Dark soliton solution of Sasa–Satsuma equation

\[ i u_t = u_{xxx} + 2 |u|^2 u + i \left( u_{xxxx} + 6 |u|^2 u_x + 3 (|u|^2)_x u \right) \]


\[ \begin{align*}
\text{Gauge transformation} & \quad u \mapsto u e^{i(kx-\omega t)} \\
\text{Galilean transformation} & \quad x \mapsto x - vt
\end{align*} \]

\[ u_t = u_{xxx} + 6 |u|^2 u_x + 3 (|u|^2)_x u \]
NLS \quad i \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \pm 2 |u|^2 u

Sign \pm cannot be changed by gauge, Galilean, scale transformations

+ : focusing equation
  bright soliton \quad u \to 0 \quad \text{as} \quad x \to \pm \infty

- : defocusing equation
  dark soliton \quad u \to A_\pm e^{i(Kx - \Omega t)} \quad \text{as} \quad x \to \pm \infty
Bright soliton \( u = \frac{g}{f} \) \((f: \text{real})\)

\( \leftarrow \) 2 component KP hierarchy

(Toda molecule equation)

Dark soliton \( u = \frac{g}{f} e^{i(k'x - \Omega t)} \) \((f: \text{real})\)

\( \leftarrow \) 1 component KP hierarchy

with negative weight time

(Toda lattice equation)

\( \uparrow \)

Regularity \( f \neq 0 \) for \( x, t : \text{real} \)

(physical solution)
• Without regularity condition, both sol. for \{focusing\} eq. from \{2-comp.\} KP
  \[\text{defocusing}\]
  \{1-comp.\}

• Selection of sign \pm by regularity condition

Motivation
Which kind of physical solution is possible for NLS type equation of sign + or -?
coupled NLS, derivative NLS, Sasa-Satsuma, ...
Usually constructing bright soliton solution (regular) is easier because of more parameters in multi-component KP.

Dark soliton is more difficult.

Multi-dark soliton for coupled system

Hu, Chaos, Solitons & Fractals 7 (1996) 211.
Sasa-Satsuma eq.

\[ u_t = u_{xxxx} + 6|u|^2 u_x + 3(1u^2)_x u \]

bright soliton with internal freedom

double hump soliton

oscillating double hump soliton

degeneration of coupled system

3-component CKP hierarchy

Dark soliton of Sasa-Satsuma eq.?
\[ U_t = U_{xxx} + 6 \varepsilon (|u|^2 - 1) U_x + 3 \varepsilon (|u|^2)_x U \]

\[ \varepsilon = \pm \]

\[ \varepsilon = + : \text{focusing} \]

\[ \varepsilon = - : \text{defocusing} \]

**Purpose**

Construct solutions for plane wave
boundary condition \(|u| \to 1\) as \(x \to \pm \infty\)

for \(\varepsilon = \pm\).
\[ u = \frac{g}{f} e^{i(\lambda x - \lambda^3 t)} \quad (f: \text{real}) \]

\[ \begin{cases} \frac{g}{f} \to 1 \quad \text{as} \quad x \to -\infty \\ \frac{g}{f} \to \alpha \quad \text{as} \quad x \to +\infty \quad (|\alpha| = 1) \end{cases} \]

\[ \begin{cases} D_x^2 f \cdot f = 4 \varepsilon (|g|^2 - f^2) \\ (D_x^3 - D_t + 3i\lambda D_x^2 + 3(2\varepsilon - \lambda^2)D_x + 6i\varepsilon \lambda) g \cdot f = 6i\varepsilon \lambda r g \\ (D_x + 2i\lambda) g \cdot g^* = 2i rf \end{cases} \]

\[ r: \text{auxiliary variable (real)} \]

\[ \lambda \neq 0 \]
• CKP hierarchy (cf. bright soliton)
• 1-component (for non-vanishing boundary condition)
• 2-different discrete shift for $g$ and $g^*$
  \[ f = \tau_{00}, \quad g = \tau_{10}, \quad g^* = \tau_{01}, \quad r = \tau_{11} \]
  (cf. $f = \tau_0, \quad g = \tau_1, \quad g^* = \tau_{-1}$ for NLS)

  \[ \text{CKP} \Rightarrow \tau_{10} = \tau_{0,-1}, \quad \tau_{01} = \tau_{-1,0} \]

• complicated reduction condition
\[ D_x f \cdot f = 4 \varepsilon (|g|^2 - f^2) \]

\[ \uparrow \]

\[ D_x D_u f \cdot f = 2 (g g^* - f^2) \]

\[ D_x D_v f \cdot f = 2 (g^* g - f^2) \]

\[ D_u + D_v = \varepsilon D_x \quad (\text{reduction}) \]

\[ \uparrow \quad f = \tau_{00}, \quad g = \tau_{10} = \tau_{0,-1}, \quad g^* = \tau_{01} = \tau_{-1,0} \]

\[ D_x D_u \tau_{00} \cdot \tau_{00} = 2 (\tau_{10} \tau_{-1,0} - \tau_{00} \tau_{00}) \]

\[ D_x D_v \tau_{00} \cdot \tau_{00} = 2 (\tau_{01} \tau_{0,-1} - \tau_{00} \tau_{00}) \]

\[ D_u + D_v = \varepsilon D_x \]
Solution \( T_{kl} = \det \left( m_{ij}^{kl} \right) \) for \( 1 \leq i, j \leq 2N \)

\[
m_{ij}^{kl} = \delta_{j,2N+1-i} + \frac{1}{p_i + p_j} \left( \frac{i\lambda - p_i}{i\lambda + p_i} \right)^k \left( \frac{i\lambda + p_i}{i\lambda - p_j} \right)^l e^{\xi_i + \xi_j}
\]

\[
\xi_i = p_i x + p_i^3 t + \frac{1}{i\lambda - p_i} u - \frac{1}{i\lambda + p_i} v + \xi_i^{(0)}
\]

\[
T_{kl} = \det \left( \delta_{j,2N+1-i} e^{-\xi_j - \xi_{2N+1-i}} + \frac{1}{p_i + p_j} \left( \frac{i\lambda - p_i}{i\lambda + p_i} \right)^k \left( \frac{i\lambda + p_i}{i\lambda - p_j} \right)^l \right)
\]

Reduction condition \((D_u + D_v = 3D_x)\), i.e., \((\partial u + \partial v) T_{kl} \)

\[
\frac{1}{i\lambda - p_i} + \frac{1}{i\lambda - p_{2N+1-i}} - \frac{1}{i\lambda + p_i} - \frac{1}{i\lambda + p_{2N+1-i}} = 3(p_i + p_{2N+1-i})
\]
\[ p_i = \delta_i + i \sqrt{\lambda^2 - \varepsilon - \delta_i^2 + \sqrt{1 - 4 \lambda^2 (\varepsilon + \delta_i^2)}} \]
\[ p_{2N+1-i} = \delta_i - i \sqrt{\ldots} \]

(1) \[ 1 - 4 \lambda^2 (\varepsilon + \delta_i^2) > 0 \]
\[ \lambda^2 - \varepsilon - \delta_i^2 + \sqrt{1 - 4 \lambda^2 (\varepsilon + \delta_i^2)} > 0 \]

(2) \[ 1 - 4 \lambda^2 (\varepsilon + \delta_i^2) > 0 \]
\[ \lambda^2 - \varepsilon - \delta_i^2 - \sqrt{1 - 4 \lambda^2 (\varepsilon + \delta_i^2)} < 0 \]

\[ \Rightarrow p_{2N+1-i} = p_i^* \]

\[ \Rightarrow p_i, p_{2N+1-i} : \text{real} \]

possible to get regular solutions for both \( \varepsilon = +1 \) and \(-1\).
$\varepsilon = -1$ defocusing
hole soliton

$\varepsilon = +1$ focusing

double hole soliton
without oscillation
1-component CKP + reduction

⇒ Sasa-Satsuma eq. and its solution for non-vanishing boundary condition for both defocusing ($\varepsilon = -1$) and focusing ($\varepsilon = +1$) cases.

- Soliton of double hole type (for $\varepsilon = -1$)
- No oscillation of double hole
  No internal freedom (Soliton is characterized by its wave number and phase parameter only.)
- Breather type solution for $\varepsilon = +1$ and $-1$.
  (homoclinic orbit solution)