The extended KP, mKP and DKP hierarchy, and generalized dressing method

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catalogue

I. Itroduction

Multi-component generalizations of KP hierarchy attract a lot of interest both from physical and mathematical point of view.

- Jimbo Miwa Date et. al. 1981-83, M. Sato et. al. 1982
- Kac, Leur 1998, 2003
- Aratyn et. al. 1998

Multi-component KP (mcKP) given by Date Jimbo et. al. 1981 contains

- Darvey-Stewartson equation
- two-dimensional Toda lattice
- three-wave resonant interaction

I. Itroduction

Several equivalent formulations of mcKP hierarchy

- Matrix pseudo-differential operator (Sato) formulation
- τ-function approach via matrix Hirota bilinear identities
- multi-component free fermion formulation

Coupled KP hierarchy was generated through Pfaffianization (Hirota,1991). This coupled KP hierarchy can be reformulated as a reduced case of the 2-component KP hierarchy (Kakei, 2000).

Another kind of mcKP equation is the KP equation with self-consistent sources (Mel'nikov, 80').

- First type of KPSCS

$$(4u_t - 12uu_x - u_{xxx})_x - 3u_{yy} + 4\sum_{i=1}^{N}(q_i r_i)_{xx} = 0, \tag{1a}$$

$$q_{i,y} = q_{i,xx} + 2uq_i, \tag{1b}$$

$$r_{i,y} = -r_{i,xx} - 2ur_i, \;\; i = 1, \cdots N. \tag{1c}$$

- Second type of KPSCS

$$(4u_t - 12uu_x - u_{xxx})_x - 3u_{yy} = 3\sum^N [q_{i,xx}r_i - q_i r_{i,xx} + (q_i r_i)_y]_x, \quad (2a)$$

$$q_{i,t} = q_{i,xxx} + 3uq_{i,x} + \frac{3}{2}q_i \int u_y dx + \frac{3}{2}q_i \sum^N q_j r_j + \frac{3}{2}u_x q_i, \quad (2b)$$

$$r_{i,t} = r_{i,xxx} + 3ur_{i,x} - \frac{3}{2}r_i \int u_y dx - \frac{3}{2}r_i \sum^N q_j r_j + \frac{3}{2}u_x r_i. \quad (2c)$$

II. The extended KP Hierarchy

The Lax equation of KP hierarchy (KPH) is given by (Sato)

$$L_{t_n} = [B_n, L]. \tag{3}$$

$$L = \partial + u_1\partial^{-1} + u_2\partial^{-2} + \cdots$$

pseudo-differential operator with infinite many potential functions u_i and $B_n = L^n_+$. Commutativity of ∂_{t_n} flows give raise to zero-curvature equations of KP hierarchy

$$B_{n,t_k} - B_{k,t_n} + [B_n, B_k] = 0.$$

II. The extended KP Hierarchy

The squared eigenfunction symmetry constraint given by

$$L^k = B_k + \sum_{i=1}^{N} q_i \partial^{-1} r_i \tag{4a}$$

$$q_{i,t_n} = B_n(q_i), \;\; r_{i,t_n} = -B_n^*(r_i), \quad i = 1, \ldots, N \tag{4b}$$

is compatible with the KP hierarchy(Konopelchenko, Chen Y).

II. The extended KP Hierarchy

We introduce the extended KP hierarchy(exKPH)

$$L_{\tau_k} = [B_k + \sum_{i=1}^{N} q_i \partial^{-1} r_i, L]. \tag{5a}$$

$$L_{t_n} = [B_n, L], \; n \neq k. \tag{5b}$$

$$q_{i,t_n} = B_n(q_i), \quad r_{i,t_n} = -B_n^*(r_i) \quad i = 1, \cdots, N. \tag{5c}$$

Lemma 1.

$$[B_n, q\partial^{-1} r]_- = B_n(q)\partial^{-1} r - q\partial^{-1} B_n^*(r)$$

Proposition 1.

The commutativity of (5a) and (5b) under (5c) give rise to zero curvature equation for exKPH

$$B_{n,\tau_k} - (B_k + \sum_{i=1}^{N} q_i \partial^{-1} r_i)_{t_n} + [B_n, B_k + \sum_{i=1}^{N} q_i \partial^{-1} r_i] = 0 \qquad (6a)$$

$$q_{i,t_n} = B_n(q_i), \qquad (6b)$$

$$r_{i,t_n} = -B_n^*(r_i), \quad i = 1, \cdots, N. \qquad (6c)$$

Under(6b) and (6c) the Lax representation for (6a)

$$\psi_{\tau_k} = (B_k + \sum_{i=1}^{N} q_i \partial^{-1} r_i)(\psi), \tag{7a}$$

$$\psi_{t_n} = B_n(\psi). \tag{7b}$$

II. The extended KP Hierarchy

Eg. 1

For $n = 2$, $k = 3$, first type of KPSCS. The Lax representation

$$\psi_y = (\partial^2 + 2u)(\psi),$$

$$\psi_t = (\partial^3 + 3u\partial + (\frac{3}{2}\int u_y + \frac{3}{2}u_x) + \sum_{i=1}^{N} q_i\partial^{-1}r_i)(\psi).$$

Eg. 2

For $n = 3$, $k = 2$, second type of KPSCS. The Lax representation

$$\psi_y = (\partial^2 + 2u + \sum^{N} q_i \partial^{-1} r_i)(\psi)$$

$$\psi_t = (\partial^3 + 3u\partial + (\frac{3}{2}\int u_y + \frac{3}{2}u_x + \frac{3}{2}\sum^{N} q_i r_i))(\psi)$$

Reductions 1: The t_n-reduction of exKPH

$$L^n = B_n \quad \text{or} \quad L^n_- = 0, \tag{8}$$

one can drop t_n dependency from exKPH $\Rightarrow$

$$B_{n,\tau_k} = [(B_n)^{\frac{k}{n}}_+ + \sum^N q_i \partial^{-1} r_i, B_n],$$

$$B_n(q_i) = \zeta_i^n q_i, \;\; B_n^*(r_i) = -\zeta_i^n r_i, \quad i = 1, \cdots, N.$$

which is Gelfand-Dickey hierarchy with self-consistent sources.

III. reductions

- For $n = 2$, $k = 3$, first type of KdVSCS

$$u_t - 3uu_x - \frac{1}{4}u_{xxx} + \sum_{i=1}^{N}(q_i r_i)_x = 0,$$

$$q_{i,xx} + 2uq_i = \zeta_i^2 q_i, \;\; r_{i,xx} + 2ur_i = -\zeta_i^2 r_i.$$

Lax representation

$$(\partial^2 + 2u)(\psi) = \lambda\psi,$$

$$\psi_t = (\partial^3 + 3u\partial + \frac{3}{2}u' + \sum_1^N q_i\partial^{-1}r_i)(\psi).$$

III. reductions

- For $n=3,\ k=2$, the first type of Boussinesq equation with sources

$$u_{tt}+\frac{1}{3}u_{xxxx}+2(u^2)_{xx}+\sum^{N}(q_{i,x}r_i-q_ir_{i,x})_{xx}+\sum^{N}(q_ir_i)_{xt}=0$$

$$q_{i,xxx}+3uq_{i,x}+q_i(\frac{3}{2}\int u_y+\frac{3}{2}u_x+\frac{3}{2}\sum^{N}q_jr_j)=\zeta_i^3q_i$$

$$r_{i,xxx}+3ur_{i,x}-r_i(\frac{3}{2}\int u_y-\frac{3}{2}u_x+\frac{3}{2}\sum^{N}q_jr_j)=\zeta_i^3r_i$$

with Lax representation

$$(\partial^3+3u_1\partial+3u_2+3u_{1,x})(\psi)=\lambda\psi,\ \psi_t=(\partial^2+2u_1+\sum_{1}^{N}q_i\partial^{-1}r_i)(\psi)$$

Reductions 2: The τ_k-reduction of exKPH

$$L^k = B_k + \sum q_i \partial^{-1} r_i,$$

dropping τ_k dependency from exKPH

$$\left(B_k + \sum^N q_i \partial^{-1} r_i \right)_{t_n} = \left[(B_k + \sum^N q_i \partial^{-1} r_i)_+^{\frac{n}{k}}, B_k + \sum^N q_i \partial^{-1} r_i \right]$$

$$q_{i,t_n} = (B_k + \sum q_i \partial^{-1} r_i)_+^{\frac{n}{k}}(q_i),$$

$$r_{i,t_n} = -(B_k + \sum q_i \partial^{-1} r_i)_+^{\frac{n}{k}*}(r_i).$$

k-constrained KP hierarchy (Konopelchenko, Cheng Yi, Dickey)

- For $k = 2$, $n = 3$, second type of KdVSCS

$$u_t = \frac{1}{4}u_{xxx} + 3uu_x + \frac{3}{2}\sum^{N}[(q_{i,xx}r_i - q_i r_{i,xx}) - \frac{3}{4}(q_i r_i)_{xx}]$$

$$q_{i,t} = q_{i,xxx} + 3uq_{i,x} + \frac{3}{2}q_i \sum^{N} q_j r_j + \frac{3}{2}u_x q_i$$

$$r_{i,t} = r_{i,xxx} + 3ur_{i,x} - \frac{3}{2}r_i \sum^{N} q_j r_j + \frac{3}{2}u_x r_i.$$

- For $k = 3$, the second type of Boussinesq equation with self-consistent sources

$$u_{tt} + \frac{1}{3}u_{xxxx} + 2(u^2)_{xx} - \frac{4}{3}\sum^{N}(q_i r_i)_{xx} = 0$$
$$q_{i,t} = q_{i,xx} + 2uq_i$$
$$r_{i,t} = -r_{i,xx} - 2ur_i, \quad i = 1, \cdots, N.$$

Remark

- The exKPH contains two time series $\{t_n\}$ and $\{\tau_k\}$ and more components by adding eigenfunctions and adjoint eigenfunctions
- The exKPH enables us to find the first type and second type of KPSCS and their Lax representations in different way from Mel'nikov, Zeng Y.B., Wang H.Y., Hu X.B.

- Reductions lead to Gelfand-Dickey hierarchy with sources and k-constrained KP hierarchy, including the first and second type of soliton equation with sources

So exKPH provides a more general approach to find (1+1) and (2+1)-dimensional soliton equations with self-consistent sources and their Lax representations.

IV. Generalized Dressing method

L can be written in the dressing form(Sato):

$$L = W\partial W^{-1},$$

where $W = 1 + w_1\partial^{-1} + w_2\partial^{-2} + \cdots$, *dressing operator*. If W satisfies

$$\partial_{t_n} W = -L_-^n W, \quad n \neq k \tag{11a}$$

$$\partial_{\tau_k} W = -L_-^k W + \sum_{i=1}^{N} q_i \partial^{-1} r_i W \tag{11b}$$

where q_i and r_i satisfy (5c), then L satisfies exKPH.

IV. Generalized Dressing method

Take

$$W = 1 + w_1\partial^{-1} + w_2\partial^{-2} + \cdots + w_N\partial^{-N}.$$

Let h_i be linearly independent functions satisfying

$$\partial_{t_n} h_i = \partial^n(h_i), \quad i = 1, \ldots, N. \tag{12}$$

$$\mathrm{Wr}(h_1, \ldots, h_N) = \begin{vmatrix} h_1 & h_2 & \cdots & h_N \\ h_1' & h_2' & \cdots & h_N' \\ \vdots & \vdots & \vdots & \vdots \\ h_1^{(N-1)} & h_2^{(N-1)} & \cdots & h_N^{(N-1)} \end{vmatrix}$$

IV. Generalized Dressing method

$$W = \frac{1}{\mathrm{Wr}(h_1, \cdots, h_N)} \begin{vmatrix} h_1 & h_2 & \cdots & h_N & 1 \\ h_1' & h_2' & \cdots & h_N' & \partial \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ h_1^{(N)} & h_2^{(N)} & \cdots & h_N^{(N)} & \partial^N \end{vmatrix} \tag{13}$$

Then W provides a dressing operator for KPH.
Let f_i, g_i satisfy

$$\partial_{t_n} f_i = \partial^n(f_i), \quad \partial_{\tau_k} f_i = \partial^k(f_i), \tag{14a}$$

$$\partial_{t_n} g_i = \partial^n(g_i), \quad \partial_{\tau_k} g_i = \partial^k(g_i). \tag{14b}$$

IV. Generalized Dressing method

Take

$$h_i = f_i + \alpha_i(\tau_k) g_i \quad i = 1, \ldots, N, \tag{15}$$

$$q_i = -\dot{\alpha}_i W(g_i), \tag{16a}$$

$$r_i = (-1)^{N-i} \frac{\mathrm{Wr}(h_1, \cdots, \hat{h}_i, \cdots, h_N)}{\mathrm{Wr}(h_1, \cdots, h_N)}. \tag{16b}$$

Proposition 2.

Let W defined by (13) and (15), $L = W\partial W^{-1}$, then W, q_i, r_i satisfy the equation for W (11).

IV. Generalized Dressing method

Lemma 2. (Oevel & Strampp)

$W^{-1} = \sum_{i=1}^{N} h_i \partial^{-1} r_i .$

Lemma 3.

$$\mathrm{Res}_{\partial} \partial^{-1} f P = P^*(f).$$

Lemma 4.

$W^*(r_i) = 0 \ , \ i = 1, \ldots, N.$

Lemma 5. (*Key lemma*)

$\partial^{-1} r_i W$ *is a pure differential operator*

$$(\partial^{-1} r_j W)(h_i) = \delta_{ij}, \quad i, j = 1, \ldots, N. \qquad (17)$$

Proof of Proposition 2.

taking ∂_{τ_k} to the identity $W(h_i) = 0$, we have

$$
\begin{aligned}
0 =& (\partial_{\tau_k} W)(h_i) + (W\partial^k)(h_i) + \dot{\alpha}_i W(g_i) \\
=& (\partial_{\tau_k} W)(h_i) + (L^k W)(h_i) - \sum_{j=1}^{N} q_j \delta_{ji} \\
=& (\partial_{\tau_k} W + L_-^k W - \sum_{j=1}^{N} q_j \partial^{-1} r_j W)(h_i).
\end{aligned}
$$

□

Theorem 1.

W defined by (13) and (15), q_i and r_i defined by (16), $L = W\partial W^{-1}$ satisfy exKPH.

Proof.

$$\begin{aligned} L_{\tau_k} =& W_{\tau_k}\partial W^{-1} - W\partial W^{-1} W_{\tau_k} W^{-1} \\ =& (-L_-^k + \sum_i q_i \partial^{-1} r_i) L + L(L_-^k - \sum_i q_i \partial^{-1} r_i) \\ =& [-L_-^k + \sum_i q_i \partial^{-1} r_i, L] = [B_k + \sum_{i=1}^N q_i \partial^{-1} r_i, L]. \end{aligned}$$

$$\begin{aligned} (W^{-1})_{t_n} &= \sum h_i^{(n)} \partial^{-1} r_i + \sum h_i \partial^{-1} r_{i,t_n} \\ &= \partial^n W^{-1} - W^{-1} B_n = \sum h_i^{(n)} \partial^{-1} r_i - \sum h_i \partial^{-1} B_n^*(r_i) \end{aligned}$$

V. Solutions for the second type of KPSCS

Let $k = 2,\ n = 3$ in (14)

$$f_i = \exp(\lambda_i x + \lambda_i^2 y + \lambda_i^3 t) := e^{\xi_i}, \quad g_i = \exp(\mu_i x + \mu_i^2 y + \mu_i^3 t) := e^{\eta_i}$$

where λ_i and μ_i (for $i = 1, \ldots, N$) are different parameters.

$$h_i = f_i + \alpha_i(y) g_i = 2\sqrt{\alpha_i} e^{\frac{\xi_i + \eta_i}{2}} \cosh(\Omega_i)$$

where $\Omega_i = \frac{\xi_i - \eta_i}{2} - \frac{1}{2}\ln(\alpha_i)$.

1. One soliton solution

$$u = \frac{(\lambda_1 - \mu_1)^2}{4}\,\mathrm{sech}^2(\Omega),$$

$$q_1 = \sqrt{\alpha_1}_y (\lambda_1 - \mu_1) e^{\frac{\xi_1 + \eta_1}{2}}\,\mathrm{sech}(\Omega_1),\ \ r_1 = \frac{1}{2\sqrt{\alpha_1}} e^{-\frac{\xi_1 + \eta_1}{2}}\,\mathrm{sech}(\Omega_1).$$

2. Two soliton solution

$$u = \partial^2 \ln \tau$$

$$q_1 = \alpha_{1,y} \frac{(\lambda_1 - \mu_1)(\lambda_2 - \mu_1)}{\tau} \left(1 + \alpha_2 \frac{(\lambda_1 - \mu_2)(\mu_2 - \mu_1)}{(\lambda_1 - \lambda_2)(\lambda_2 - \mu_1)} e^{\chi_2} \right) e^{\eta_1}$$

$$q_2 = \alpha_{2,y} \frac{(\lambda_2 - \mu_2)(\lambda_1 - \mu_2)}{\tau} \left(1 + \alpha_1 \frac{(\lambda_2 - \mu_1)(\mu_1 - \mu_2)}{(\lambda_2 - \lambda_1)(\lambda_1 - \mu_2)} e^{\chi_1} \right) e^{\eta_2}$$

$$r_1 = \frac{1 + \alpha_2 e^{\chi_2}}{(\lambda_1 - \lambda_2)\tau} e^{-\xi_1}, \quad r_2 = \frac{1 + \alpha_1 e^{\chi_1}}{(\lambda_2 - \lambda_1)\tau} e^{-\xi_2}$$

where $\chi_i = \eta_i - \xi_i$ $(i = 1, 2)$ and

$$\tau = 1 + \alpha_1 \frac{\lambda_2 - \mu_1}{\lambda_2 - \lambda_1} e^{\chi_1} + \alpha_2 \frac{\mu_2 - \lambda_1}{\lambda_2 - \lambda_1} e^{\chi_2} + \alpha_1 \alpha_2 \frac{\mu_2 - \mu_1}{\lambda_2 - \lambda_1} e^{\chi_1 + \chi_2}.$$

VI. The extended mKP hierarchy and gauge transformation

The Lax equations of mKP hierarchy

$$L_{t_n} = [B_n, L] \quad n \geq 1, \quad B_n = L^n_{\geq 1}. \tag{18}$$

$$L = \partial + v_0 + v_1\partial^{-1} + v_2\partial^{-2} + \cdots$$

The commutativity of ∂_{t_n} and ∂_{t_k} flows gives the zero curvature equation

$$B_{n,t_k} - B_{k,t_n} + [B_n, B_k] = 0.$$

When $n = 2$ and $k = 3$, *mKP equation*:

$$4v_t - v_{xxx} + 6v^2 v_x - 3(D^{-1}v_{yy}) - 6v_x(D^{-1}v_y) = 0. \tag{19}$$

VI(A). Extended mKP hierarchy (exmkPH)

The *extended mKP hierarchy* (exmKPH) is defined by

$$L_{\tau_k} = [B_k + \sum_{i=1}^{N} q_i \partial^{-1} r_i \partial, L], \tag{20a}$$

$$L_{t_n} = [B_n, L], \quad n \neq k, \tag{20b}$$

$$q_{i,t_n} = B_n(q_i), \quad i = 1, \cdots, N \tag{20c}$$

$$r_{i,t_n} = -(\partial B_n \partial^{-1})^*(r_i). \tag{20d}$$

VI(A). Extended mKP hierarchy (exmkPH)

The compatibility of (20a) and (20b) leads to the zero curvature equation for exmKPH

$$B_{k,t_n} - B_{n,\tau_k} + [B_k, B_n] + \sum_{i=1}^{N}[q_i\partial^{-1}r_i\partial, B_n]_{\geq 1} = 0 \tag{21a}$$

$$q_{i,t_n} = B_n(q_i), \quad i = 1, \cdots, N \tag{21b}$$

$$r_{i,t_n} = -(\partial^{-1}B_n^*\partial)(r_i). \tag{21c}$$

the Lax pair for (21)

$$\Psi_{t_n} = B_n(\Psi), \quad \Psi_{\tau_k} = (B_k + \sum^{N} q_i\partial^{-1}r_i\partial)(\Psi).$$

VI(A). Extended mKP hierarchy (exmkPH)

When $n = 2$ and $k = 3$, the first type of mKP with self-consistent sources

$$4v_t - v_{xxx} + 6v^2 v_x - 3D^{-1}v_{yy} - 6v_x D^{-1} v_y + 4\sum_{i=1}^{N}(q_i r_i)_x = 0,$$

$$q_{i,y} = q_{i,xx} + 2vq_{i,x},$$

$$r_{i,y} = -r_{i,xx} + 2vr_{i,x}$$

VI(A). Extended mKP hierarchy (exmkPH)

When $n = 3$ and $k = 2$, the second type of mKPSCS

$$4v_t - v_{xxx} + 6v^2 v_x - 3D^{-1}v_{yy} - 6v_x D^{-1}v_y + \sum_{i=1}^{N}[3(q_i r_{i,xx} - q_{i,xx} r_i) - 3(q_i r_i)_y - 6(v q_i r_i)_x] = 0, \quad (22a)$$

$$q_{i,t} = q_{i,xxx} + 3v q_{i,xx} + \frac{3}{2}(D^{-1}v_y) q_{i,x} + \frac{3}{2} v_x q_{i,x} + \frac{3}{2} v^2 q_{i,x} + \frac{3}{2}\sum_{j=1}^{N}(q_j r_j) q_{i,x}, \quad (22b)$$

$$r_{i,t} = r_{i,xxx} - 3v r_{i,xx} + \frac{3}{2}(D^{-1}v_y) r_{i,x} - \frac{3}{2} v_x r_{i,x} + \frac{3}{2} v^2 r_{i,x} + \frac{3}{2}\sum_{j=1}^{N}(q_j r_j) r_{i,x}, \quad (22c)$$

VI(A). Extended mKP hierarchy (exmkPH)

Proposition 3.

Let $N = 2$, $q_1 := q$, $r_1 = 1$, $q_2 = -1$ and $r_2 = r$, then we obtain the non-standard exmKPH:

$$
\begin{aligned}
& B_{k,t_n} - B_{n,\tau_k} \\
& + [B_k, B_n] + [q - r + \partial^{-1} r_x, B_n]_{\geq 1} = 0 && \text{(23a)} \\
& q_{t_n} = B_n(q), && \text{(23b)} \\
& r_{t_n} = -(\partial^{-1} B_n^* \partial)(r). && \text{(23c)}
\end{aligned}
$$

1. The t_n-reduction

$$L^n_{\leq 0} = 0, \quad or \quad L^n = B_n. \tag{24}$$

the t_n-reduction for the exmKPH,

$$\mathcal{L}_{\tau_k} = [\mathcal{L}^{k/n}_{\geq 1} + \sum_{i=1}^{N} q_i \partial^{-1} r_i \partial, \mathcal{L}] \tag{25a}$$

$$\lambda_i^n q_i = \mathcal{L}(q_i) \tag{25b}$$

$$\lambda_i^n r_i = -\partial^{-1} \mathcal{L}^* \partial(r_i), \quad i = 1, \ldots, N \tag{25c}$$

where $\mathcal{L} = L^n = \partial^n + V_{n-2}\partial^{n-1} + \cdots + V_0\partial$. which can be regarded as (1+1)-dimensional integrable soliton hierarchy with self-consistent sources.

2. The τ_k-reduction

$$L^k_{\leq 0} = \sum q_i \partial^{-1} r_i \partial$$
$$q_{i,t_n} = B_n(q_i)$$
$$r_{i,t_n} = -(\partial^{-1} B_n^* \partial)(r_i)$$

exmKPH reduces to k-constrained mKPH

$$(L_k)_{t_n} = [(L_k)^{n/k}_{\geq 1}, L_k], \tag{26a}$$
$$q_{i,t_n} = B_n(q_i), \tag{26b}$$
$$r_{i,t_n} = -(\partial^{-1} B_n^* \partial)(r_i) \tag{26c}$$

where $L_k = L^k_{\geq 1} + \sum_i q_i \partial^{-1} r_i \partial$, $B_n = (L_k)^{n/k}_{\geq 1}$.

VI(C).Gauge Transformation between exKPH and exmKPH

Theorem 2.

Suppose L, q_i, r_i satisfy exKPH, f is a particular eigenfunction for Lax pair for exKPH, then

$$\tilde{L} := f^{-1}Lf, \quad \tilde{q}_i := f^{-1}q_i, \quad \tilde{r}_i := -D^{-1}(fr_i)$$

satisfies the exmKP hierarchy.

VI(C). Gauge Transformation between exKPH and exmKPH

$$f = W(1) = (-1)^N \frac{\mathrm{Wr}(h_1', \cdots, h_N')}{\mathrm{Wr}(h_1, \cdots, h_N)}$$

the Wronskian solution for exmKP hierarchy is

$$\tilde{L} = \frac{\mathrm{Wr}(h_1, \cdots, h_N, \partial)}{\mathrm{Wr}(h_1', \cdots, h_N')} \partial \left[\frac{\mathrm{Wr}(h_1, \cdots, h_N, \partial)}{\mathrm{Wr}(h_1', \cdots, h_N')} \right]^{-1} \quad (27a)$$

$$\tilde{q}_i = -\dot{\alpha}_i \frac{\mathrm{Wr}(h_1, \cdots, h_N, g_i)}{\mathrm{Wr}(h_1', \cdots, h_N')}, \quad (27b)$$

$$\tilde{r}_i = \frac{\mathrm{Wr}(h_1', \cdots, \widehat{h_i'}, \cdots, h_N')}{\mathrm{Wr}(h_1, \cdots, h_N)}. \quad (27c)$$

VI(D). Solutions for exmKPH

one soliton solution of mKPSCS with $N = 1$ is

$$u = \frac{(\lambda_1 - \mu_1)^2}{4} \operatorname{sech}^2(\Omega),$$

$$q_1 = \sqrt{\alpha_1}_y (\lambda_1 - \mu_1) e^{\frac{\xi_1 + \eta_1}{2}} \operatorname{sech}(\Omega_1),$$

$$r_1 = \frac{1}{2\sqrt{\alpha_1}} e^{-\frac{\xi_1 + \eta_1}{2}} \operatorname{sech}(\Omega_1).$$

two soliton solution for mKPSCS with $N = 2$ is

$$v = \partial \ln f,$$

$$\tilde{q}_1 = \alpha_{1,y} \frac{(\lambda_1 - \mu_1)(\lambda_2 - \mu_1)}{\lambda_1 \lambda_2 \tilde{\tau}} \left(1 + \alpha_2 \frac{(\lambda_1 - \mu_2)(\mu_2 - \mu_1)}{(\lambda_1 - \lambda_2)(\lambda_2 - \mu_1)} e^{\chi_2} \right) e^{\eta_1},$$

$$\tilde{q}_2 = \alpha_{2,y} \frac{(\lambda_2 - \mu_2)(\lambda_1 - \mu_2)}{\lambda_1 \lambda_2 \tilde{\tau}} \left(1 + \alpha_1 \frac{(\lambda_2 - \mu_1)(\mu_1 - \mu_2)}{(\lambda_2 - \lambda_1)(\lambda_1 - \mu_2)} e^{\chi_1} \right) e^{\eta_2},$$

$$\tilde{r}_1 = \frac{\lambda_2 + \alpha_2 \mu_2 e^{\chi_2}}{(\lambda_2 - \lambda_1)\tau} e^{-\xi_1}, \quad \tilde{r}_2 = \frac{\lambda_1 + \alpha_1 \mu_1 e^{\chi_1}}{(\lambda_2 - \lambda_1)\tau} e^{-\xi_2}.$$

VII(A). New extended DKP hierarchy

the shift and the difference operators Γ and Δ,

$$\Gamma(f(l)) = f(l+1) = f^{(1)}(l),$$

$$\Delta(f(l)) = f(l+1) - f(l).$$

$$\Delta^j f = \sum_{i=0}^{\infty} \begin{pmatrix} j \\ i \end{pmatrix} (\Delta^i (f(l+j-i)))\Delta^{j-i}, \quad \begin{pmatrix} j \\ i \end{pmatrix} = \frac{j(j-1)\cdots(j-i+1)}{i!}. \tag{28}$$

adjoint operator Δ^*

$$\Delta^*(f(l)) = (\Gamma^{-1} - I)(f(l)) = f(l-1) - f(l), \tag{29}$$

$$\Delta^{*j} f = \sum_{i=0}^{\infty} \begin{pmatrix} j \\ i \end{pmatrix} (\Delta^{*i}(f(l+i-j)))\Delta^{*j-i}. \tag{30}$$

VII(A). New extended DKP hierarchy

The Lax equation of the DKP hierarchy is

$$L_{t_n} = [B_n, L], \tag{31}$$

$$L = \Delta + f_0 + f_1\Delta^{-1} + f_2\Delta^{-2} + \cdots$$

$B_n = L^n_+$ stands for the difference part of L^n. The zero-curvature equations for DKP hierarchy:

$$B_{n,t_m} - B_{m,t_n} + [B_n, B_m] = 0. \tag{32}$$

with the Lax pair given by

$$\psi_{t_n} = B_n(\psi), \quad \psi_{t_m} = B_m(\psi). \tag{33}$$

VII(A). New extended DKP hierarchy

For $n = 2$, $m = 1$, (32) gives rise to the DKP equation

$$\Delta(f_{0t_2} + 2f_{0t_1} - 2f_0 f_{0t_1}) = (\Delta + 2) f_{0t_1t_1}. \tag{34}$$

The new exDKPH

$$L_{t_n} = [B_n, L], \tag{35a}$$

$$L_{\tau_k} = [B_k + \sum_{i=1}^{N} \psi_i \Delta^{-1} \phi_i, L], \tag{35b}$$

$$\psi_{i,t_n} = B_n(\psi_i), \ \phi_{i,t_n} = -B_n^*(\phi_i), \ i = 1, \cdots, N. \tag{35c}$$

VII(A). New extended DKP hierarchy

Proposition 4. The zero-curvature representation for exDKPH

$$B_{n,\tau_k} - (B_k + \sum_{i=1}^{N} \psi_i \Delta^{-1} \phi_i)_{t_n} + [B_n, B_k + \sum_{i=1}^{N} \psi_i \Delta^{-1} \phi_i] = 0, \tag{36a}$$

$$\psi_{i,t_n} = B_n(\psi_i), \ \phi_{i,t_n} = -B_n^*(\phi_i), \tag{36b}$$

$$\Psi_{t_n} = B_n(\Psi), \quad \Psi_{\tau_k} = (B_k + \sum_{i=1}^{N} \psi_i \Delta^{-1} \phi_i)(\Psi). \tag{37}$$

VII(A). New extended DKP hierarchy

Eg. 1 $n = 1,\ k = 2$,the first type of DKPESCS

$$\Delta(f_{0\tau_2} + 2f_{0t_1} - 2f_0 f_{0t_1}) = (\Delta + 2)f_{0t_1t_1} - \Delta^2 \sum_{i=1}^{N} (\psi_i \phi_i^{(-1)}), \tag{38a}$$

$$\psi_{i,t_1} = \Delta(\psi_i) + f_0\psi_i,\ \phi_{i,t_1} = -\Delta^*(\phi_i) - f_0\phi_i. \tag{38b}$$

Its Lax representation is

$$\Psi_{t_1} = (\Delta + f_0)(\Psi) \tag{39a}$$

$$\Psi_{\tau_2} = (\Delta^2 + (f_0 + f_0^{(1)})\Delta + \Delta(f_0) + f_1^{(1)} + f_1 + f_0^2 + \sum_{i=1}^{N} \psi_i \Delta^{-1} \phi_i)(\Psi). \tag{39b}$$

VII(A). New extended DKP hierarchy

Eg. 1 $n = 2,\ k = 1$, the second type of DKPESCS

$$\Delta(f_{0t_2} + 2f_{0\tau_1} - 2f_0 f_{0\tau_1}) = (\Delta + 2)f_{0\tau_1\tau_1} + \sum_{i=1}^{N}[\Delta^2((f_0 + f_0^{-1} - 2)\psi_i\phi_i^{-1}) + \Delta(\psi_i^{(2)}\phi_i - \psi_i\phi_i^{(-2)}) + \Delta((\Gamma + 1)(\psi_i\phi_i^{(-1)})_{\tau_1})], \quad (40a)$$

$$\psi_{i,t_2} = \Delta^2(\psi_i) + (f_0 + f_0^{(1)})\Delta(\psi_i) + (\Delta(f_0) + f_1^{(1)} + f_1 + f_0^2)\psi_i, \quad (40b)$$

$$\phi_{i,t_2} = -\Delta^{*2}(\psi_i) - \Delta^*((f_0 + f_0^{(1)})\psi_i) - (\Delta(f_0) + f_1^{(1)} + f_1 + f_0^2)\psi_i. \quad (40c)$$

Its Lax representation is

$$\Psi_{t_2} = (\Delta^2 + (f_0 + f_0^{(1)})\Delta + \Delta(f_0) + f_1^{(1)} + f_1 + f_0^2)(\Psi) \quad (41a)$$

$$\Psi_{\tau_1} = (\Delta + f_0 + \sum_{i=1}^{N}\psi_i\Delta^{-1}\phi_i)(\Psi). \quad (41b)$$

1. The t_n- reduction

$$L^n = B_n \quad or \quad L^n_- = 0 \tag{42}$$

drop t_n dependency from (36) and obtain

$$B_{n,\tau_k} = [(B_n)_+^{\frac{k}{n}} + \sum_{i=1}^{N} \psi_i \Delta^{-1} \phi_i, B_n], \tag{43a}$$

$$B_n(\psi_i) = \lambda_i^n \psi_i, \; B_n^*(\phi_i) = \lambda_i^n \phi_i, \; i = 1, 2, \cdots, N, \tag{43b}$$

with the Lax pair given by

$$\Psi_{\tau_k} = ((B_n)_+^{\frac{k}{n}} + \sum_{i=1}^{N} \psi_i \Delta^{-1} \phi_i)(\Psi), \; B_n(\Psi) = \lambda^n \Psi.$$

VII(B). Reductions

Eg. 1 When $n = 2,\ k = 1,$

$$2\Delta(f_{0\tau_1} - f_0 f_{0\tau_1}) = (\Delta + 2) f_{0\tau_1\tau_1} + \sum_{i=1}^{N} [\Delta^2 (f_0 + f_0^{(-1)} - 2)\psi_i \phi_i^{-1} + \Delta(\psi_i^{(2)}\phi_i - \psi_i \phi_i^{(-2)}) + \Delta(\Gamma + 1)(\psi_i \phi_i^{(-1)})_{\tau_1}] \tag{44a}$$

$$\Delta^2(\psi_i) + (f_0 + f_0^{(1)})\Delta(\psi_i) + (\Delta(f_0) + f_1^{(1)} + f_1 + f_0^2)\psi_i = \lambda_i^2 \psi_i, \tag{44b}$$

$$\Delta^{*2}(\psi_i) + \Delta^*((f_0 + f_0^{(1)})\psi_i) + (\Delta(f_0) + f_1^{(1)} + f_1 + f_0^2)\psi_i = \lambda_i^2 \phi_i, \tag{44c}$$

which can be transformed to the first type of Veselov-Shabat equation with self-consistent sources (VSESCS).

2. The τ_k- reduction

$$L^k = B_k + \sum_{i=1}^{N} \psi_i \Delta^{-1} \phi_i.$$

By dropping τ_k dependency , we obtain

$$(B_k + \sum_{i=1}^{N} \psi_i \Delta^{-1} \phi_i)_{t_n} = [(B_k + \sum_{i=1}^{N} \psi_i \Delta^{-1} \phi_i)_+^{\frac{n}{k}}, B_k + \sum_{i=1}^{N} \psi_i \Delta^{-1} \phi_i], \tag{45a}$$

$$\psi_{i,t_n} = (B_k + \sum_{i=1}^{N} \psi_i \Delta^{-1} \phi_i)_+^{\frac{n}{k}}(\psi_i), \tag{45b}$$

$$\phi_{i,t_n} = -(B_k + \sum_{i=1}^{N} \psi_i \Delta^{-1} \phi_i)_+^{\frac{n}{k}*}(\phi_i), \;\; i = 1, 2, \cdots, N, \tag{45c}$$

which is the k-constrained DKP hierarchy.

VII(B). Reductions

Eg. 1 When $n = 1, \ k = 2,$

(45) leads to

$$2\Delta(f_{0t_1} - f_0 f_{0t_1}) = (\Delta + 2) f_{0t_1t_1} + \Delta^2 \sum_{i=1}^{N} (\psi_i \phi_i^{(-1)}), \tag{46a}$$

$$\psi_{i,t_1} = \Delta(\psi_i) + f_0\psi_i, \ \phi_{i,t_1} = -\Delta^*(\phi_i) - f_0\phi_i, \ i = 1, 2, \cdots, N, \tag{46b}$$

which can be transformed to the second type of VSESCS.

VII(C). Dressing method for exDKPH

Assume that

$$L = W\Delta W^{-1}, \tag{47}$$

$$W = \Delta^N + w_1\Delta^{N-1} + w_2\Delta^{N-2} + \cdots + w_N.$$

if W satisfies

$$W_{t_n} = -L_-^n W, \tag{48}$$

then L satisfies DKPH.

VII(C). Dressing method for exDKPH

Assume that h_i satisfies

$$h_{i,t_n} = \Delta^n(h_i), \;\; i = 1, \cdots, N \tag{49}$$

$$W = \frac{1}{Wrd(h_1, \cdots, h_N)} \begin{vmatrix} h_1 & h_2 & \cdots & h_N & 1 \\ \Delta(h_1) & \Delta(h_2) & \cdots & \Delta(h_N) & \Delta \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Delta^N(h_1) & \Delta^N(h_2) & \cdots & \Delta^N(h_N) & \Delta^N \end{vmatrix} \tag{50}$$

$$Wrd(h_1, \cdots, h_N) = \begin{vmatrix} h_1 & h_2 & \cdots & h_N \\ \Delta(h_1) & \Delta(h_2) & \cdots & \Delta(h_N) \\ \vdots & \vdots & \vdots & \vdots \\ \Delta^{N-1}(h_1) & \Delta^{N-1}(h_2) & \cdots & \Delta^{N-1}(h_N) \end{vmatrix}$$

Proposition 5.

W is dressing operator for DKPH.

VII(C). Dressing method for exDKPH

Lemma 6.

If W satisfies (48) and

$$W_{\tau_k} = -L_-^k W + \sum_{i=1}^{N} \psi_i \Delta^{-1} \phi_i W \tag{51}$$

then L satisfies exDKPH.

Let g_i, $\bar{g}_i$ satisfy

$$g_{i,t_n} = \Delta^n(g_i), \quad g_{i,\tau_k} = \Delta^k(g_i) \tag{52a}$$

$$\bar{g}_{i,t_n} = \Delta^n(\bar{g}_i), \quad \bar{g}_{i,\tau_k} = \Delta^k(\bar{g}_i), \ \ i = 1, \ldots, N. \tag{52b}$$

VII(C). Dressing method for exDKPH

$$h_i = g_i + \alpha_i(\tau_k)\bar{g}_i \quad i = 1, \dots, N, \tag{53}$$

$$\psi_i = -\dot{\alpha}_i W(\bar{g}_i),$$

$$\phi_i = (-1)^{N-i} \frac{\mathrm{Wr}(\Gamma h_1, \cdots, \hat{\Gamma h_i}, \cdots, \Gamma h_N)}{\mathrm{Wr}(\Gamma h_1, \cdots, \Gamma h_N)} \tag{54}$$

Proposition 6.

W, L, ψ_i, ϕ_i satisfy the equation for W and exDKPH.

N-soliton solution for the first type of DKPSCS

Let

$$\delta_i = e^{\lambda_i} - 1, \;\; kp_i = e^{\mu_i} - 1,$$

$$g_i := \exp(l\lambda_i + \delta_i t_1 + \delta_i^2 \tau_2) = e^{\xi_i}, \quad \bar{g}_i := \exp(l\mu_i + \kappa_i t_1 + \kappa_i^2 \tau_2) = e^{\eta_i}$$

$$h_i := g_i + \alpha_i(\tau_2)\bar{g}_i = 2\sqrt{\alpha_i}\exp(\frac{\xi_i + \eta_i}{2})\cosh(\Omega_i), \;\; \Omega_i = \frac{1}{2}(\xi_i - \eta_i - \ln\alpha_i). \quad (55)$$

Since $L = W\Delta W^{-1} = \Delta + f_0 + f_1\Delta^{-1} + \cdots$, we have

$$f_0 = Res_\Delta(W\Delta W^{-1}\Delta^{-1}) \qquad (56)$$

then f_0, ψ_i and ϕ_i given by (54) gives rise to the N-soliton solution for first type of DKPSCS.

VII(D). N-soliton solutions for exDKPH

Example. $N = 1$, 1-soliton solution for first type of DKPSCS

$$f_0 = \exp(\frac{\lambda_1 + \mu_1}{2}) \left(\frac{\cosh(\Omega_1 + 2\theta_1)}{\cosh(\Omega_1 + \theta_1)} - \frac{\cosh(\Omega_1 + \theta_1)}{\cosh \Omega_1} \right),$$

$$\theta_1 = \frac{\lambda_1 - \mu_1}{2},$$

$$\psi_1 = -\frac{d\sqrt{\alpha_1}}{d\tau_2}(e^{\mu_1 - \lambda_1}) \exp \frac{\xi_1 + \eta_1}{2} \operatorname{sech} \Omega_1,$$

$$\phi_1 = \frac{e^{-(\lambda_1 + \mu_1)/2} \exp(-\frac{\xi_1 + \eta_1}{2})}{2\sqrt{\alpha_1}} \operatorname{sech}(\Omega_1 + \theta_1).$$

VII(D). N-soliton solutions for exDKPH

Example. $N=2$, 2-soliton solution for first type of DKPSCS

$$f_0 = -\Delta(w_1) = (e^{\lambda_1} + e^{\lambda_2})\Delta(\frac{v_1}{v}),$$

$$\psi_1 = -\frac{\dot{\alpha}_1}{v}\left(1 + \alpha_2 \frac{(e^{\mu_2} - e^{\lambda_1})(e^{\mu_1} - e^{\mu_2})}{(e^{\lambda_2} - e^{\lambda_1})(e^{\mu_1} - e^{\lambda_2})} e^{\chi_2}\right)(e^{\mu_1} - e^{\lambda_1})(e^{\mu_1} - e^{\lambda_2})e^{\eta_1},$$

$$\psi_2 = -\frac{\dot{\alpha}_2}{v}\left(1 + \alpha_1 \frac{(e^{\mu_1} - e^{\lambda_2})(e^{\mu_1} - e^{\mu_2})}{(e^{\lambda_2} - e^{\lambda_1})(e^{\mu_2} - e^{\lambda_1})} e^{\chi_2}\right)(e^{\mu_2} - e^{\lambda_2})(e^{\mu_2} - e^{\lambda_1})e^{\eta_2},$$

$$\phi_1 = \Gamma\left(\frac{1 + \alpha_2 e^{\chi_2}}{(e^{\lambda_1} - e^{\lambda_2})v} e^{-\xi_1}\right), \quad \phi_2 = \Gamma\left(\frac{1 + \alpha_1 e^{\chi_1}}{(e^{\lambda_2} - e^{\lambda_1})v} e^{-\xi_2}\right),$$

with

$$v = 1 + \alpha_1 \frac{e^{\lambda_2} - e^{\mu_1}}{e^{\lambda_2} - e^{\lambda_1}} e^{\chi_1} + \alpha_2 \frac{e^{\mu_2} - e^{\lambda_1}}{e^{\lambda_2} - e^{\lambda_1}} e^{\chi_2} + \alpha_1\alpha_2 \frac{e^{\mu_2} - e^{\mu_1}}{e^{\lambda_2} - e^{\lambda_1}} e^{\chi_1 + \chi_2},$$

$$v_1 = 1 + \alpha_1 \frac{e^{2\lambda_2} - e^{2\mu_1}}{e^{\lambda_2} - e^{\lambda_1}} e^{\chi_1} + \alpha_2 \frac{e^{2\mu_2} - e^{2\lambda_1}}{e^{\lambda_2} - e^{\lambda_1}} e^{\chi_2} + \alpha_1\alpha_2 \frac{e^{2\mu_2} - e^{2\mu_1}}{e^{\lambda_2} - e^{\lambda_1}} e^{\chi_1 + \chi_2}.$$

N-soliton solution for the second type of DKPSCS

$$g_i := \exp(l\lambda_i + \delta_i\tau_1 + \delta_i^2 t_2) = e^{\xi_i},$$

$$\bar{g}_i := \exp(l\mu_i + \kappa_i\tau_1 + \kappa_i^2 t_2) = e^{\eta_i}$$

$$h_i := g_i + \alpha_i(\tau_1)\bar{g}_i = 2\sqrt{\alpha_i}\exp(\frac{\xi_i + \eta_i}{2})\cosh(\Omega_i).$$

Then

$$f_0 = Res_\Delta(W\Delta W^{-1}\Delta^{-1}),\ f_1 = Res_\Delta(W\Delta W^{-1})$$

together with ψ_i and ϕ_i given by (54) presents the N-soliton solution for (40).

Thank you

for your attention!