Virasoro constraints and W-constraints for the q-KP hierarchy

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Abstract

Based on the Adler-Shiota-van Moerbeke (ASvM) formula, the Virasoro constraints for the p-reduced q-deformed Kadomtsev-Petviashvili (q-KP) hierarchy are established, and then the Virasoro constraint generators are obtained. The another main purpose of this article is to give the W-constraints for the p-reduced q-KP hierarchy constrained by the string equation.

- q-derivative
- q-KP hierarchy
- Virasoro constraints for the p-reduced q-KP hierarchy
- W-constraints for the p-reduced q-KP hierarchy

q-derivative

The q-derivative ∂_q is defined by

$$\partial_q(f(x)) = \frac{f(qx) - f(x)}{x(q-1)} \qquad (1)$$

and the q-shift operator is $\theta(f(x)) = f(qx)$, for $0 < q < 1$.
Let ∂_q^{-1} denote the formal inverse of ∂_q.
In general the following q-deformed Leibniz rule holds:

$$\partial_q^n \circ f = \sum_{k\geq 0} \binom{n}{k}_q \theta^{n-k}(\partial_q^k f)\partial_q^{n-k}, \qquad n \in Z \qquad (2)$$

where the q-number and the q-binomial are defined by

$$(n)_q = \frac{q^n - 1}{q - 1}$$

$$\binom{n}{k}_q = \frac{(n)_q(n-1)_q \cdots (n-k+1)_q}{(1)_q(2)_q \cdots (k)_q}, \qquad \binom{n}{0}_q = 1,$$

- For a q-pseudo-differential operator(q-PDO) of the form $P=\sum_{i=-\infty}^{n} p_i\partial_q^i$, we separate P into the differential part $P_+=\sum_{i\geq 0} p_i\partial_q^i$ and the integral part $P_-=\sum_{i\leq -1} p_i\partial_q^i$.
- The q-exponent $e_q(x)$ is defined as follows

$$e_q(x)=\sum_{n=0}^{\infty}\frac{x^n}{(n)_q!},\quad (n)_q!=(n)_q(n-1)_q(n-2)_q\cdots(1)_q.$$

Its equivalent expression is of the form

$$e_q(x)=exp(\sum_{k=1}^{\infty}\frac{(1-q)^k}{k(1-q^k)}x^k). \tag{3}$$

q-KP Hierarchy

Let L be one q-PDO given by

$$L = \partial_q + u_0 + u_{-1}\partial_q^{-1} + u_{-2}\partial_q^{-2} + \cdots , \tag{4}$$

which is called Lax operator of q-KP hierarchy. There exist infinite number of q-partial differential equations relating to dynamical variables $\{u_i(x, t_1, t_2, t_3, \cdots ,), i = 0, -1, -2, -3, \cdots\}$ and can be deduced from generalized Lax equation,

$$\frac{\partial L}{\partial t_n} = [B_n, L], n = 1, 2, 3, \cdots , \tag{5}$$

which are called q-KP hierarchy. Here $B_n = (L^n)_+ = \sum_{i=0}^{n} b_i \partial_q^i$ means the positive part of q-PDO, and we will use $L_-^n = L^n - L_+^n$ denote the negative part.

L in (5) can be generated by dressing operator $S = 1 + \sum_{k=1}^{\infty} s_k \partial_q^{-k}$ in the following way

$$L = S \circ \partial_q \circ S^{-1}, \tag{6}$$

Dressing operator S satisfies Sato equation

$$\frac{\partial S}{\partial t_n} = -(L^n)_- S, \quad n = 1, 2, 3, \cdots . \tag{7}$$

The wave function for q-KP hierarchy is defined by

$$w_q(x, t; z) = S e_q(xz) \exp(\sum_{i=1}^{\infty} t_i z^i)$$

$$L w_q = z w_q$$

The adjoint wave function is

$$w_q^*(x,t;z) = (S^*)^{-1}|_{x/q} e_{1/q}(-xz) \exp(-\sum_{i=1}^{\infty} t_i z^i),$$

and the notation $P|_{x/t} = \sum_i P_i(x/t) t^i \partial_q^i$ is used for $P = \sum_i p_i(x) \partial_q^i$.

Theorem

(Iliev P) There exists the tau function of q−KP hierarchy i.e. τ_q, *s.t.*

$$w_q = \frac{\tau_q(x; t - [z^{-1}])}{\tau_q(x; t)} e_q(xz) exp(\sum_{i=1}^{\infty} t_i z^i), \tag{8}$$

$$w_q^* = \frac{\tau_q(x; t + [z^{-1}])}{\tau_q(x; t)} e_{1/q}(-xz) exp(-\sum_{i=1}^{\infty} t_i z^i), \tag{9}$$

here

$$[z] = (z, \frac{z^2}{2}, \frac{z^3}{3}, \cdots).$$

Theorem

(Iliev P) If τ is the tau function of the KP hierarchy, then

$$\tau_q(x;t) = \tau(t + [x]_q) \tag{10}$$

is the tau function of $q-$KP hierarchy, where

$$[x]_q = (x, \frac{(1-q)^2}{2(1-q^2)}x^2, \frac{(1-q)^3}{3(1-q^3)}x^3, \cdots).$$

Additional symmetries of the q-KP hierarchy

- Define Γ_q and M (Tu M.H.)

$$\Gamma_q = \sum_{i=1}^{\infty}[it_i + \frac{(1-q)^i}{(1-q^i)}x^i]\partial_q^{i-1} \tag{11}$$

$$M \equiv S\Gamma_q S^{-1} \tag{12}$$

- Dressing $[\partial_k - \partial_q^k, \Gamma_q] = 0$ gives:

$$\partial_k M = [B_k, M] \tag{13}$$

$$\partial_k(M^m L^n) = [B_k, M^m L^n] \tag{14}$$

- Define the additional flows

$$\frac{\partial S}{\partial t^*_{m,n}} = -(M^m L^n)_- S \tag{15}$$

or equivalently

$$\frac{\partial L}{\partial t^*_{m,n}} = -[(M^m L^n)_-, L] \tag{16}$$

- The additional flows $\partial^*_{mn} = \dfrac{\partial}{\partial t^*_{m,n}}$ commute with the hierarchy, i.e. $[\partial^*_{mn}, \partial_k] = 0$ but do not commute with each other, they are the additional symmetries of the q-KP hierarchy.

String equation of the q-KP hierarchy

Theorem

When $L^p = (L^p)_+$ and $\partial^*_{1,-p+1} S = 0$, the string equation of the p-reduced q-KP hierarchy is

$$[L^p, \frac{1}{p}(ML^{-p+1})_+] = 1, p = 2, 3, \cdots . \tag{17}$$

Process

OLD

- Additional flows
  $\Downarrow$
  String equation ($\partial^*_{-n+1,1}$)
  $\Downarrow$
  Virasoro constraint ($L_{-n}, n = 1, 2, 3, \cdots$)
- Additional flows $\longrightarrow \hat{w} \longrightarrow \tau$

This paper

- Virasoro constraints and W-constraints by Adler-Shiota-van Moerbeke (ASvM) formula

Vertex operator

The vertex operator $X_q(\mu,\lambda)$ is introduced by

$$X_q(\mu,\lambda)=e_q(x\mu)e_q^{-1}(x\lambda)exp(\sum_{i=1}^{\infty}t_i(\mu^i-\lambda^i))exp(-\sum_{i=1}^{\infty}\frac{\mu^{-i}-\lambda^{-i}}{i}\partial_i). \tag{18}$$

Vertex operator $X_q(\mu,\lambda)$ also can be denoted as

$$X_q(\mu,\lambda)=: exp(\alpha(\lambda)-\alpha(\mu)): \tag{19}$$

where $\alpha(\lambda)=\sum\alpha_n\cdot\frac{\lambda^{-n}}{n}$, and
$\alpha_n=\partial_n$ for $n>o$, $\alpha_0=0$
$\alpha_n=|n|t_{|n|}+\frac{(1-q)^{|n|}}{1-q^{|n|}}x^{|n|}$ for $n<o$, .
The Symbol :: means that we keep t_i be always the right side of ∂_j.

Taylor expansion of the $X_q(\mu,\lambda)$ on μ at the point of λ is

$$X_q(\mu,\lambda)=\sum_{m=0}^{\infty}\frac{(\mu-\lambda)^m}{m!}\sum_{n=-\infty}^{\infty}\lambda^{-m-n}W_n^{(m)},$$

here

$$\sum_{n=-\infty}^{\infty}\lambda^{-m-n}W_n^{(m)}=\partial_\mu^m X_q(\mu,\lambda)|_{\mu=\lambda}.$$

The first items of $W_n^{(m)}$ are

$$W_n^{(o)} = \delta_{n,0},$$

$$W_n^{(1)} = \alpha_n,$$

$$W_n^{(2)} = (-n-1)\alpha_n + \sum_{i+j=n} :\alpha_i\alpha_j:$$

$$W_n^{(3)} = (n+1)(n+1)\alpha_n + \sum_{i+j+k=n} :\alpha_i\alpha_j\alpha_k: -\frac{3}{2}(n+2)\sum_{i+j=n} :\alpha_i\alpha_j:$$

ASvM Formula

There is ASvM formula for q-KP hierarchy:

$$X_q(\mu, \lambda) w_q(x, t; z) = (\lambda - \mu) Y_q(\mu, \lambda) w_q(x, t; z). \tag{20}$$

where the operator $Y_q(\mu, \lambda)$ is the generators of additional symmetry of q-KP hierarchy as

$$Y_q(\mu, \lambda) = \sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \sum_{n=-\infty}^{\infty} \lambda^{-m-n-1} (M^m L^{m+n})_-. \tag{21}$$

ASvM formula is equivalent to the following equation

$$\partial^*_{n+m,m} \tau_q = \frac{W_n^{(m+1)}(\tau_q)}{m+1} \tag{22}$$

Consider the condition $\partial^*_{n+m,m}\hat{w}_q = 0$, we have

$$\hat{w}_q(G(z)-1)\frac{\partial^*_{n+m,m}\tau_q}{\tau_q} = 0. \qquad (23)$$

- Using the ASvM formula we get

$$\left(\frac{W_n^{(m+1)}}{m+1} - c\right)\tau_q = 0, \;\; m = 0, 1, 2, 3\cdots \qquad (24)$$

This equation is the key part of this method.

Virasoro constraints for p-reduced q-KP hierarchy

For $m = 0$,

$$(\frac{W_n^{(1)}}{1} - c)\tau_q = 0. \qquad (25)$$

It is just the condition $L^p = (L^p)_+$ for p-reduced q-KP hierarchy.
For $m = 1$, it is

$$(\frac{W_n^{(2)}}{2} - c)\tau_q = 0,$$

i.e.

$$(\frac{1}{2}\sum_{i+j=n} : \alpha_i\alpha_j : -c)\tau_q = 0 \qquad (26)$$

Let $n = kp$, and denote $\tilde{t}_i = t_i + \frac{(1-q)^i}{i(1-q^i)} x^i, \ i = 1, 2, 3, \cdots$.
Virasoro constraints of the p-reduced q-KP hierarchy are

$$L_n \tau_q = 0, \quad n = -1, 0, 1, 2, 3, \cdots ,$$

here

$$L_{-1} \equiv \frac{1}{p} \sum_{\substack{n = p+1 \\ n \neq 0 (\mathrm{mod} p)}}^{\infty} n\tilde{t}_n \frac{\partial}{\partial \tilde{t}_{n-p}} + \frac{1}{2p} \sum_{i+j=p} ij \tilde{t}_i \tilde{t}_j,$$

$$L_0 \equiv \frac{1}{p} \sum_{\substack{n = 1 \\ n \neq 0 (\mathrm{mod} p)}}^{\infty} n\tilde{t}_n \frac{\partial}{\partial \tilde{t}_n} + \left(\frac{p}{24} - \frac{1}{24p}\right),$$

$$L_k \equiv \frac{1}{p} \sum_{\substack{n=1 \\ n \neq 0 (\mathrm{mod} p)}}^{\infty} n\tilde{t}_n \frac{\partial}{\partial \tilde{t}_{n+kp}} + \frac{1}{2p} \sum_{\substack{i+j=kp \\ i,j \neq 0 (\mathrm{mod} p)}} ij\tilde{t}_i \tilde{t}_j .$$

L_n satisfy Virasoro algebra commutation relations

$$[L_n, L_m] = (n-m)L_{(n+m)}, \;\; m, n = -1, 0, 1, 2, 3, \cdots , \tag{27}$$

W-constraints for the p-reduced q-KP hierarchy

For $m = 2$, it is

$$(\frac{W_n^{(3)}}{3} - c)\tau_q = 0,$$

i.e.

$$(\frac{1}{3}\sum_{i+j+h=kp} :\alpha_i\alpha_j\alpha_h: - c)\tau_q = 0 \tag{28}$$

W-constraints for the p-reduced q-KP hierarchy

Let

$$w_m \equiv \sum_{\substack{i+j+h=mp \\ i,j,h \neq 0 (\mathrm{mod} p)}} :\alpha_i \alpha_j \alpha_h:, \quad m \geq -2,$$

which are the W-constraints for the p-reduced q-KP hierarchy, i.e.

$$w_m \tau_q = 0, m \geq -2,$$

and they satisfy following algebra commutation relations

$$[L_n, w_m] = (2n-m) w_{n+m}, n \geq -1, m \geq -2$$

Summary and Discussion

- $m = 1 \Rightarrow$ Virasoro constraints
  $m = 2 \Rightarrow$ W-constraints
  $m = 3, 4, 5, \cdots \Rightarrow$ higher algebra constraints in a similar manner
- The q-KP hierarchy tends to the classical KP hierarchy when $q \to 1$ and $u_0 = 0$, and then $t_1 + x \to x$. The result of this paper is consistent with the classical ones given by L. A. Dickey and S. Panda, S.Roy.
- We have known the Virasoro constraints and W-constraints for the KP, BKP(Tu M.H.), q-KP hierarchy,
  What about CKP hierarchy?

- string theory: Matrix Models、Seiberg-Witten theory

  partition function $\sim$ tau function (Douglas M.)
  solving Virasoro constraints in matrix models(h-th/0412205)
  Kontsevich-Hurwitz partition function(arXiv:0807.2843)

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Thank you!