Does the Supersymmetric Integrability Imply the Integrability of Bosonic Sector?

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Plan:

1.) Integrability.

2.) Classification of 2 component KdV like system.
   a.) Generalized symmetries
   b.) Recursion operator
   c.) Lax operator

3.) New system of interacted two KdV equations and its Lax operator.

4.) Manin-Radul N=1 Supersymmetric Kadomtsev – Pietvisahvilli hierarchy.
   a.) $MRSKP_{3,0}$ as supersymmetric Sawada – Kotera equation
   b.) $MRSKP_{5,0}$ the bosonic sector gives us this new system
Korteweg de Vries equation

\[ u_t = u_{xxx} + 6uu_x. \]

Many different generalization to 2-component system.

I.) Svinolupov 1991

\[ u_t^i = u_{xxx}^i + a_{j,k}^i u_j u^k_x, \]

where the constants \( a \) satisfy

\[ a^n_{j,k} ( a^j_{n,k} a^r_{m,s} - a^i_{m,r} a^r_{n,s} ) + \text{cyclic} ( j, k, m ) = 0 \]

and hence they are the structural constants of Jordan algebra.

It has the infinite number of generalized symmetries

II.) Gűrses and Karasu 1998

\[ u_t^i = b_j^i u_{xxx}^i + s_{j,k}^i u_j u^k_x, \]

where \( b, s \) are constants
Investigated conditions on \( b, s \) in order to find the recursion operator of second and fourth order.

Decomposable hereditary operator Wen – Xiu Ma, Fordy, Antonowicz

\[
\begin{pmatrix}
    u_t \\
    v_t
\end{pmatrix} = \begin{pmatrix}
    0 & a_0 \partial \\
    a_0 \partial & a_1 \partial
\end{pmatrix} \begin{pmatrix}
    \frac{\delta H}{\delta u} \\
    \frac{\delta H}{\delta v}
\end{pmatrix} = \begin{pmatrix}
    0 & M_0 \\
    M_0 & M_1
\end{pmatrix} \begin{pmatrix}
    \frac{\delta H}{\delta u} \\
    \frac{\delta H}{\delta v}
\end{pmatrix}
\]

where \( M_i = c_1 \partial^3 + d_i (\partial u_i + u_i \partial) \)

Definition of Integrability of a system of equations:

A.) If possesses the recursion operator.

B.) Has infinite number of conserved quantities and generalized symmetries.

C.) Has the Lax representation.
Foursov in 2003 investigated 2-component kdV system

\[ u_t = F[u,v] \]
\[ v_t = G[u,v] \]

where F and G are polynomial functions of \( u, v, u_x, v_x \)

Definition: A system of t-independent evolution equation

\[ u_t = Q_1[u,v] \]
\[ v_t = Q_2[u,v] \]

Is said to be a generalized symmetry if their flows commute

\[ D_K(Q) - D_Q(K) = 0 \]

where \( Q = (Q_1, Q_2) \quad K[u,v] = (F[u,v], G[u,v]) \) and \( D_K \) is a Frechet derivative
Foursov using CA found 5 systems with finite number of generalized symmetries and conserved quantities, 3 of them are known to be integrable

A.) Hirota – Satsuma

\[ u_t = u_{xxx} + 6 u u_x - 12 v v_x \]
\[ v_t = -2 v_{xxx} - 6 u u_x \]

B.) Ito system

\[ u_t = u_{xxx} + 3 u u_x + 3 v v_x \]
\[ v_t = u_x v + u v_x \]

C.) Drinfeld – Sokolov

\[ u_t = u_{xxx} + 2 v u_x + u v_x \]
\[ v_t = u u_x \]
Two of them are new

D.)

\[
\begin{align*}
    u_t &= u_{xxx} + v_{xxx} + 2\, vu_x + 2\, uv_x \\
    v_t &= v_{xxx} - 9\, uu_x + 6\, vu_x + 3\, uv_x + 2\, vv_x 
\end{align*}
\]

This system has been also found by Meshkov.

E.

\[
\begin{align*}
    u_t &= 4\, u_{xxx} + 3\, v_{xxx} + 4\, uu_x + vu_x + 2\, uv_x \\
    v_t &= 3\, u_{xxx} + v_{xxx} - 4\, uu_x - 2\, uv_x - 2\, vv_x 
\end{align*}
\]

Foursov found conserved densities of weight 2, 4, 8, 10, 12, 14 and generalized symmetries of weight 9, 11, 13, 15 and 19.

We show that the last system is integrable because it has the Lax representation and has the Bi–Hamiltonian structure.
There is a possibility to construct 2 component system of interacted KdV equations with the nonlocal generalized symmetries

\[
m_t = m_{xxx} - 3 \, n_{xxx} - 3 \, m_x (4 \, m - 9 \, n) + 3 \, n_x (8 \, m - 15 \, n)
\]
\[
n_t = -3 \, m_{xxx} + 4 \, n_{xxx} + 12 \, m_x \, n + 6 \, n_x (m - 4 \, n)
\]

The Lax operator classification of 3 system:

i.) Hirota –i Satsuma

\[ L = (\partial^2 + u + v)(\partial^2 + u - v) = L_{KdV} L_{KdV} \quad \Rightarrow \quad L_t = [L_{4}^{\frac{3}{4}}, L] \]

Recursion operator of 4 order

ii.) Drinfeld – Sokolov

\[ L = (\partial^3 + u \partial + \frac{u_x}{2})(\partial^3 - v \partial - \frac{v_x}{2}) = L_{KK} L_{KK} \quad \Rightarrow \quad L_t = [L_{2}^{\frac{1}{2}}, L] \]

Recursion operator of 6 order

iii.) New equation

\[ L = (\partial^3 + \frac{2}{3} u \partial + \frac{1}{3} u_x)(\partial^2 - \frac{1}{3} v) = L_{KK} L_{KdV} \quad \Rightarrow \quad L_t = [L_{5}^{\frac{3}{5}}, L] \]

Recursion operator of 10 order
Bi - Hamiltonian structure of new equation:

\[
\begin{pmatrix}
L_{KdV} \\
L_{KK}
\end{pmatrix}
= \begin{pmatrix}
Hirota – Satsuma & new equation \\
new equation & Drinfeld – Sokolov
\end{pmatrix}
\]

\[
\begin{pmatrix}
\frac{\delta H}{\delta u} \\
\frac{\delta H}{\delta v}
\end{pmatrix}
= \begin{pmatrix}
3 \partial^3 + \partial u + u \partial & 0 \\
0 & 3 \partial^3 - 2 \partial v - 2 v \partial
\end{pmatrix}
\begin{pmatrix}
\frac{\delta H}{\delta u} \\
\frac{\delta H}{\delta v}
\end{pmatrix}
\]

\[
H = \int dx \left( v^2 + 4 u^2 + 6 uv \right).
\]
The recursion operator has been found using the method of Gurses, Karasu and Sokolov.

\[ L_{t_3} = [(L^5)^+, L], \quad L_{t_{13}} = [(L^2 L^5)^+, L], \quad (L^2 L^5)_\Delta = (L^2 (L^5)^+ + L^2 (L^5)_\Delta)_\Delta \]

\[ (L^2 (L^5)^+)_\Delta = Q = A_9 \partial^9 + \ldots + A_0 \]

\[ L_{t_{13}} = L^2 L_{t_3} + [Q, L] \]

\[ R = \begin{pmatrix} -\frac{18}{125} \partial^{10} + 268 \text{ terms}, & -\frac{11}{375} \partial^{10} + 268 \text{ terms} \\ -\frac{11}{375} \partial^{10} + 268 \text{ terms}, & -\frac{7}{375} \partial^{10} + 268 \text{ terms} \end{pmatrix} \]

and we can obtain the next Hamiltonian structure factorizing \( R = P Z \) where

\[ Z \begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} \frac{6}{125} \partial^7 + 74 \text{ terms}, & -\frac{11}{375} \partial^7 + 104 \text{ terms} \\ -\frac{11}{375} \partial^7 + 104 \text{ terms}, & -\frac{7}{375} \partial^7 + 104 \text{ terms} \end{pmatrix} \begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} \frac{\delta H}{\delta u} \\ \frac{\delta H}{\delta v} \end{pmatrix} \]

and

\[ H = \int dx \left( 21 u_{10x} u + 26 v_{10x} u + 8 v_{10x} v + 158 \text{ terms} \right) \]
The supersymmetric Integrability for N=1.

The Manin – Radul supersymmetric N=1 KP hierarchy $SKP_{r,m}$

$$L := D^r + \sum_{i=0}^{r-2} v_i D^i + \sum_{j=1}^{m} \Phi_j D^{-j}$$

where $D = \partial_\theta + \theta \partial$ and $v$ and $\Phi$ are the superbosonic or superfermionic superfields

For even $r$ it is well known hierarchy.

A. For $r=3$ and $m=0$ we have

$$\Lambda = D^3 + \Phi \Rightarrow L_{t_k} = 9 [ \Lambda, \Lambda_3^{k+3} ]$$

where $\Phi = \zeta + \theta u$ and $\xi$ is a fermionic, $u$ is bosonic function.

$$\Phi_{t,2} = \Phi_x$$

$$\Phi_{r,7} = \left( (D \Phi)_{xx} + \frac{1}{2} (D \Phi)^2 + 2 \Phi \Phi_x \right)$$

$$\Phi_{t,10} = \Phi_{5x} + 5 \Phi_{xxx} (D \Phi) + 5 \Phi_{xx} \Phi_x + 5 \Phi_x (D \Phi)^2$$
There are several interesting observations:

A.) $t$ is usual time while $\mathcal{T}$ is an odd time

B.) for $k=10$ we have the susy N=1 Sawada–Kotera equation and in components it is

\[
\begin{align*}
    u_t &= u_{5x} + 5u_{xxx}u + 5u_xu_{xx} + 5u^2u_x - 5\zeta_{xxx}\zeta_x \\
    \zeta_t &= \zeta_{5x} + 5u\zeta_{xxx} + 5u_x\zeta_{xx} + 5u^2\zeta_x
\end{align*}
\]

and has been recently considered by Tian and Liu.

C.) The trace formula for Lax operator gives us conserved quantities which however are not reduced to the classical conserved charges of Sawada–Kotera equation.

\[
\begin{align*}
    H_1 &= \int d\theta dx \Phi\Phi_x = 2\int dx u\zeta_x \\
    H_2 &= \int d\theta dx 3(D\Phi_{xx})D\Phi_x + 2(D\Phi)^3 = 6\int dx (\zeta_{xxx}u + \zeta_xu^2)
\end{align*}
\]

D.) The odd time hierarchy is generated by even hamiltonian structure

\[
\Phi_{\tau,7} = P \frac{\delta H_1}{\delta \Phi} = (D^5 + 2\partial\Phi + 2\Phi\partial + D\Phi D)\frac{\delta H_1}{\delta \Phi}
\]

$P$ is the usual hamiltonian operator of the supersymmetric KdV equation which is connected with the N-1 supersymmetrical Virasoro algebra.
E. Second Hamiltonian Structure of susy Sawada – Kotera equation is odd

\[ \Omega = (D \partial^2 + 2 \partial \Phi + 2 \Phi \partial + D \Phi D) \partial^{-1} (D \partial^2 + 2 \partial \Phi + 2 \Phi \partial + D \Phi D), \]

\[ \Phi_t = \Omega \frac{\delta H_1}{\delta \Phi}. \]

Proof:
A.) We have to check the Jacobi identity

\[ <\alpha, P'_{(P\beta)\gamma}> + <\beta, P'_{(P\gamma)\alpha}> + <\gamma, P'_{(P\alpha)\beta}> = 0, \]

\[ <\alpha, \beta> = \int dx \, d\theta \alpha \beta. \]

where \( P'_{\alpha} \) denote the Gâteaux derivative along the vector \( \alpha \).

B.) R- matrix approach to Bi-Hamiltonian systems

\[ \frac{\partial L}{\partial t} = \Gamma_1 \Delta F = (L(\Delta F))_+ - (\Delta F)_+L, \]

\[ \frac{\partial L}{\partial t} = \Gamma_2 \Delta F = L((\Delta F)L)_+ - (\Delta F)_+L, \]

\[ \frac{\partial L}{\partial t} = \Gamma_3 \Delta F = L(L(\Delta F)L)_+ - (L(\Delta F)L)_+L - L((\Delta F)L)_+L + L(L(\Delta F))_+L \]
where gradient is parametrized as

\[ \Delta H = \sum_{k>0} \partial^{-k-1}(-D \frac{\delta H}{\delta a_k} + \frac{\delta H}{\delta b_k}), \]

\[ L = \sum_{k>0} (a_k + b_k D) \partial^k. \]

We have to use the Dirac reduction, because we need to embed the Lax operator into a larger space.

Ad 1. For \( \Gamma_1 \) it is impossible to carry out such reduction.

Ad 2. For \( \Gamma_2 \) we have 3 dimensional matrix and Dirac reduction gives us \( P \) operator which generate odd time flows.

Ad 3. For \( \Gamma_3 \) we have 6 dimensional matrix and Dirac reduction gives us \( \Omega \) which generates the even time flows.

Second Hamiltonian structure. Is obtained from the factorization of the recursion operator which was derived by Tian and Liu

\[ J = \partial^2 + (D \Phi) - \partial^{-1}(D \Phi)_x + \partial^{-1} \Phi_x D + \Phi_x \partial^{-1} D \]

\[ J \Phi_t = \frac{\delta H}{\delta \Phi} \]
The SKP hierarchy for $r=5$ and $m=0$. Results

$$L = D^5 + \frac{1}{3}(\partial Z + \partial Z) - \frac{1}{3}D P D$$

where $Z = \zeta_1 + \theta z_1$, $P = p_1 + \theta \zeta_2$ are superfermionic and superbosonic supermultiplet.

Computing

$$\frac{\partial L}{\partial t} = 5 [L, (L^5)_+]$$

we obtain

$$Z_t = 4 Z_{xxx} + 3 P_{xxx} - 2 Z_x (DZ + DP) + Z (6DZ_x + 2DP_x) + P (3DZ_x + DP_x) - P_x DP$$

$$P_t = 3 Z_{xxx} + P_{xxx} + 8Z_x DZ - Z (8DZ_x + 6DP_x) + P_x (4DZ + DP) - P_x (4DZ_x + 3DP_x)$$

The bosonic sector in which

$$Z = \theta u, \quad P = \theta v.$$

gives us new 2-component interacted KdV equations.