On the different kinds of KP equation with self-consistent sources

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Outline

1 Backgrounds and the SGP

Soliton equations with self-consistent sources (SESCSs)

∗ An integrable coupled generalization of soliton equations

∗ Important physical applications: plasma physics, hydrodynamics and solid-state physics

∗ Interactions between different solitary waves

⋇ The KPESCS and the KdVESCS

⋆ *V.K. Mel'nikov, Lett. Math. Phys. 7 (1983)129.*

⋆ *E.V. Doktorov and R.A. Vlasov, Opt. Acta 30 (2) (1983) 223.*

⋆ *V.K. Mel'nikov, Commun. Math. Phys., 112 (1987) 639; 112 (1987) 639; 126 (1989) 201.*

⋆ *C. Claude, A. Latifi and J. Leon, J. Math. Phys. 32 (12) (1991) 3321.*

Examples

- The KP equation: $(4u_t - u_{xxx} - 6uu_x)_x - 3u_{yy} = 0$

※The KPESCS:

$$\begin{cases} (4u_t - u_{xxx} - 6uu_x)_x - 3u_{yy} = -\sum_{j=1}^{K} (\phi_j \psi_j)_{xx}, \\ \phi_{j,y} = \phi_{j,xx} + u\phi_j, \quad j = 1, 2, \cdots, K \\ -\psi_{j,y} = \psi_{j,xx} + u\psi_j, \quad j = 1, 2, \cdots, K \end{cases}$$

- The KdV equation: $u_t + 6uu_x + u_{xxx} = 0$

※The KdVESCS:

$$\begin{cases} u_t + u_{xxx} + 6uu_x = -\sum_{j=1}^{K} (\phi_j^2)_x, \\ \phi_{j,xx} + (\lambda_j + u)\phi_j = 0, \quad j = 1, 2, \cdots, K \end{cases}$$

⋆ *V. K. Mel' nikov, Inverse Problems, 6 (1990) 233.*

⋆ J. *Leon and A. Latifi, J. Phys. A, 23 (1990) 1385.*

⋆ *Y.B. Zeng, Phys. A, 262 (1999) 405.*

A systematic way to construct a SESCS

Restricted flows of a soliton equation $\Longleftrightarrow$ The stationary case of the SESCS.

$\Downarrow$

Lax pair for the SESCS

$\Downarrow$

Nonlinear form of the SESCS

Methods to solve a SESCS

—Inverse scattering transform,

—Darboux transformation

—Hirota's bilinear method

. . . .

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⋆ *A.B. Khasanov, G.U. Urazboev, J. Math. Phys. Anal. Geom. 2 (3) (2006) 287.*

⋆ *H.Y. Wang, X.B. Hu and Gegenhasi, J. Comp.Appl.Math. 202 (2007) 133.*

Source generation procedure (SGP)

- Motivation —— looking for a direct method to construct and solve a SESCS
- Basic idea —— Hirota's bilinear form of original equations without sources
  —— Determinant or pfaffian solutions of equations without sources

⋆ *X.B. Hu, H.Y. Wang, Inverse Problems 22 (2006) 1903.*

- Steps of the procedure:

Step 1. Give the exact solution of an original bilinear equation: determinant or pfaffian form including some arbitrary constants c_{ij}.

Step 2. Generalize some constants c_{ij} into some functions of one independent variable, e.g., $c_{ij}(x)$, then we get the generalized determinant (pfaffian).

(x: some independent variable)

Step 3. Seek new coupled equations $\xrightarrow{solution}$ Generalized determinant (pfaffian)

2 Construction of the KPESCS

§2.1 Elementary Properties of Pfaffian

1. An N-th order Pfaffian can be defined by the determinant of a $2N \times 2N$ antisymmetric matrix

$$A = \det(a_{i,j})_{2N\times 2N}, \quad a_{i,j} = -a_{j,i}$$

$$A = \mathrm{pf}(1,2,\cdots,2N)^2, \quad \mathrm{pf}(i,j) = a_{i,j}, \quad and \quad \mathrm{pf}(i,j) = -\mathrm{pf}(j,i), \quad 1 \leqslant i,j \leqslant 2N.$$

Examples:

$$N = 1, A = (a_{1,2})^2 = \mathrm{pf}(1,2)^2, \text{ then } \mathrm{pf}(1,2) = a_{12}$$

$$N = 2, A = (a_{1,2}a_{3,4} - a_{1,3}a_{2,4} + a_{1,4}a_{2,3})^2 = \mathrm{pf}(1,2,3,4)^2.$$

That is

$$\mathrm{pf}(1,2,3,4) = a_{1,2}a_{3,4} - a_{1,3}a_{2,4} + a_{1,4}a_{2,3}$$

2. An N-th order Pfaffian can be expanded as

$$\mathrm{pf}(1,2,\cdots,2N)=\sum_{j=1}^{2N}(-1)^{j}(1,j)(2,3,\cdots,\hat{j},\cdots,2N)$$

or

$$\mathrm{pf}(1,2,\cdots,2N)=\sum_{1\leqslant i<j\leqslant 2N}(-1)^{i+j-1}\mathrm{pf}(i,j)\mathrm{pf}(1,\cdots,\hat{i},\cdots,\hat{j},\cdots,2N),$$

where the hat on the letter denotes the deletion of the letter under it.

3. An N-th order determinant

$$B=\det(b_{i,j})_{1\leqslant i,j\leqslant N}$$

can be expressed as an N-th order Pfaffian

$$B=\mathrm{pf}(1,2,\cdots,N,N^*,\cdots,2^*,1^*),$$

where Pfaffian entries are defined by

$$\mathrm{pf}(i,j^*)=b_{i,j},\ \ \mathrm{pf}(i,j)=\mathrm{pf}(i^*,j^*)=0.$$

4. Pfaffian identities:

$$\begin{aligned}&\mathrm{pf}(a_1,a_2,a_3,a_4,1,\cdots,2N)\mathrm{pf}(1,2,\cdots,2N)\\&=\sum_{j=2}^{4}(-1)^j\mathrm{pf}(a_1,a_j,1,\cdots,2N)\mathrm{pf}(a_2,\cdots,\hat{a}_j,\cdots,a_4,1,\cdots,2N),\end{aligned}$$

$$\begin{aligned}&\mathrm{pf}(a_1,a_2,a_3,1,\cdots,2N-1)\mathrm{pf}(1,2,\cdots,2N)\\&=\mathrm{pf}(a_1,1,\cdots,2N-1)\mathrm{pf}(a_2,a_3,1,2,\cdots,2N)\\&\quad-\mathrm{pf}(a_2,1,\cdots,2N-1)\mathrm{pf}(a_1,a_3,1,2,\cdots,2N)\\&\quad+\mathrm{pf}(a_3,1,\cdots,2N-1)\mathrm{pf}(a_1,a_2,1,2,\cdots,2N).\end{aligned}$$

The Jacobi determinant identity and the Plucker relation are special examples of the above pfaffian identities.

[1] *R. Hirota, Direct method in soliton theory (In English), (Edited and Translated by A. Nagai, J. Nimmo and C. Gilson Cambridge University Press, 2004. 6).*

§2.2 Construction of KPESCS

$$\textit{KP equation} \qquad (4u_t - u_{xxx} - 6uu_x)_x - 3u_{yy} = 0, \tag{1}$$

$$\Big\Downarrow \; u = 2(\ln\tau)_{xx},$$

$$\textit{Bilinear KP} \qquad (D_x^4 - 4D_xD_t + 3D_y^2)\tau\cdot\tau = 0, \tag{2}$$

- Hirota bilinear operator [1]:

$$D_x^m D_t^n f(x,t)\cdot g(x,t) = \frac{\partial^m}{\partial y^m}\frac{\partial^n}{\partial s^n} f(x+y,t+s)g(x-y,t-s)|_{s=0,y=0},$$

Step 1. Grammian determinant solution of Eq. (2):

$$\tau = \det(\beta_{ij} + \int^{x} f_i \tilde{f}_j \mathrm{d}x)_{1\leqslant i,j\leqslant N}, \quad \beta_{ij} = constant, \tag{3}$$

f_i and $\tilde{f}_j$ satisfy the dispersion relation:

$$\frac{\partial f_i}{\partial x_n} = \frac{\partial^n f_i}{\partial x^n}, \quad \frac{\partial \tilde{f}_i}{\partial x_n} = (-1)^{n-1}\frac{\partial^n \tilde{f}_i}{\partial x^n}, \quad (x_1 = x, \ \ x_2 = y, \ \ x_3 = t). \tag{4}$$

Step 2. Change the function τ in (3) into the following form:

$$f = \det(a_{ij})_{1\leqslant i,j\leqslant N} = \mathrm{pf}(1,2,\cdots,N,N^*,\cdots,1^*) = \mathrm{pf}(\cdot), \tag{5}$$

Pfaffian elements:

$$\mathrm{pf}(i,j^*) = a_{ij} = \beta_{ij}(t) + \int^x f_i\tilde{f}_j \mathrm{d}x, \quad \mathrm{pf}(i,j) = \mathrm{pf}(i^*,j^*) = 0, \quad i,j = 1,2,\cdots,N,$$

$\beta_{ij}(t)$ satisfies the following condition:

$$\beta_{ij}(t) = \begin{cases} \beta_i(t), & i = j \text{ and } 1 \leqslant i \leqslant K \leqslant N, \ \ K,N \in Z^+, \\ \beta_{ij}, & \text{otherwise}, \end{cases}$$

Step 3. Introduce other new functions:

$$g_i = 2\sqrt{2\dot{\beta}_i(t)}\mathrm{pf}(d_0^*, 1, \cdots, N, N^*, \cdots, \hat{i^*}, \cdots, 1^*), \;\; i = 1, 2, \cdots, K, \tag{6}$$

$$h_i = 2\sqrt{2\dot{\beta}_i(t)}\mathrm{pf}(d_0, 1, \cdots, \hat{i}, \cdots, N, N^*, \cdots, 1^*), \;\; i = 1, 2, \cdots, K \tag{7}$$

$\dot{\beta}_i(t)$: the derivative of $\beta_i(t)$ with respect to t.

New pfaffian elements:

$$\mathrm{pf}(d_m^*, i) = \frac{\partial^m}{\partial x^m} f_i, \quad \mathrm{pf}(d_n, j^*) = \frac{\partial^n}{\partial x^n} \tilde{f}_j,$$

$$\mathrm{pf}(d_m^*, d_n) = \mathrm{pf}(d_m^*, d_n^*) = \mathrm{pf}(d_m, d_n) = \mathrm{pf}(d_m^*, j^*) = \mathrm{pf}(d_m, i) = 0.$$

$$m, n \in Z, \;\; i, j = 1, 2 \cdots, N$$

f, g_i and h_i in (5)-(7) satisfy the bilinear equations:

$$(D_x^4 - 4D_xD_t + 3D_y^2)f \cdot f = \sum_{i=1}^{K} g_i h_i, \tag{8}$$

$$(D_y + D_x^2)f \cdot g_i = 0, \quad i = 1, 2, \cdots, K, \tag{9}$$

$$(D_y + D_x^2)h_i \cdot f = 0, \quad i = 1, 2, \cdots, K. \tag{10}$$

$\Downarrow$

Bilinear form of KPESCS

- Proof of equations (8)-(10)

► According to the formula of the derivative of a determinant, we have

$$
\begin{aligned}
f_t = &\sum_{j=1}^{K} \dot{\beta}_j(t)\mathrm{pf}(1, 2, \cdots, \hat{j}, \cdots, N, N^*, \cdots, \hat{j}^*, \cdots, 1^*) \\
&+\mathrm{pf}(d_2, d_0^*, \bullet) - \mathrm{pf}(d_1, d_1^*, \bullet) + \mathrm{pf}(d_0, d_2^*, \bullet), \\
f_{xt} = &\sum_{j=1}^{K} \dot{\beta}_j(t)\mathrm{pf}(d_0, d_0^*, 1, 2, \cdots, \hat{j}, \cdots, N, N^*, \cdots, \hat{j}^*, \cdots, 1^*) \\
&+\mathrm{pf}(d_3, d_0^*, \bullet) - \mathrm{pf}(d_0, d_0^*, d_1, d_1^*, \bullet) + \mathrm{pf}(d_0, d_3^*, \bullet),
\end{aligned}
$$

where the dot in the above pfaffians denotes the set $\{1, 2, \cdots, N, N^*, \cdots, 1^*\}$.

Substituting the above formulas into (8), we get the expression

$$\begin{aligned}0 = 3[&\mathrm{pf}(d_0, d_0^*, d_1, d_1^*, \bullet)\mathrm{pf}(\bullet) - \mathrm{pf}(d_0, d_0^*, \bullet)\mathrm{pf}(d_1, d_1^*, \bullet) + \mathrm{pf}(d_0, d_1^*, \bullet)\mathrm{pf}(d_1, d_0^*, \bullet)] \\ &-\sum_{j=1}^{K}\dot{\beta}_j(t)[\mathrm{pf}(d_0, d_0^*, 1, \cdots, \hat{j}, \cdots, N, N^*, \cdots, \hat{j}^*, \cdots, 1^*)\mathrm{pf}(\bullet) \\ &-\mathrm{pf}(d_0, d_0^*, \bullet)\mathrm{pf}(1, \cdots, \hat{j}, \cdots, N, N^*, \cdots, \hat{j}^*, \cdots, 1^*) \\ &+\mathrm{pf}(d_0^*, 1, \cdots, N, N^*, \cdots, \hat{j}^*, \cdots, 1^*)\mathrm{pf}(d_0, 1, \cdots, \hat{j}, \cdots, N, N^*, \cdots, 1^*)].\end{aligned}$$

which is a linear combination of $K + 1$ pfaffian identities. So equation (8) holds.

▶ According to the expression of g_i, we have

$$\begin{aligned} g_{j,x} &= 2\sqrt{2\dot{\beta}_j(t)}\mathrm{pf}(d_1^*, \star), \\ g_{j,xx} &= 2\sqrt{2\dot{\beta}_j(t)}[\mathrm{pf}(d_2^*, \star) + \mathrm{pf}(d_0, d_0^*, d_1^*, \star)], \\ g_{j,y} &= 2\sqrt{2\dot{\beta}_j(t)}[\mathrm{pf}(d_2^*, \star) - \mathrm{pf}(d_0, d_0^*, d_1^*, \star)], \end{aligned}$$

where $\star$ denotes the set $\{1, \cdots, N, N^*, \cdots, \hat{j}^*, \cdots, 1^*\}$.

Substituting the formulas into eqs. (9), we also get an identity of determinants:

$$\begin{aligned} &\mathrm{pf}(d_0, d_1^*, \bullet)\mathrm{pf}(d_0^*, 1, \cdots, N, N^*, \cdots, \hat{j}^*, \cdots, 1^*) \\ &\qquad +\mathrm{pf}(\bullet)\mathrm{pf}(d_0, d_0^*, d_1^*, 1, \cdots, N, N^*, \cdots, \hat{j}^*, \cdots, 1^*) \\ &\qquad -\mathrm{pf}(d_0, d_0^*, \bullet)\mathrm{pf}(d_1^*, 1, \cdots, N, N^*, \cdots, \hat{j}^*, \cdots, 1^*) = 0. \end{aligned}$$

So eqs. (9) holds.

Bilinear form of the KPESCS (8)-(10)

$$u = 2(\ln f)_{xx} \quad \Big\Downarrow \quad g_i = \phi_i f, \quad h_i = \psi_i f$$

Nonlinear form of the KPESCS

$$\begin{aligned}
&(4u_t - u_{xxx} - 6uu_x)_x - 3u_{yy} + \sum_{i=1}^{K}(\phi_i\psi_i)_{xx} = 0,\\
&\phi_{i,y} = \phi_{i,xx} + u\phi_i, \ \ i = 1, 2, \cdots, K\\
&-\psi_{i,y} = \psi_{i,xx} + u\psi_i, \ \ i = 1, 2, \cdots, K
\end{aligned} \tag{11}$$

⋆ *V. K. Mel'nikov, Commun. Math.Phys.* ***126*** *(1989) 201-215;*

⋆ *S.F. Deng, D.Y. Chen and D.J. Zhang, J. Phys. Soc. Jpn, 72 (2003) 2184;*

⋆ *T. Xiao and Y.B. Zeng, J. Phys. A.* **37** *(2004) 7143-7162.*

3 New types of KPESCS

- Recall:

$$\left\{\begin{array}{l} The\ KP\ equation \\ Determinant\ solution \end{array}\right. \xrightarrow{c_{ij} \to C_{ij}(t)} \left\{\begin{array}{l} The\ KPESCS \\ Determinant\ solution \end{array}\right.$$

(t — temporal variable)

- Question 1:

$$\left\{\begin{array}{l} The\ KP\ equation \\ Determinant\ solution \end{array}\right. \xrightarrow{c_{ij} \to C_{ij}(y)} \text{What kind of coupled system?}$$

(y — spatial variable)

- Question 2:

$$\left\{\begin{array}{l} The\ KP\ equation \\ Determinant\ solution \end{array}\right. \xrightarrow{c_{ij} \to C_{ij}(X)} \text{What kind of coupled system?}$$

(X — linear combination of t and y)

§3.1 A new type of KPESCS:

⋆ *X.B. Hu, H.Y. Wang, Inverse Problems, 23 (2007) 1433-1444.*

step 1: the same as the first step in the normal KPESCS
step 2: Introduce a new function:

$$f = \det(a_{ij})_{1\leqslant i,j\leqslant N} \equiv \mathrm{pf}(1, 2, \cdots, N, N^*, \cdots, 1^*), \tag{12}$$

Pfaffian elements:

$$a_{ij} = \mathrm{pf}(i, j^*) = \beta_{ij}(y) + \int^x f_i \tilde{f}_j \mathrm{d}x, \quad \mathrm{pf}(i, j) = \mathrm{pf}(i^*, j^*) = 0, \quad i, j = 1, 2, \cdots, N,$$

$\beta_{ij}(y)$ satisfies the following condition:

$$\beta_{ij}(y) = \begin{cases} \beta_i(y), & i = j \text{ and } 1 \leqslant i \leqslant K \leqslant N, \ \ K, N \in Z^+, \\ \beta_{ij}, & \text{otherwise,} \end{cases}$$

step 3: Introduce other new functions:

$$g_i = \sqrt{\dot{C}_i(y)}\mathrm{pf}(d_0^*, 1, \cdots, N, N^*, \cdots, \hat{i}^*, \cdots, 1^*), \tag{13}$$

$$h_i = \sqrt{\dot{C}_i(y)}\mathrm{pf}(d_0, 1, \cdots, \hat{i}, \cdots, N, N^*, \cdots, 1^*), \tag{14}$$

New pfaffian elements:

$$\mathrm{pf}(d_m^*, i) = \frac{\partial^m f_i}{\partial x^m}, \quad \mathrm{pf}(d_m, j^*) = \frac{\partial^m \tilde{f}_j}{\partial x^m},$$

$$\mathrm{pf}(d_m^*, d_l^*) = \mathrm{pf}(d_m, d_l) = \mathrm{pf}(d_m^*, d_l) = \mathrm{pf}(d_m^*, j^*) = \mathrm{pf}(d_m, i) = 0, \quad m, l \in \mathbb{Z}.$$

Besides, we introduce some auxiliary functions:

$$k_i = \dot{C}_i(y)\mathrm{pf}(1, \cdots, \hat{i}, \cdots, N, N^*, \cdots, \hat{i}^*, \cdots, 2^*, 1^*), \quad i = 1, 2, \cdots, M \tag{15}$$

$$\begin{aligned} P_i = & \frac{C_i''(y)}{2\sqrt{\dot{C}_i(y)}}\mathrm{pf}(d_0^*, 1, \cdots, N, N^*, \cdots, \hat{i}^*, \cdots, 1^*) \\ & + \sqrt{\dot{C}_i(y)}[\sum_{1\leqslant i<j\leqslant M} \dot{C}_j(y)\mathrm{pf}(d_0^*, 1, \cdots, \hat{j}, \cdots, N, N^*, \cdots, \hat{j}^*, \cdots, \hat{i}^*, \cdots, 1^*) \\ & - \sum_{1\leqslant j<i\leqslant M} \dot{C}_j(y)\mathrm{pf}(d_0^*, 1, \cdots, \hat{j}, \cdots, N, N^*, \cdots, \hat{i}^*, \cdots, \hat{j}^*, \cdots, 1^*)], \end{aligned} \tag{16}$$

$$\begin{aligned} Q_i = & \frac{C_i''(y)}{2\sqrt{\dot{C}_i(y)}}\mathrm{pf}(d_0, 1, \cdots, \hat{i}, \cdots, N, N^*, \cdots, 1^*) \\ & + \sqrt{\dot{C}_i(y)}[\sum_{1\leqslant i<j\leqslant M} \dot{C}_j(y)\mathrm{pf}(d_0, 1, \cdots, \hat{i}, \cdots, \hat{j}, \cdots, N, N^*, \cdots, \hat{j}^*, \cdots, 1^*) \\ & - \sum_{1\leqslant j<i\leqslant M} \dot{C}_j(y)\mathrm{pf}(d_0, 1, \cdots, \hat{j}, \cdots, \hat{i}, \cdots, N, N^*, \cdots, \hat{j}^*, \cdots, 1^*)], \end{aligned} \tag{17}$$

Then those functions satisfy bilinear equations:

$$(D_x^4 - 4D_xD_t + 3D_y^2)f \cdot f = 6\sum_{i=1}^{M}(D_yk_i \cdot f - D_xg_i \cdot h_i), \tag{18}$$

$$D_xk_i \cdot f + g_ih_i = 0, \tag{19}$$

$$(D_y - D_x^2)g_i \cdot f = P_if - g_i\sum_{j=1}^{M}k_j, \tag{20}$$

$$(D_y - D_x^2)f \cdot h_i = h_i\sum_{j=1}^{M}k_j - fQ_i, \tag{21}$$

$$(D_x^3 + 3D_xD_y - 4D_t)g_i \cdot f = 3D_x[P_i \cdot f - g_i \cdot (\sum_{j=1}^{M}k_j)], \tag{22}$$

$$(D_x^3 + 3D_xD_y - 4D_t)f \cdot h_i = 3D_x[(\sum_{j=1}^{M}k_j) \cdot h_i - f \cdot Q_i]. \tag{23}$$

Through the dependent variable transformations:

$$u = 2(\ln f)_{xx}, \;\; \phi_i = h_i/f, \quad \psi_i = g_i/f,$$

$$\chi_i = k_i/f \quad \phi_{1,i} = P_i/f, \quad \phi_{2,i} = Q_i/f,$$

bilinear Eqs.(18)-(22) can be finally transformed into the following coupled system:

$$\begin{cases} (u_{xxx} + 6uu_x - 4u_t)_x + 3u_{yy} = 6\sum_{i=1}^{M}[\phi_{i,xx}\psi_i - \phi_i\psi_{i,xx} - (\phi_i\psi_i)_y]_x, \\ 4\phi_{i,xxx} - 4\phi_{i,t} + 6u\phi_{i,x} + 3u_x\phi_i - 3\phi_i\int_{-\infty}^{x} u_y \mathrm{d}x = 6\phi_i\sum_{j=1}^{M}\phi_j\psi_j, \\ 4\psi_{i,xxx} - 4\psi_{i,t} + 6u\psi_{i,x} + 3u_x\psi_i + 3\psi_i\int_{-\infty}^{x} u_y \mathrm{d}x = -6\psi_i\sum_{j=1}^{M}\phi_j\psi_j. \end{cases} \tag{24}$$

$$\Downarrow$$

New type of KPESCS

$\star$ *V. K. Mel'nikov, On equations for wave interactions, Lett. Math. Phys. 7 (1983) 129.*

• The new KP equation with one source ($M = 1$):

$$\begin{cases} (u_{xxx} + 6uu_x - 4u_t)_x + 3u_{yy} = 6[\phi_{xx}\psi - \phi\psi_{xx} - (\phi\psi)_y]_x, \\ 4\phi_{xxx} - 4\phi_t + 6u\phi_x + 3u_x\phi - 3\phi \int_{-\infty}^{x} u_y \mathrm{d}x = 6\phi^2\psi, \\ 4\psi_{xxx} - 4\psi_t + 6u\psi_x + 3u_x\psi + 3\psi \int_{-\infty}^{x} u_y \mathrm{d}x = -6\phi\psi^2. \end{cases} \tag{25}$$

► Exact solutions:

$$u = 2(\ln f)_{xx}, \quad \phi = \frac{h}{f}, \quad \psi = \frac{g}{f}, \tag{26}$$

According to formulas in (12)-(14):

$$f = det(a_{ij})_{1\leqslant i,j\leqslant N}, \quad a_{ij} = \beta_{ij}(y) + \int^x f_i\tilde{f}_j \mathrm{d}x, \tag{27}$$

with $\beta_{ij}(y) = \begin{cases} C_1(y), & i=j=1, \\ \text{constant}, & \text{otherwise}, \end{cases}$

$$g = -\sqrt{\dot{C}_1(y)} \begin{vmatrix} f_1 & a_{12} & \cdots & a_{1N} \\ f_2 & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & & \vdots \\ f_N & a_{N2} & \cdots & a_{NN} \end{vmatrix}, \quad h = \sqrt{\dot{C}_1(y)} \begin{vmatrix} \tilde{f}_1 & \tilde{f}_2 & \cdots & \tilde{f}_N \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{vmatrix} \tag{28}$$

where f_1, $\tilde{f}_i$ satisfy

$$\frac{\partial f_i}{\partial x_n} = \frac{\partial^n f_i}{\partial x^n}, \quad \frac{\partial \tilde{f}_i}{\partial x_n} = (-1)^{n-1}\frac{\partial^n \tilde{f}_i}{\partial x^n}, \quad (x_1 = x, \ \ x_2 = y, \ \ x_3 = t).$$

▶ one-soliton solution:

Take $N = 1$ and

$$\beta_{11} = C_1(y) = \frac{1}{p+q} e^{2\alpha(y)},$$

$$f_1 = e^{\xi} = e^{px+p^2y+p^3t}, \quad \tilde{f}_1 = e^{\eta} = e^{qx-q^2y+q^3t}, \quad p, q \in \mathbb{R},$$

where $\alpha(y)$ is an arbitrary function of the variable y.

1-soliton solution:

$$u = 2\frac{\partial^2}{\partial x^2} \ln(1 + e^{\xi+\eta-2\alpha(y)}),$$

$$\psi = \frac{\sqrt{2(p+q)\dot{\alpha}(y)}e^{\xi-\alpha(y)}}{1 + e^{\xi+\eta-2\alpha(y)}}, \quad \phi = \frac{\sqrt{2(p+q)\dot{\alpha}(y)}e^{\eta-\alpha(y)}}{1 + e^{\xi+\eta-2\alpha(y)}}.$$

▶ two-soliton solution:

Take $N = 2$ and

$$\beta_{11} = C_1(y) = \frac{1}{p_1 + q_1} e^{2\alpha(y)}, \quad \beta_{12} = \beta_{21} = 0, \quad \beta_{22} = \frac{1}{p_2 + q_2},$$
$$f_i = e^{\xi_i} = e^{p_i x + p_i^2 y + p_i^3 t}, \quad \tilde{f}_i = e^{\eta_i} = e^{q_i x - q_i^2 y + q_i^3 t}, \quad p_i, q_i \in \mathbb{R}, \quad i = 1, 2$$

then

$$u = 2\frac{\partial^2}{\partial x^2} \ln[1 + e^{\xi_1 + \eta_1 - 2\alpha(y)} + e^{\xi_2 + \eta_2} + A e^{\xi_1 + \eta_1 + \xi_2 + \eta_2 - 2\alpha(y)}],$$
$$\phi = \frac{\sqrt{2(p_1 + q_1)\dot{\alpha}(y)} e^{\eta_1 - \alpha(y)} [1 + b_1 e^{\xi_2 + \eta_2}]}{1 + e^{\xi_1 + \eta_1 - 2\alpha(y)} + e^{\xi_2 + \eta_2} + A e^{\xi_1 + \eta_1 + \xi_2 + \eta_2 - 2\alpha(y)}},$$
$$\psi = -\frac{\sqrt{2(p_1 + q_1)\dot{\alpha}(y)} e^{\xi_1 - \alpha(y)} [1 + a_1 e^{\xi_2 + \eta_2}]}{1 + e^{\xi_1 + \eta_1 - 2\alpha(y)} + e^{\xi_2 + \eta_2} + A e^{\xi_1 + \eta_1 + \xi_2 + \eta_2 - 2\alpha(y)}},$$

$$where\ b_1 = \frac{q_1 - q_2}{p_2 + q_1}, \quad a_1 = \frac{p_1 - p_2}{p_1 + q_2}, \quad A = \frac{(p_1 - p_2)(q_1 - q_2)}{(p_1 + q_2)(p_2 + q_1)}.$$

§3.2 Hybrid type of KPESCS

step 1: $\cdots\ \cdots$
step 2: Introduce a set of new functions:

$$f = \det(a_{ij})_{1\leqslant i,j\leqslant N} \equiv \mathrm{pf}(1,2,\cdots,N,N^*,\cdots,1^*), \tag{29}$$

Pfaffian elements:

$$a_{ij} = \mathrm{pf}(i,j^*) = \beta_{ij}(X) + \int^x f_i\tilde{f}_j \mathrm{d}x, \quad \mathrm{pf}(i,j) = \mathrm{pf}(i^*,j^*) = 0, \quad i,j = 1,2,\cdots,N,$$

$\beta_{ij}(X)$ satisfies the following condition：

$$\beta_{ij}(X) = \begin{cases} \beta_i(X) \equiv \beta_i(ay+bt), & i=j \text{ and } 1\leqslant i\leqslant K\leqslant N,\ \ K,N\in Z^+, \\ \beta_{ij}, & \text{otherwise}, \end{cases}$$

where a and b are real constants and they cannot be zeroes at the same time.

$$g_i = \sqrt{\dot{C}_i(X)}\mathrm{pf}(d_0^*, 1, \cdots, N, N^*, \cdots, \hat{i}^*, \cdots, 1^*), \tag{30}$$

$$h_i = \sqrt{\dot{C}_i(X)}\mathrm{pf}(d_0, 1, \cdots, \hat{i}, \cdots, N, N^*, \cdots, 1^*), \tag{31}$$

New pfaffian elements:

$$\mathrm{pf}(d_m^*, i) = \frac{\partial^m f_i}{\partial x^m}, \quad \mathrm{pf}(d_m, j^*) = \frac{\partial^m \tilde{f}_j}{\partial x^m}, \quad m, l \ \in Z$$

$$\mathrm{pf}(d_m^*, d_l^*) = \mathrm{pf}(d_m, d_l) = \mathrm{pf}(d_m^*, d_l) = \mathrm{pf}(d_m^*, j^*) = \mathrm{pf}(d_m, i) = 0.$$

step 3: Introduce other auxiliary functions:

$$k_i = \dot{C}_i(X)\mathrm{pf}(1, \cdots, \hat{i}, \cdots, N, N^*, \cdots, \hat{i}^*, \cdots, 2^*, 1^*),$$

$$P_i = \frac{C_i''(X)}{2\sqrt{\dot{C}_i(X)}}\mathrm{pf}(d_0^*, 1, \cdots, N, N^*, \cdots, \hat{i}^*, \cdots, 1^*)$$
$$+\sqrt{\dot{C}_i(X)}[\sum_{1\leqslant i<j\leqslant M} \dot{C}_j(X)\mathrm{pf}(d_0^*, 1, \cdots, \hat{j}, \cdots, N, N^*, \cdots, \hat{j}^*, \cdots, \hat{i}^*, \cdots, 1^*)$$
$$- \sum_{1\leqslant j<i\leqslant M} \dot{C}_j(X)\mathrm{pf}(d_0^*, 1, \cdots, \hat{j}, \cdots, N, N^*, \cdots, \hat{i}^*, \cdots, \hat{j}^*, \cdots, 1^*)],$$

$$Q_i = \frac{C_i''(X)}{2\sqrt{\dot{C}_i(X)}}\mathrm{pf}(d_0, 1, \cdots, \hat{i}, \cdots, N, N^*, \cdots, 1^*)$$
$$+\sqrt{\dot{C}_i(X)}[\sum_{1\leqslant i<j\leqslant M} \dot{C}_j(X)\mathrm{pf}(d_0, 1, \cdots, \hat{i}, \cdots, \hat{j}, \cdots, N, N^*, \cdots, \hat{j}^*, \cdots, 1^*)$$
$$- \sum_{1\leqslant j<i\leqslant M} \dot{C}_j(X)\mathrm{pf}(d_0, 1, \cdots, \hat{j}, \cdots, \hat{i}, \cdots, N, N^*, \cdots, \hat{j}^*, \cdots, 1^*)].$$

$C_i''(X)$: the second order of $C_i(X)$ with respect to X

These new functions satisfy the bilinear equations:

$$(D_x^4 - 4D_xD_t + 3D_y^2)f \cdot f = 8b\sum_{i=1}^{M} g_ih_i + 6a\sum_{i=1}^{M}(D_yk_i \cdot f - D_xg_i \cdot h_i), \tag{32}$$

$$D_xk_i \cdot f + g_ih_i = 0, \tag{33}$$

$$(D_y - D_x^2)g_i \cdot f = a[P_if - g_i\sum_{j=1}^{M} k_j], \tag{34}$$

$$(D_y - D_x^2)f \cdot h_i = a[h_i\sum_{j=1}^{M} k_j - fQ_i], \tag{35}$$

$$(D_x^3 + 3D_xD_y - 4D_t)g_i \cdot f = (3aD_x - 4b)[P_i \cdot f - g_i \cdot (\sum_{j=1}^{M} k_j)], \tag{36}$$

$$(D_x^3 + 3D_xD_y - 4D_t)f \cdot h_i = (3aD_x - 4b)[(\sum_{j=1}^{M} k_j) \cdot h_i - f \cdot Q_i] \tag{37}$$

Applying the dependent variable transformations:

$$u = 2(\ln f)_{xx}, \quad \phi_i = h_i/f, \quad \psi_i = g_i/f,$$

$$\chi_i = k_i/f \quad \phi_{1,i} = P_i/f, \quad \phi_{2,i} = Q_i/f,$$

Bilinear equations (32)-(37) are finally transformed into the nonlinear system:

$$\begin{aligned}
&(u_{xxx} + 6uu_x - 4u_t)_x + 3u_{yy} \\
&\qquad = 8b \sum_{i=1}^{M} (\phi_i \psi_i)_{xx} + 6a \sum_{i=1}^{M} [\phi_{i,xx}\psi_i - \phi_i \psi_{i,xx} - (\phi_i \psi_i)_y]_x, \\
&a[4\phi_{i,xxx} - 4\phi_{i,t} + 6u\phi_{i,x} + 3u_x \phi_i \\
&\qquad -3\phi_i \int_{-\infty}^{x} u_y \mathrm{d}x] + 4b[\phi_{i,y} + \phi_{i,xx} + u\phi_i] = 6a^2 \phi_i \sum_{j=1}^{M} \phi_j \psi_j, \\
&a[4\psi_{i,xxx} - 4\psi_{i,t} + 6u\psi_{i,x} + 3u_x \psi_i \\
&\qquad +3\psi_i \int_{-\infty}^{x} u_y \mathrm{d}x] + 4b[\psi_{i,y} - \psi_{i,xx} - u\psi_i] = -6a^2 \psi_i \sum_{j=1}^{M} \phi_j \psi_j.
\end{aligned} \tag{38}$$

$$\Downarrow$$

Hybrid type of KPESCS

- If $a = 0,\ b = \frac{1}{8}$, the system(38) is reduced to the general KPESCS (11)

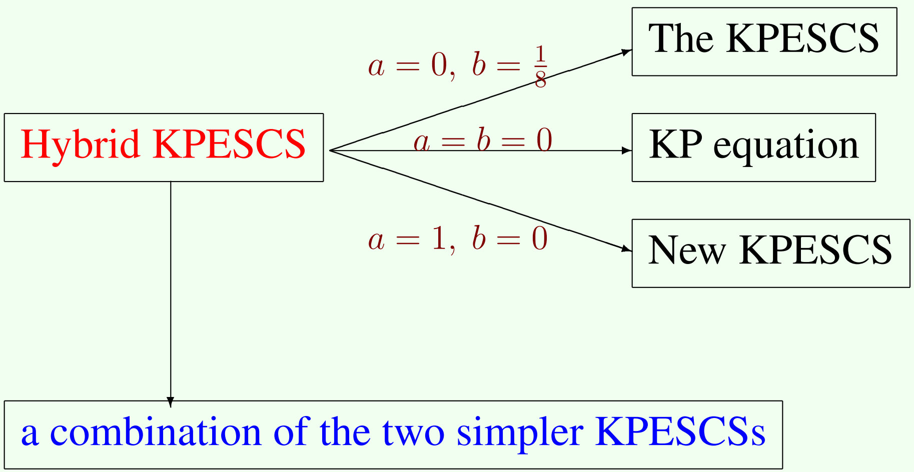
$$
\begin{aligned}
&u_{xxx} + 6uu_x - 4u_t + 3\int_{-\infty}^{x} u_{yy}\mathrm{d}x = \sum_{i=1}^{M}(\phi_i\psi_i)_x,\\
&\phi_{i,y} + \phi_{i,xx} + u\phi_i = 0,\ \ i = 1, 2, \cdots, M\\
&\psi_{i,y} - \psi_{i,xx} - u\psi_i = 0,\ \ i = 1, 2, \cdots, M
\end{aligned}
$$

- If $a = 1,\ b = 0$, the system (38) is reduced to the first new type KPESCS (24)

$$
\begin{aligned}
&u_{xxx} + 6uu_x - 4u_t + 3\int_{-\infty}^{x} u_{yy}\mathrm{d}x = 6\sum_{i=1}^{M}[\phi_{i,xx}\psi_i - \phi_i\psi_{i,xx} - (\phi_i\psi_i)_y],\\
&4\phi_{i,xxx} - 4\phi_{i,t} + 6u\phi_{i,x} + 3u_x\phi_i - 3\phi_i\int_{-\infty}^{x} u_y\mathrm{d}x = 6\phi_i\sum_{j=1}^{M}\phi_j\psi_j,\\
&4\psi_{i,xxx} - 4\psi_{i,t} + 6u\psi_{i,x} + 3u_x\psi_i + 3\psi_i\int_{-\infty}^{x} u_y\mathrm{d}x = -6\psi_i\sum_{j=1}^{M}\phi_j\psi_j,
\end{aligned}
$$

Characteristic:

- Considering the forms of equations



- Why we call this kind of system a hybrid type?
  - ▶ some hybrid-type equations in the soliton theory

※ The generalized nonlinear Schrödinger equation

$$q_t = iq_{xx} + a(q^2q^*)_x + ibq^2q^*,$$

$$a = 0,\ b \neq 0: \quad the\ NLS\ equation$$
$$a \neq 0,\ b = 0: \quad the\ derivative\ NLS\ equation$$

※ a hybrid differential-difference equation

$$\frac{\mathrm{d}w_n}{\mathrm{d}t} = [1 + bw_n + cw_n^2](w_{n-1} - w_{n+1}),$$

$$u_n = 1 + w_n,\ b = 1,\ c = 0: \quad Lotka - Volterra\ equation$$
$$u_n = w_n,\ b = 0,\ c = 1: \quad Self - dual\ nonlinear\ network\ equation$$

⋆ *M.J. Ablowitz and P.A.Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering, 1991, Cambridge: Cambridge University Press.*

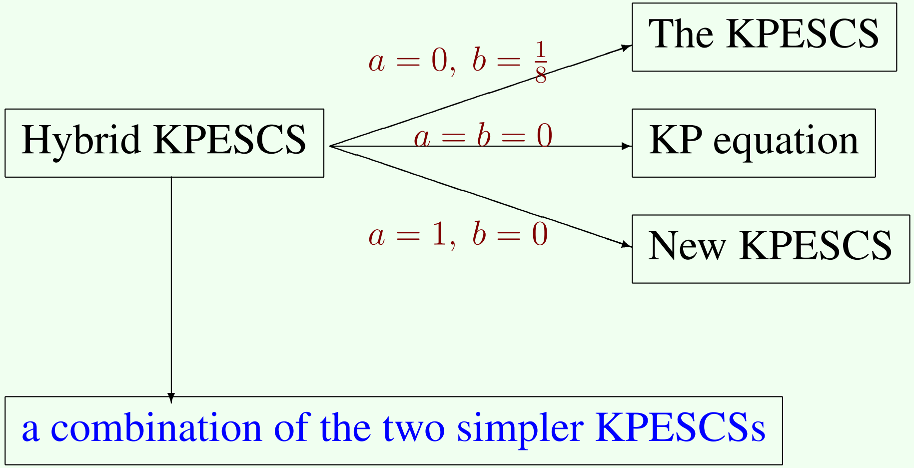
⋆ *A.S. Carstea, Phys. Lett. A 233 (1997) 378-382.*

⋆ *R. Hirota and M. Iwao, CRM Proceedings and Lecture Notes 25 (2000) 217-243.*

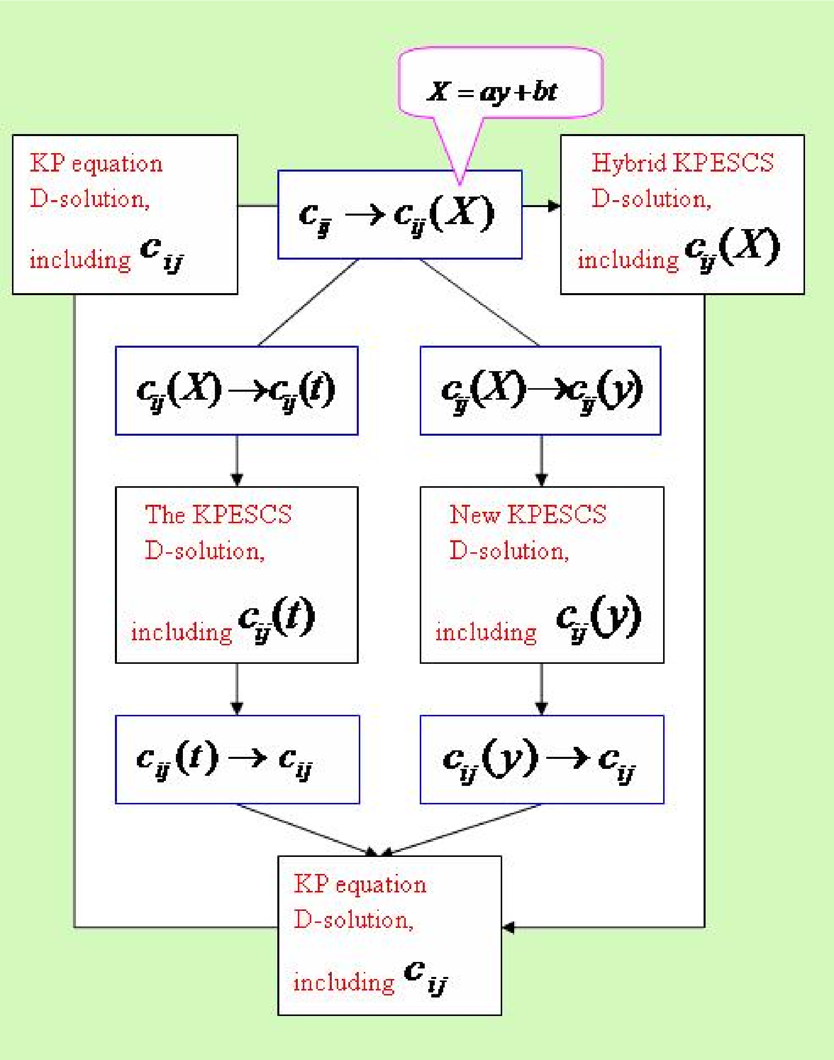
4 Summary

• The source generation procedure is applied to the KP equation, and three kinds of KPESCSs are given and their determinant solutions are derived respectively.
• The relationship among the three coupled systems

▶ From the point of equations



▶ From the point of exact solutions (determinant solutions)


$X = ay + bt$
KP equation D-solution, including c_{ij}
$c_{ij} \to c_{ij}(X)$
Hybrid KPESCS D-solution, including $c_{ij}(X)$
$c_{ij}(X) \to c_{ij}(t)$
$c_{ij}(X) \to c_{ij}(y)$
The KPESCS D-solution, including $c_{ij}(t)$
New KPESCS D-solution, including $c_{ij}(y)$
$c_{ij}(t) \to c_{ij}$
$c_{ij}(y) \to c_{ij}$
KP equation D-solution, including c_{ij}

Thank you!

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