Discrete Toda lattice and Elliptic orthogonal polynomials

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Laurent biorthogonal polynomials (LBP)

Let $\mathcal{L}$ be a linear functional.

Moments

$$c_n = \mathcal{L}\{z^n\}, \quad n = 0, \pm 1, \pm 2 \ldots$$

The functional $\mathcal{L}$ is defined on the space of Laurent polynomials $\mathcal{P}(z) = \sum_{n=-N_1}^{N_2} a_n z^n$:

$$\mathcal{L}\{\mathcal{P}(z)\} = \sum_{n=-N_1}^{N_2} a_n c_n.$$

The monic LBP $P_n(z)$ are defined by the determinant

$$P_n(z) = (\Delta_n)^{-1} \begin{vmatrix} c_0 & c_1 & \dots & c_n \\ c_{-1} & c_0 & \dots & c_{n-1} \\ \dots & \dots & \dots & \dots \\ c_{1-n} & c_{2-n} & \dots & c_1 \\ 1 & z & \dots & z^n \end{vmatrix},$$

where Δ_n is defined as the Toeplitz determinant

$$\Delta_n = \begin{vmatrix} c_0 & c_1 & \dots & c_{n-1} \\ c_{-1} & c_0 & \dots & c_{n-2} \\ \dots & \dots & \dots & \dots \\ c_{1-n} & c_{2-n} & \dots & c_0 \end{vmatrix}.$$

Orthogonality property

$$\mathcal{L}\{P_n(z)z^{-k}\} = h_n\delta_{kn}, \quad 0 \le k \le n,$$

where the normalization constants h_n are

$$h_0 = c_0, \quad h_n = \Delta_{n+1}/\Delta_n.$$

In what follows we will assume that

$$\Delta_n \neq 0, \quad n = 1, 2, \ldots$$

and

$$\Delta_n^{(1)} \neq 0, \quad n = 1, 2, \ldots$$

where

$$\Delta_0^{(j)} = 1;$$

$$\Delta_n^{(j)} = \begin{vmatrix} c_j & c_{j+1} & \cdots & c_{n+j-1} \\ c_{j-1} & c_j & \cdots & c_{n+j-2} \\ \cdots & \cdots & \cdots & \cdots \\ c_{1+j-n} & c_{2+j-n} & \cdots & c_j \end{vmatrix}.$$

This orthogonality property can be rewritten as the biorthogonal relation

$$\mathcal{L}\{P_n(z)Q_m(1/z)\} = h_n \delta_{nm},$$

where the polynomials $Q_n(z)$ are defined by the formula

$$Q_n(z) = (\Delta_n)^{-1} \begin{vmatrix} c_0 & c_{-1} & \dots & c_{-n} \\ c_1 & c_0 & \dots & c_{1-n} \\ \dots & \dots & \dots & \dots \\ c_{n-1} & c_{n-2} & \dots & c_{-1} \\ 1 & z & \dots & z^n \end{vmatrix}.$$

Polynomials $Q_n(z)$ are again LBP with moments $c_n^{\{Q\}} = c_{-n}$.

$P_n(z)$ satisfy the recurrence relation

$$P_{n+1}(z) + (d_n - z)P_n(z) = zb_nP_{n-1}(z), \quad n \geq 1$$

Recurrence coefficients

$$d_n = \frac{T_{n+1}\Delta_n}{T_n\Delta_{n+1}} \neq 0,$$
$$b_n = \frac{T_{n+1}\Delta_{n-1}}{T_n\Delta_n} \neq 0,$$

with $T_n = \Delta_n^{(1)}$.

LBP and Relativistic Toda

There is a connection between the LBP and the restricted relativistic Toda chain. Assume that LBP $P_n(z;t)$ depend on an additional "time" parameter t. Assume

$$\dot{P}_n(z) = -\frac{b_n}{d_n} P_{n-1}(z).$$

This Ansatz leads to equations

$$\dot{d}_n = \frac{b_{n+1}}{d_{n+1}} - \frac{b_n}{d_{n-1}},$$
$$\dot{b}_n = b_n \left(\frac{1}{d_n} - \frac{1}{d_{n-1}} \right).$$

For the corresponding moments $c_n(t)$ we have very simple relation

$$\dot{c}_n = c_{n-1}, \quad n = 0, \pm 1, \pm 2, \dots$$

The (restricted) "discrete-time" relativistic Toda chain corresponds to the following Ansatz for the moments

$$c_n(t+h) = c_{n+1}(t), \quad n = 0, \pm 1, \pm 2, \ldots.$$

For Laurent biorthogonal polynomials

$$P_n(z; t+h) = P_n(z; t) + b_n(t) P_{n-1}(z; t)$$

and

$$(d_n - b_n)\, P_n(z; t-h) = z P_n(z; t) - P_{n+1}(z; t)$$

For recurrence coefficients

$$d_n(t+h) = d_{n-1} \frac{b_{n+1} - d_n}{b_n - d_{n-1}}, \quad b_n(t+h) = b_n \frac{b_{n+1} - d_n}{b_n - d_{n-1}}$$

Frobenius determinant

Assume that $v_i, u_i,\ i = 0, 1, \ldots$ be two arbitrary sequences of complex numbers. Let

$$H_n = \det ||g_{ij}||_{i,j=0..n-1},$$

where

$$g_{ij} = \frac{\sigma(u_i + v_j + \beta)}{\sigma(u_i + v_j)\sigma(\beta)} \exp(\gamma_1 u_i + \gamma_2 v_j)$$

Then

$$H_n = \frac{\sigma(U + V + \beta)\prod_{i>j}\sigma(u_i - u_j)\sigma(v_i - v_j)}{\sigma(\beta)\prod_{i,j}\sigma(u_i + v_j)} \times$$

$$\exp(\gamma_1 U + \gamma_2 V)$$

where $U = \sum_{i=0}^{n-1} v_i, V = \sum_{i=0}^{n-1} w_i$.

To biorthogonal functions

Let $\phi_k(x), \psi_k(x), k = 0, 1, \dots$ (initial conditions $\phi_0 = \psi_0 = 1$) be two sets of functions. Assume that there exists a linear functional $\mathcal{L}$ (we call it the "Frobenius functional") such that

$$\langle \mathcal{L}, \phi_j(x)\psi_i(x)\rangle = g_{ij}$$

Functional $\mathcal{L}$ is defined for bilinear combinations

$$f(x) = \sum_{i,k=0} c_{ik}\phi_i(x)\psi_k(x)$$

with arbitrary coefficients c_{ik}.

Introduce the following functions

$$P_n(x) = \frac{1}{\Delta_n} \begin{vmatrix} g_{00} & g_{01} & \cdots & g_{0n} \\ g_{10} & g_{11} & \cdots & g_{1n} \\ \dots & \dots & \dots & \dots \\ g_{n-1,0} & g_{n-1,1} & \cdots & g_{n-1,n} \\ \phi_0(x) & \phi_1(x) & \dots & \phi_n(x) \end{vmatrix}.$$

where

$$\Delta_n = H_n = \det ||g_{ij}||_{i,j=0..n-1}$$

Explicitly

$$P_n(x) = \sum_{k=0}^{n} p_{nk} \phi_k(x)$$

where the coefficients p_{nk} are calculated as ratio of two determinants:

$$p_{nk} = \frac{H_n(k)}{\Delta_n}$$

where

$$H_n(k) = \det ||g_{ij}(k)||_{i,j=0..n-1}$$

where

$$g_{ij}(k) = \frac{\sigma(u_i + v_j(k) + \beta)}{\sigma(u_i + v_j(k))\sigma(\beta)} \exp(\gamma_1 u_i + \gamma_2 v_j(k))$$

Thus the determinant $H_n(k)$ is obtained from the determinant H_n by replacing sequence v_i with the sequence $v_i(k)$. (By definition $H_n(n) = H_n$ and $v_i(n) = v_i$). Hence we can calculate all the determinant $H_n(k)$ explicitly:

$$p_{nk} = e^{\gamma_2(v_n - v_k)} \frac{\sigma(U + V + v_n - v_k + \beta)}{\sigma(U + V + \beta)} \times$$

$$\begin{bmatrix} n \\ k \end{bmatrix} \prod_{i=0}^{n-1} \frac{\sigma(u_i + v_k)}{\sigma(u_i + v_n)}$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{\prod_{i=0}^{n-1} \sigma(v_n - v_i)}{\prod_{i=0}^{k-1} \sigma(v_k - v_i) \prod_{i=k+1}^{n} \sigma(v_i - v_k)}$$

are "generalized binomial coefficients". In case when the sequence v_j is *linear* with respect t j: $v_j = wj + \xi$ we obtain the conventional "elliptic binomial coefficients"

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!} = (-1)^k \frac{[-n]_k}{[1]_k},$$

where $[x] = \sigma(wx)/\sigma(w)$ is so-called "elliptic number" and $[x]_k = [x][x+1]\dots[x+k-1]$ is elliptic Pochhammer symbol.

Introduce also functions

$$P_n^*(x) = \frac{1}{\Delta_n} \begin{vmatrix} g_{00} & g_{10} & \cdots & g_{n0} \\ g_{01} & g_{11} & \cdots & g_{n1} \\ \cdots & \cdots & \cdots & \cdots \\ g_{0,n-1} & g_{1,n-1} & \cdots & g_{n,n-1} \\ \psi_0(x) & \psi_1(x) & \ldots & \psi_n(x) \end{vmatrix}.$$

Then functions $P_n(x)$ and $P_n^*(x)$ are biorthogonal with respect to Frobenius functional

$$\langle \mathcal{L}, P_n(x) P_m^*(x) \rangle = 0, \quad n \neq m$$

LBP from Frobenius

Put

$$\gamma_1 = \gamma_2 = \gamma,\ u_i = -iw + \alpha,\ v_j = jw,$$

where w is an arbitrary real parameter which is incompatible with the real period $2\omega_1$:

$$wN_1 \neq \omega_1 N_2$$

Then

$$g_{ij} = e^{\gamma w(j-i)+\gamma\alpha} \frac{\sigma(w(j-i)+\beta+\alpha)}{\sigma(w(j-i)+\alpha)\sigma(\beta)}$$

This matrix has the Toeplitz form. Monic "Frobenius" Laurent biorthogonal polynomials

$$P_n(z) = \frac{1}{\Delta_n} \begin{vmatrix} c_0 & c_1 & \dots & c_n \\ c_{-1} & c_0 & \dots & c_{n-1} \\ \dots & \dots & \dots & \dots \\ c_{-n+1} & c_{-n+2} & \dots & c_1 \\ 1 & z & \dots & z^n \end{vmatrix},$$

where

$$c_n = g_{0,n} = e^{\gamma wn+\gamma\alpha} \frac{\sigma(wn+\beta+\alpha)}{\sigma(wn+\alpha)\sigma(\beta)}$$

We slightly modify definition of elliptic numbers and elliptic Pochhammer symbol

$$[x] = \sigma(x),$$

the elliptic Pochhammer symbol

$$[x]_n = [x][x+w]\dots[x+w(n-1)]$$

The elliptic hypergeometric function

$${}_{r+1}G_r\left(\begin{matrix}\vec{a}\\ \vec{b}\end{matrix}; z\right) = \sum_{s=0}^{\infty} \frac{[a_1]_s[a_2]_s\dots[a_{r+1}]_s}{[w]_s[b_1]_s[b_2]_s\dots[b_r]_s} e^{Ms(s-1)}\, z^s,$$

where

$$M = \frac{\eta_1}{2\omega_1} w^2 \left(w + \sum_{i=1}^{r} b_i - \sum_{i=1}^{r+1} a_i\right)$$

Proposition 1 *The "Frobenius" Laurent biorthogonal polynomials are expressed in terms of the elliptic hypergeometric function:*

$$P_n(z) = B_n \, {}_3G_2\left(\begin{matrix} -nw, \alpha_1, -\alpha_1 n - \beta + w \\ \alpha_1 - nw, -\alpha_1 n - \beta \end{matrix}; ze^{-\gamma w}\right),$$

where $\alpha_1 = \alpha + w$ *and*

$$B_n = e^{\gamma w n} \frac{[-\alpha]_n}{[\alpha + w]_n} \frac{[\alpha n + \beta + wn]}{[\alpha n + \beta]}$$

Determinant $\Delta_n^{(1)}$ is obtained from $\Delta_n^{(0)}$ by the shift of the parameter $\alpha \to \alpha + w$ because $c_{n+1}(\alpha) = c_n(\alpha + w)$. Thus in general

$$\Delta_n^{(j)}(\alpha) = \Delta_n(\alpha + jw)$$

From these formulas we find recurrence coefficients

$$d_n = \frac{h_n^{(1)}}{h_n} = e^{\gamma w} \frac{[\alpha - wn][\beta + \alpha_1(n+1)][\beta + \alpha n]}{[\alpha + w(n+1)][\beta + \alpha_1 n][\beta + \alpha(n+1)]}$$

and

$$b_n = -\frac{h_n^{(1)}}{h_{n-1}} =$$

$$-e^{\gamma w} \frac{[wn]^2[\beta + \alpha_1(n+1)][\beta + \alpha(n-1)]}{[\beta + \alpha_1 n][\beta + \alpha n][\alpha + wn][\alpha + w(n+1)]}$$

We thus obtained a new explicit example of the Laurent biorthogonal polynomials which have both explicit expression in terms of the elliptic hypergeometric function ${}_3G_2(z)$ and explicit recurrence coefficients.

There are 5 free parameters: α, β, γ, w and elliptic modulus k (equivalently ratio ω'/ω). The parameter γ is not essential - it describes the scaling transformation of the argument: $P_n(z) \to \kappa^n P_n(z/\kappa)$. Nevertheless, the parameter γ is important in finding of explicit orthogonality measure.

As a by-product, we have also obtained a new explicit solution of the discrete-time relativistic Toda chain or, equivalently, a new explicit solution of the two-point QD-algorithm.

How to find explicit (bi)orthogonality relation for these polynomials? We need the Fourier expansion of the pseudoelliptic functions!

Explicit biorthogonal relation

We have

$$c_n = f(wn),$$

where $f(z)$ is the pseudoelliptic function

$$f(z) = \frac{\sigma(z+\alpha+\beta)}{\sigma(\beta)\sigma(z+\alpha)} e^{\gamma z}$$

Assume first that the parameter γ is chosen to provide the periodicity of the function $f(z)$ with period $2\omega_1 j$, $j = 1, 2, \ldots$. Then from Fourier expansion

$$c_n = \sum_{s=-\infty}^{\infty} A_s \exp\left(\frac{i\pi swn}{j\omega_1}\right) = \sum_{s=-\infty}^{\infty} A_s z_s^n,$$

where

$$z_s = \exp\left(\frac{i\pi sw}{j\omega_1}\right), \quad s = 0, \pm 1, \pm 2, \ldots$$

$$z_s = \exp\left(\frac{i\pi s w}{j\omega_1}\right), \quad s = 0, \pm 1, \pm 2, \ldots$$

is an infinite set of points belonging to the unit circle $|z_s| = 1$. These points are distinct $z_s \neq z_t$ if $t \neq s$ and hence they are *dense* on the unit circle.

Thus the moments c_n are expressible in terms of the Lebesgue integral

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} e^{i\theta n} d\mu(\theta)$$

over the unit circle $|z| = 1$, where the measure $\mu(\theta)$ is a (complex) function of a bounded variation on the interval $[0, 2\pi]$ consisting only from discrete jumps A_s localized in the points θ_s.

Thus we found explicit realization of the moments c_n and hence obtain biorthogonality relation for our Laurent biorthogonal polynomials

$$\sum_{s=-\infty}^{\infty} A_s\, P_n(z_s) Q_m(1/z_s) = h_n \delta_{nm},$$

where $Q_n(z)$ are biorthogonal partners with respect to polynomials $P_n(z)$. The Fourier coefficients A_s play the role of discrete weights in this biorthogonality relation. Hence we obtained

Proposition 2 *In the periodic case* $f(z+2\omega_1 j) = f(z)$ *the elliptic Frobenius polynomials* $P_n(z)$ *are biorthogonal on the unit circle* $|z|=1$ *with respect to a dense point measure with weights* A_s.

Positivity of measure

Proposition 3 *The Fourier coefficients of the pseudoelliptic $2j\omega_1$- periodic function are positive (up to inessential common factor) if and only if the real parts of parameters α, β satisfy conditions (1) and (2). In this case the expression for the Fourier coefficients can be presented in the form*

$$A_n = \kappa_0 \frac{h^{-2\nu k}}{1+\kappa_1 h^{-2k}}, \quad n = m + jk,\ k = 0, \pm 1, \pm 2, \ldots,$$

and $A_n = 0$ if $n \neq m \mod(j)$ where $\kappa_1 = e^{\frac{\pi \Im(\beta)}{\omega_1}}$ is a positive parameter.

For positivity of A_n one should have $2\pi i R_0 = \kappa_0$, where κ_0 is a positive parameter, and for the real part of α we have the conditions

$$\alpha_0 = 2J_0\omega_1,\ J_0 = 0, \pm 1, \pm 2, \dots \tag{1}$$

and

$$Re(\beta) = (2J_1 + 1)\omega_1, \quad J_1 = 0, \pm 1, \pm 2, \dots \tag{2}$$

When the measure is a positive nondecreasing function on the unit circle then biorthogonal polynomials become the **orthogonal** polynomials on the unit circle (Szegö, Geronimus, Ahiezer, Krein, B.Simon...)

$$\int_0^{2\pi} P_n(e^{i\theta})\bar{P}_n(e^{-i\theta})d\mu(\theta) = h_n\delta_{nm}, \quad h_n > 0$$

Summary

* Laurent biorthogonal polynomials from the Frobenius determinant

$$P_n(z) = B_n \, {}_3G_2\left(\begin{matrix} -nw, \alpha_1, -\alpha_1 n - \beta + w \\ \alpha_1 - nw, -\alpha_1 n - \beta \end{matrix}; ze^{-\gamma w}\right)$$

* Explicit biorthogonality relation
* Positivity of the measure and polynomials orthogonal on the unit circle

Thank you!