Integral transformation and Darboux transformation of Heun’s differential equation

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References

Heun’s differential equation

\[
\frac{d^2y}{dz^2} + \left( \frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-t} \right) \frac{dy}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z-t)} y = 0,
\]

with the condition

\[
\gamma + \delta + \epsilon = \alpha + \beta + 1.
\]

Four singularities \(\{0, 1, t, \infty\}\).

Three singularities: Hypergeometric equation

\[
\frac{d^2y}{dz^2} + \left( \frac{\gamma}{z} + \frac{\alpha + \beta - \gamma + 1}{z-1} \right) \frac{dy}{dz} + \frac{\alpha\beta}{z(z-1)} y = 0,
\]

which has been studied very well.

It is much harder to study Heun’s equation.

**Known solutions of Heun’s equation**

- Heun polynomial (Quasi-exact solvability)
- Heun function (Approximation)
- Algebraic solutions (Finite monodromy)
- Finite-gap integration

We now change variables.
Elliptic functions

\( \wp(x) \) : Weierstrass elliptic function.

\[
\wp(x) = \frac{1}{x^2} + \sum_{(m,n) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0,0)\}} \left( \frac{1}{(x - 2m\omega_1 - 2n\omega_3)^2} - \frac{1}{(2m\omega_1 + 2n\omega_3)^2} \right).
\]

Double-periodicity:

\[
\wp(x) = \wp(x + 2\omega_1) = \wp(x + 2\omega_3).
\]

Set \( \omega_2 = -\omega_1 - \omega_3 \), \( e_i = \wp(\omega_i) \) (\( i = 1, 2, 3 \)).

Half periods: 0(\( = \omega_0 \)), \( \omega_1, \omega_2, \omega_3 \).

Relations:

\[
e_1 + e_2 + e_3 = 0,
\]

\[
(\wp'(x))^2 = 4(\wp(x) - e_1)(\wp(x) - e_2)(\wp(x) - e_3),
\]

\[
\frac{\wp''(z)}{\wp'(z)^2} = \frac{1}{2} \sum_{i=1}^{3} \frac{1}{\wp(x) - e_i},
\]

\[
\wp(x + \omega_1) = e_1 + \frac{(e_1 - e_2)(e_1 - e_3)}{\wp(x) - e_1}, \text{ etc.}
\]
Elliptic representation

Heun’s differential equation

\[ \frac{d^2y}{dz^2} + \left( \frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-t} \right) \frac{dy}{dz} + \frac{\alpha \beta z - q}{z(z-1)(z-t)} y = 0, \]

\( q \): accessory parameter.

By setting

\[ z = \frac{\varphi(x) - e_1}{e_2 - e_1}, \quad t = \frac{e_3 - e_1}{e_2 - e_1}, \]

\[ y\tilde{\Phi}(z) = f(x), \quad \tilde{\Phi}(z) = z^{-\frac{l_0}{2}} (z-1)^{-\frac{l_1}{2}} (z-t)^{-\frac{l_2}{2}}, \]

Heun’s equation is transformed to

\[ \left( -\frac{d^2}{dx^2} + \sum_{i=0}^{3} l_i(l_i + 1) \varphi(x + \omega_i) \right) f(x) = Ef(x). \]

Correspondence

\[ 0 \leftrightarrow \omega_1, \quad 1 \leftrightarrow \omega_2, \quad t \leftrightarrow \omega_3, \quad \infty \leftrightarrow \omega_0 (= 0), \]

\[ l_0 = \beta - \alpha - 1/2, \quad l_1 = -\gamma + 1/2, \quad l_2 = -\delta + 1/2, \]

\[ E = -4q(e_2 - e_1) + (*), \quad l_3 = -\epsilon + 1/2. \]

The case \( l_1 = l_2 = l_3 = 0 \) (\( \gamma = \delta = \epsilon = 1/2 \)): Lamé’s differential equation
\[ H^{(l_0,l_1,l_2,l_3)} = -\frac{d^2}{dx^2} + \sum_{i=0}^{3} l_i(l_i + 1)\wp(x + \omega_i). \]

Finite-gap integration is applicable for the case \( l_0, l_1, l_2, l_3 \in \mathbb{Z}, \) all eigenvalues \( E. \)

\( \Rightarrow \) Monodromy formulas by hyperelliptic integral, Hermite-Krichever Ansatz.

Set

\[ H_1 = -\frac{d^2}{dx^2} + 6\wp(x), \quad (l_0 = 2, \ l_1 = l_2 = l_3 = 0) \]

\[ H_2 = -\frac{d^2}{dx^2} + 2\wp(x) + 2\wp(x + \omega_1) + 2\wp(x + \omega_2), \quad (l_0 = l_1 = l_2 = 1, \ l_3 = 0). \]

It is shown that monodromy formulas for \( H_1 \) coincide with the ones for \( H_2. \)

We explain this phenomena by Darboux transformation.
Moreover, we establish that eigenfunctions of

\[- \frac{d^2}{dx^2} + 2l(2l + 1) \varphi(x)\]

with eigenvalue \( E \) is isomonodromic to the ones of

\[- \frac{d^2}{dx^2} + l(l + 1) \varphi(x) + l(l + 1) \varphi(x + \omega_1) + l(l + 1) \varphi(x + \omega_2) + (l - 1)l \varphi(x + \omega_3)\]

by generalized Darboux transformation.
Darboux transformation

Set

\[ H = -\frac{d^2}{dx^2} + q(x) \]

and assume that \( \phi_0(x) \) satisfies

\[ H\phi_0(x) = E_0\phi_0(x). \]

Then \( q(x) = \frac{\phi_0''(x)}{\phi_0(x)} + E_0 \). Set

\[ L = \frac{d}{dx} - \frac{\phi_0'(x)}{\phi_0(x)}, \quad L^\dagger = -\frac{d}{dx} - \frac{\phi_0'(x)}{\phi_0(x)}. \]

We have

\[ L^\dagger L = \left( -\frac{d}{dx} - \frac{\phi_0'(x)}{\phi_0(x)} \right) \left( \frac{d}{dx} - \frac{\phi_0'(x)}{\phi_0(x)} \right) \]
\[ = -\frac{d^2}{dx^2} + \left( \frac{\phi_0'(x)}{\phi_0(x)} \right)' + \left( \frac{\phi_0'(x)}{\phi_0(x)} \right)^2 \]
\[ = H - E_0, \]

\[ LL^\dagger = -\frac{d^2}{dx^2} - \left( \frac{\phi_0'(x)}{\phi_0(x)} \right)' + \left( \frac{\phi_0'(x)}{\phi_0(x)} \right)^2 \]
\[ = H - 2 \left( \frac{\phi_0'(x)}{\phi_0(x)} \right)' - E_0. \]
Set

\[ \tilde{H} = -\frac{d^2}{dx^2} + q(x) + 2 \left( \frac{\phi_0'(x)}{\phi_0(x)} \right)' , \]

then

\[ H = L^\dagger L + E_0, \quad \tilde{H} = LL^\dagger + E_0, \]

\[ LH = LL^\dagger L + E_0L = \tilde{H}L, \]
\[ L^\dagger \tilde{H} = L^\dagger LL^\dagger + E_0L^\dagger = HL^\dagger. \]

If \( f(x) \) is an eigenfunction of \( H \) with the eigenvalue \( E \), then \(Lf(x)\) is an eigenfunction of \( \tilde{H} \) with the eigenvalue \( E \), because

\[ \tilde{H}(Lf(x)) = LHf(x) = L(Ef(x)) = E(Lf(x)). \]

Note that the operator \( L \left( = \frac{d}{dx} - \frac{\phi_0'(x)}{\phi_0(x)} \right) \) annihilates the 1-dimensional space \( \mathbb{C}\phi_0(x) \).
Generalized Darboux transf.

\[ H = -\frac{d^2}{dx^2} + q(x). \]

\( U: \) \( n \)-dimensional space of functions

\[ L = \left( \frac{d}{dx} \right)^n + \sum_{i=1}^{n} c_i(x) \left( \frac{d}{dx} \right)^{n-i} \]

is the operator that annihilates any elements in \( U \), i.e., \( Lf(x) = 0 \) for all \( f(x) \in U \). Set

\[ \tilde{H} = -\frac{d^2}{dx^2} + q(x) + 2c'_1(x). \]

**Proposition 1.** (c.f. Crum 1955 Aoyama, Sato, Tanaka 2001)

If the space \( U \) is invariant under the action of \( H \), then we have

\[ \tilde{H}L = LH. \]

We call \( L \) Crum-Darboux transformation (the generalized Darboux transformation).

The case \( n = 1 \).

\( U = \mathbb{C} \phi_0(x), \) \( L = \frac{d}{dx} - \frac{\phi'_0(x)}{\phi_0(x)}, \) \( 2c'_1(x) = 2 \left( \frac{\phi'_0(x)}{\phi_0(x)} \right)'. \)

We reproduce Darboux transformation.
Quasi-solvability of Heun’s equation

\[ H(l_0, l_1, l_2, l_3) = -\frac{d^2}{dx^2} + \sum_{i=0}^{3} l_i(l_i + 1) \wp(x + \omega_i). \]

**Proposition 2.** (Quasi-solvability)
\[ \alpha_i = -l_i \text{ or } l_i + 1 \quad (i = 0, 1, 2, 3), \quad d = -\sum_{i=0}^{3} \alpha_i/2. \]
Assume \( d \in \mathbb{Z}_{\geq 0} \).
Let \( V_{\alpha_0, \alpha_1, \alpha_2, \alpha_3} \) be the \( d+1 \)-dimensional space spanned by
\[ \left\{ \hat{\Phi}(\wp(x))\wp(x)^n \right\}_{n=0,\ldots,d}, \]
where
\[ \hat{\Phi}(z) = (z - e_1)^{\alpha_1/2}(z - e_2)^{\alpha_2/2}(z - e_3)^{\alpha_3/2}. \]
Then the operator \( H(l_0, l_1, l_2, l_3) \) preserves the space \( V_{\alpha_0, \alpha_1, \alpha_2, \alpha_3} \).

**Proposition 3.** Write the minimal annihilation operator \( L_{\alpha_0, \alpha_1, \alpha_2, \alpha_3} \) of \( V_{\alpha_0, \alpha_1, \alpha_2, \alpha_3} \) as
\[ L_{\alpha_0, \alpha_1, \alpha_2, \alpha_3} = \left( \frac{d}{dx} \right)^{d+1} + \sum_{i=1}^{d+1} c_i(x) \left( \frac{d}{dx} \right)^{d+1-i}. \]
Then
\[ c_1(x) = -\frac{d + 1}{4} \left( \sum_{i=1}^{3} \frac{2\alpha_i + d}{\wp(x) - e_i} \right) \wp'(x), \]
and \( c_i(x) \) \( (i = 1, \ldots, d+1) \) are doubly-periodic.
Crum-Darboux transformation for Heun’s equation

Theorem 4.

\[ \alpha_i = -l_i \text{ or } l_i + 1 \ (i = 0, 1, 2, 3), \]
\[ d = -\sum_{i=0}^{3} \alpha_i / 2. \text{ Assume } d \in \mathbb{Z}_{\geq 0}. \]

Let \( L_{\alpha_0,\alpha_1,\alpha_2,\alpha_3} \) be the operator defined in Proposition 3. Then we have

\[
H^{(\alpha_0+d,\alpha_1+d,\alpha_2+d,\alpha_3+d)} L_{\alpha_0,\alpha_1,\alpha_2,\alpha_3} = L_{\alpha_0,\alpha_1,\alpha_2,\alpha_3} H^{(l_0,l_1,l_2,l_3)}.
\]

Proof. It follows from Proposition 1 that

\[
(H^{(l_0,l_1,l_2,l_3)} + 2c_1'(x)) L_{\alpha_0,\alpha_1,\alpha_2,\alpha_3} = L_{\alpha_0,\alpha_1,\alpha_2,\alpha_3} H^{(l_0,l_1,l_2,l_3)}.
\]

It is shown that

\[
H^{(l_0,l_1,l_2,l_3)} + 2c_1'(x) = H^{(\alpha_0+d,\alpha_1+d,\alpha_2+d,\alpha_3+d)}.
\]

If \( d \equiv 0 \) (the case of Darboux transformation), then the theorem was essentially obtained by Khare and Sukhatme (2005).
Monodromy

\( f_1(x, E), f_2(x, E) \): a basis of solutions to \((H^{(l_0, l_1, l_2, l_3)} - E)f(x) = 0.\)

\( f_1(x + 2\omega_k, E), f_2(x + 2\omega_k, E) \) \((k = 1, 3)\) are also solutions to the differential equation, and

\[
(f_1(x + 2\omega_k, E) \ f_2(x + 2\omega_k, E)) = (f_1(x, E) \ f_2(x, E)) \begin{pmatrix} a_{11}^{(k)} & a_{12}^{(k)} \\ a_{21}^{(k)} & a_{22}^{(k)} \end{pmatrix}.
\]

Set \( \tilde{f}_i(x, E) = L_{\alpha_0, \alpha_1, \alpha_2, \alpha_3} f_i(x, E) \) \((i = 1, 2)\). Then

\[
H^{(\alpha_0 + d, \alpha_1 + d, \alpha_2 + d, \alpha_3 + d)} \tilde{f}_i(x, E) = E \tilde{f}_i(x, E).
\]

Since \( L_{\alpha_0, \alpha_1, \alpha_2, \alpha_3} \) is doubly-periodic, we have

\[
(\tilde{f}_1(x + 2\omega_k, E) \ \tilde{f}_2(x + 2\omega_k, E)) = (\tilde{f}_1(x, E) \ \tilde{f}_2(x, E)) \begin{pmatrix} a_{11}^{(k)} & a_{12}^{(k)} \\ a_{21}^{(k)} & a_{22}^{(k)} \end{pmatrix}.
\]

**Proposition 5.** The monodromy structure of \( H^{(l_0, l_1, l_2, l_3)} \) coincides with the one of \( H^{(\alpha_0 + d, \alpha_1 + d, \alpha_2 + d, \alpha_3 + d)} \).

Namely, the operator \( L_{\alpha_0, \alpha_1, \alpha_2, \alpha_3} \) defines an isomonodromic transformation from \( H^{(l_0, l_1, l_2, l_3)} \) to \( H^{(\alpha_0 + d, \alpha_1 + d, \alpha_2 + d, \alpha_3 + d)} \).
Example

The case $l_0 = 2l$ ($l \in \mathbb{Z}_{\geq 1}$), $l_1 = l_2 = l_3 = 0$.

Set $\alpha_0 = -2l$, $\alpha_1 = \alpha_2 = 1$, $\alpha_3 = 0$.
Then $d = -(\alpha_0 + \cdots + \alpha_3)/2 = l - 1$.

$$
H^{(-l-1,l,l,l-1)} L_{-2l,1,1,0} = L_{-2l,1,1,0} H^{(2l,0,0,0)},
$$

$$
H^{(-l-1,l,l,l-1)} = H^{(l,l,l,l-1)}.
$$

$H^{(2l,0,0,0)}$ is isomonodromic to $H^{(l,l,l,l-1)}$.

If $l = 1$, then $d = 0$,

$$
H^{(1,1,1,0)} L_{-2,1,1,0} = L_{-2,1,1,0} H^{(2,0,0,0)},
$$

and the operator $L_{-2,1,1,0}$ is written as

$$
L_{-2,1,1,0} = \frac{d}{dx} - \frac{\varphi'(x)}{2(\varphi(x) - e_1)} - \frac{\varphi'(x)}{2(\varphi(x) - e_2)}.
$$

Hence $H_1 = -\frac{d^2}{dx^2} + 6\varphi(x)$ is isomonodromic to $H_2 = -\frac{d^2}{dx^2} + 2\varphi(x) + 2\varphi(x + \omega_1) + 2\varphi(x + \omega_2)$. 
Application to finite-gap integration

A feature of finite-gap integration:
Existence of an differential operator $\tilde{A}$ s.t.
$[\tilde{A}, H] = 0$ ($H = -d^2/dx^2 + v(x)$) and
deg($\tilde{A}$) is odd.
($\Leftrightarrow$ the potential $v(x)$ satisfies stationary higher order KdV equation)

**Theorem 6.**

*If* $l_0, l_1, l_2, l_3 \in \mathbb{Z}$, *then we can construct an odd-order differential operator $\tilde{A}$ such that* 
$[\tilde{A}, H^{(l_0,l_1,l_2,l_3)}] = 0$ *by composing four Crum-Darboux transformations.*

If $l_0 = 2$, $l_1 = l_2 = l_3 = 0$, then
$\tilde{A} = L_{2,-1,-1,0}L_{1,-2,1,0}L_{0,2,-1,-1}L_{-2,0,0,0}$. 
Integral transformation of Heun’s equation

Middle convolution for \(2 \times 2\) Fuchsian system with four singularities \(\{0, 1, t, \infty\}\)

\[
\frac{dY}{dz} = \left( \frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_t}{z-t} \right) Y, \quad Y = \begin{pmatrix} y_1(z) \\ y_2(z) \end{pmatrix}.
\]

⇒ Integral transformation of \(2 \times 2\) Fuchsian system with four singularities \(\{0, 1, t, \infty\}\) (T. JMAA 2008).
⇒ Integral transformation of Heun’s equation (T. SIGMA 2009).

But it was already established by Kazakov and Slavyanov (1996) by another method.

Theorem 7. ([KS1996]) Set

\[
\{\mu - (2 - \alpha)\}\{\mu - (2 - \beta)\} = 0, \quad \gamma' = \gamma + \mu - 1,
\]

\[
\delta' = \delta + \mu - 1, \quad \epsilon' = \epsilon + \mu - 1,
\]

\[
\alpha' = \mu, \quad \beta' = 2\mu + \alpha + \beta - 3,
\]

\[
q' = q + (1 - \mu)(\epsilon + \delta t + (\gamma - \mu)(t + 1)).
\]

Let \(y(w)\) be a solution to

\[
\frac{d^2y}{dw^2} + \left( \frac{\gamma}{w} + \frac{\delta}{w-1} + \frac{\epsilon}{w-t} \right) \frac{dy}{dw} + \frac{\alpha\beta w - q}{w(w-1)(w-t)} y = 0.
\]

Then the functions \((i \in \{0, 1, t, \infty\})\)

\[
\tilde{y}(z) = \int_{[\alpha_i, \alpha_i]} y(w)(z - w)^{-\mu} dw
\]

are solutions to

\[
\frac{d^2\tilde{y}}{dz^2} + \left( \frac{\gamma'}{z} + \frac{\delta'}{z-1} + \frac{\epsilon'}{z-t} \right) \frac{d\tilde{y}}{dz} + \frac{\alpha'\beta' z - q'}{z(z-1)(z-t)} \tilde{y} = 0.
\]
Elliptic representation of integral transformation

\( \sigma(x) \): Weierstrass sigma function,
\( \sigma_i(x) (i = 1, 2, 3) \): Weierstrass co-sigma function which has a zero at \( x = \omega_i \).

Let \( \alpha_0 \in \{-l_0, l_0 + 1\} \) and set

\[
\begin{align*}
\eta &= \frac{\alpha_0 + l_1 + l_2 + l_3 + 3}{2}, \\
l'_0 &= \frac{-\alpha_0 + l_1 + l_2 + l_3 + 1}{2}, \\
l'_1 &= \frac{-\alpha_0 + l_1 - l_2 - l_3 - 1}{2}, \\
l'_2 &= \frac{-\alpha_0 - l_1 + l_2 - l_3 - 1}{2}, \\
l'_3 &= \frac{-\alpha_0 - l_1 - l_2 + l_3 - 1}{2}.
\end{align*}
\]

**Proposition 8.** If \( f(x) \) is a solution of

\[
\left(-\frac{d^2}{dx^2} + \sum_{i=0}^{3} l_i(l_i + 1)\varphi(x + \omega_i) - E\right) f(x) = 0,
\]

then the function

\[
\begin{align*}
\tilde{f}(x) &= \sigma(x)^{-l'_0} \sigma_1(x)^{-l'_1} \sigma_2(x)^{-l'_2} \sigma_3(x)^{-l'_3} \int_{I_l} f(y)\sigma(y)^{-\alpha_0+1} \\
\sigma_1(y)^{l_1+1} \sigma_2(y)^{l_2+1} \sigma_3(y)^{l_3+1} (\sigma(x + y)\sigma(x - y))^{-\eta} dy
\end{align*}
\]

(\( l \in \{0, 1, 2, 3\} \)) is a solution of

\[
\left(-\frac{d^2}{dx^2} + \sum_{i=0}^{3} l'_i(l'_i + 1)\varphi(x + \omega_i) - E\right) \tilde{f}(x) = 0.
\]

\( I_l \ (l = 0, 1, 2, 3) \): suitable cycle on \( \mathbb{C} \) with variable \( y \).

If \( \eta \in \mathbb{Z}_{\geq 1} \), then we essentially recover generalized Darboux transformation.
Application to monodromy

$f_1(x, E), f_2(x, E)$: independent solutions of

$$\left(-\frac{d^2}{dx^2} + \sum_{i=0}^{3} l_i(l_i + 1)\varphi(x + \omega_i) - E\right) f(x) = 0, \quad (1)$$

$M_{2\omega_k}(E) \ (k \in \{1, 3\})$: $2 \times 2$ matrix;
Monodromy matrix of Eq.(1) w.r.t. $x \rightarrow x + 2\omega_k$

$$(f_1(x + 2\omega_k, E) \ f_2(x + 2\omega_k, E)) = (f_1(x, E) \ f_2(x, E)) M_{2\omega_k}(E)$$

$$\left(-\frac{d^2}{dx^2} + \sum_{i=0}^{3} l'_i(l'_i + 1)\varphi(x + \omega_i) - E\right) \tilde{f}(x) = 0, \quad (2)$$

$M'_{2\omega_k}(E)$: Monod. matrix of Eq.(2) w.r.t. $x \rightarrow x + 2\omega_k$.

**Theorem 9.**

$$\text{tr} M'_{2\omega_k}(E) = \text{tr} M_{2\omega_k}(E), \quad (k = 1, 3)$$

**Corollary 10.** Let $k \in \{1, 3\}$.

$\exists f(x, E)$: solution of Eq.(1)

s.t. $f(x + 2\omega_k, E) = C_k(E) f(x, E)$

$\Rightarrow \exists \tilde{f}(x, E)$: solution of Eq.(2)

s.t. $\tilde{f}(x + 2\omega_k, E) = C_k(E) \tilde{f}(x, E)$.

In other word, periodicity is preserved by the integral transformation.
Summary

Isomonodromic transformations for Heun’s equation by Crum-Darboux transformations.

Relationship to finite-gap integration. Construction of a commuting operator.

Integral transformations for Heun’s equation: A generalization of Crum-Darboux transformation.

Invariance of monodromy