé equation, and it is given as follows:

$$\begin{aligned}
&t(t-1)H_{\mathrm{VI}}\left({\theta_0, \theta_1 \atop \theta_t, \kappa_1}; t; q, p\right) \\
&\quad = q(q-1)(q-t)p^2 \\
&\qquad -\{\theta_0(q-1)(q-t)+\theta_1(q-t)q+\theta_t q(q-1)\}p \\
&\qquad +\kappa_1(\kappa_1+\theta_0+\theta_1+\theta_t)(q-t) \\
&\qquad +(t-1)\theta_0\theta_t+t\theta_1\theta_t.
\end{aligned}$$

§4.2. Hamiltonian of the type 31,22,22,1111

Eigen values of each matrices are given as

$$0, 0, \theta^0, \theta^0; \quad 0, 0, \theta^1, \theta^1; \quad 0, 0, 0, \theta^t; \quad \kappa_1, \kappa_2, \kappa_3, \kappa_4,$$

$$2\theta^0 + 2\theta^1 + \theta^t + \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 = 0 \text{ (Fuchs)}$$

The hamltonian H can be written as

$$t(t-1)H \;\; =$$

§4.3. Hamiltonian of the type 22,22,22,211

Eigen values of each matrices are given as

$$0, 0, \theta^0, \theta^0; \quad 0, 0, \theta^1, \theta^1; \quad 0, 0, \theta^t, \theta^t; \quad \kappa_1, \kappa_1, \kappa_2, \kappa_3,$$

$$2\theta^0 + 2\theta^1 + 2\theta^t + 2\kappa_1 + \kappa_2 + \kappa_3 = 0 \text{ (Fuchs)}$$

The hamltonian H can be written as

$$
\begin{aligned}
t(t-1)H \;=\; & \frac{t(t-1)}{2} H_{\mathrm{VI}}\left(\begin{matrix}-2\theta^0, -2\theta^1 \\ -2\theta^t, -2\kappa_1\end{matrix}; q_1, p_1\right) \\
& -\frac{1}{2}(\theta + (3q_1 - t - 1)p_1)p_1 q_2 \\
& -(1-2q_1)(q_2^2 p_2^2 + (q_2 p_2 - \theta - \kappa_1 - \kappa_2)^2) \\
& -2(q_1(q_1-1) - q_2)(q_1 - t)p_2(q_2 p_2 - \theta - \kappa_2 - \kappa_2) \\
& -\{(t+1)\theta^0 + t\theta^1 + \theta^t + 2\kappa_1 q_1 + ((1+t)q_1 - t)p_1 - p_1 q_2\} \\
& \quad \times (2q_2 p_2 - \theta - \kappa_1 - \kappa_2),
\end{aligned}
$$

where $\theta = \theta^0 + \theta^1 + \theta^t$.

Particular solution

On the condition that $\theta + \kappa_1 + \kappa_2 = 0$ ($\kappa_2 = \kappa_3$), $q_2 = 0$ is a solution because $\frac{d}{dt} q_2 = 0$.

In this case,

$$\frac{d}{dt} q_1 = \frac{1}{2} \frac{\partial H_{\mathrm{VI}}}{\partial p_1}, \qquad \frac{d}{dt} p_1 = -\frac{1}{2} \frac{\partial H_{\mathrm{VI}}}{\partial q_1}, \qquad \frac{d}{dt} q_2 = 0,$$

$$\begin{aligned} t(t-1)\frac{d}{dt} p_2 = {} & 2q_1(q_1-1)(q_1-t)p_2^2 \\ & + 2\{(t+1)\theta_0 + t\theta_1 + \theta_t + 2\kappa_1 q_1 + ((1+t)q_1 - t)p_1\} p_2 \\ & + \frac{1}{2}(\theta + (3q_1 - t - 1)p_1)p_1. \end{aligned}$$

If q_1, p_1 is a solution of the sixth Painlevé equation, $q_2 = 0$, p_2 is a solution of the Riccati equation, whose coefficients are written by q_1 and p_1, then it is a particular solution.