Classification of local and quasi-local symmetries to fourth-order nonlinear evolution equations

Changzheng Qu

joint work with Q. Huang and R. Zhdanov

Center for Nonlinear Studies, Northwest University, Xi'an
Department of Mathematics, Northwest University, Xi'an

Outline

- Introduction
- Classification of Lie point symmetries
- Classification of quasi-local symmetries
- Invariant sets

1. Introduction

Consider the general nonlinear evolution equation

$$u_t = F(t, x, u, u_x, \cdots, u_n), \tag{1}$$

where u_i denote the ith derivative of u with respect to x.

1.1. The second-order evolution equations

♠ Lie point symmetries

Definition 1.1. A vector field

$$V = \tau(t, x, u)\frac{\partial}{\partial t} + \xi(t, x, u)\frac{\partial}{\partial x} + \eta(t, x, u)\frac{\partial}{\partial u}, \tag{2}$$

is a *Lie point symmetry* of (1) iff

$$V^{(2)}(u_t - F)|_E = 0,$$

where E is the solution manifold of (1).

Prop. 1.1. The most general invariant group of (1) is generated by

$$V = \tau(t)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u},$$

where $\tau(t)$, $\xi(t,x,u)$ and $\eta(t,x,u)$ and F satisfy the determining equation

$$\eta^t - \tau F_t - \xi F_x - \eta F_u - \eta^x F_{u_x} - \cdots - \eta^{(n)} F_{u_n} = 0,$$

where and hereafter $\eta^{(j)}$ is the jth-prolongation of η.

Difficulty: F is too general.

- Nonlinear diffusion equations (Ovsianikov, 1959, 1978)

$$u_t = (D(u)u_x)_x,$$

- Nonlinear diffusion equations with convection term (Yung, Verburg and Baveye, 1994)

$$u_t = (D(u)u_x)_x + P(u)u_x.$$

- Nonlinear diffusion equations with source term (Dorodnitsyn, 1994; Galaktionov et al, 1988)

$$u_t = (D(u)u_x)_x + Q(u).$$

- Burgers equation with variable coefficients (Qu, 1997)

$$u_t = A(t,x)u_{xx} + B(t,x)uu_x.$$

- Nonlinear diffusion equations with convection and source terms (Cherniha and Serov, 1998)

$$u_t = (D(u)u_x)_x + P(u)u_x + Q(u).$$

- Nonlinear diffusion equations with convection and source terms (Gandarias, 1996)

$$u_t = (D(u)u_x)_x + g(x)u^m u_x + f(x)u^s.$$

- Linear diffusion equations with source term (Chou and Qu, 2002; Zhdanov and Lahno, 1999)

$$u_t = u_{xx} + F(t, x, u, u_x).$$

- Nonlinear diffusion equations (Bluman, 1988; Akhatov, Gazizov, Ibragimov, 1987)

$$u_t = D(u_x)u_{xx}.$$

- General nonlinear diffusion equations (Gazizov, Ibragimov, 1987)

$$u_t = D(u_{xx}).$$

- The curve evolution flows (Chou, Li, 2002; Qu, Huang, 2007, 2008)

$$\gamma_t = f(\kappa)\vec{n}.$$

Question: How to classify (1) with general F?
Equations (1) have been classified within four-dimensional symmetries (Zhdanov, Lahno, Basarab-Horwath, 2001, 2007).

Remark:
The group-invariant solutions associated with the Lie point symmetries are useful to study the behavior of general solutions of the considered equations.

♠ Conditional symmetries

Definition 1.2. A vector field

$$V = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}, \tag{3}$$

is a *conditional symmetry* of (1) iff

$$V^{(2)}(u_t - F)|_{E\cap M} = 0,$$

where E is the solution manifold of (1), where M denotes the set of equations $D_x^i(\eta - \xi u_x - \tau u_t) = 0$, $i = 1, 2, \cdots,$.

- Linear heat equation (Bluman and Cole, 1969)

$$u_t = u_{xx}.$$

- Burgers equation (Arrigo, Broadbridge and Hill, 1993; Pucci, 1992)

$$u_t = u_{xx} - uu_x.$$

- Fitzhugh-Nagumo equation (Nucci, Clarkson, 1992)

$$u_t = u_{xx} + u(1-u)(u-a).$$

- Linear heat equation with source term (Clarkson and Mansfield, 1993)

$$u_t = u_{xx} + Q(u),$$

- Nonlinear diffusion equation with source term (Clarkson and Mansfield, 1994; Arrigo and Hill, 1995)

$$u_t = (D(u)u_x)_x + Q(u),$$

- Nonlinear diffusion equation with convection and source terms (Cherniha and Serov, 1998)

$$u_t = (D(u)u_x)_x + P(u)u_x + Q(u).$$

Question: How to classify (1) for general F?

♠ Lie Bäcklund symmetries

Let

$$V = \eta[u]\frac{\partial}{\partial u}, \tag{4}$$

be an evolutionary vector field with $[u] = (t, x, u, u_x, \cdots,)$

Definition 1.3. The evolutionary vector field (4) is said to be a *Lie Bäcklund symmetry* (LBS) of equation (1) iff

$$V^{(2)}(u_t - F)|_E = 0,$$

• Scalar case (Svinolupov, Sokolov, 2002).

• System case (Mikhailov, Shabat, Sokolov, Yamilov, Olver, Foursov, Wang JP, Svinolupov, Qu CZ, Kang J, Anco, Wolf et al).

Remark : The existence of higher-order Lie Bäcklund symmetries is associated with the linearization of the nonlinear equations (Fokas, Yortsos, 1982).

♠ Conditional Lie-Bäcklund symmetries

Definition 1.4. *(Zhdanov, 1995) The evolutionary vector field (14) is said to be a conditional Lie Bäcklund symmetry (CLBS) of (1) iff*

$$V(u_t - F)|_{E\cap W} = 0,$$

where W *denotes the set of* $D_x^j\eta = 0,\ j = 0, 1, 2, \cdots$.

Prop. 1.2. *Equation (1) admits the CLBS* η *if there exists a function* $G(t, x, u, \eta)$ *such that*

$$\frac{\partial\eta}{\partial t} = [E, \eta] + G(t, x, u, \eta), \quad G(t, x, u, 0) = 0,$$

where $[E, \eta] = E'\eta - \eta' E$.

- Porous medium equation (Qu, 1999)

$$u_t = [D(u)u_x]_x.$$

- Nonlinear diffusion equation with convection and source terms (Qu, 1997; Qu and Ji, 2007)

$$u_t = [D(u)(u_x)^n]_x + P(u)u_x + Q(u).$$

- Nonlinear diffusion equation with x-dependent convection and source terms (Qu, 2004)

$$u_t = [D(u)u_x]_x + P(x,u)u_x + Q(x,u).$$

- Higher-dimensional nonlinear diffusion equation (Ji and Qu, 2007,2008)

$$u_t = \mathrm{div}\,(|x|^s D(u)|\bigtriangledown u|^{m-1} \bigtriangledown u) + Q(x,u).$$

- Reduction of initial value problem of nonlinear diffusion equations (Zhdanov, 2004).

Assume that equation (1) with $n = 2$ admits the CLBS with the characteristic

$$\eta = u_{xx} - g(x, u, u_x), \tag{5}$$

then the functions satisfy

$$A_x + A_u u_x + g A_{u_x} - g_t - F g_u - A g_{u_x} = 0,$$

where

$$A \equiv F_x + F_u u_x + g F_{u_x} + F_{u_x}(g_x + g_u u_x + g g_{u_x}).$$

Assume that equation (1) with $n = 2$ admits the CLBS with the characteristic

$$\eta = u_{xxx} - g(x, u, u_x, u_{xx}), \tag{6}$$

then the functions satisfy

$$\begin{aligned} & B_x + B_u u_x + B_{u_x} u_{xx} + B_{u_{xx}} g \\ & -g_t - F g_u - A g_{u_x} - B g_{u_{xx}} = 0, \end{aligned}$$

where

$$\begin{aligned} A &= F_x + F_u u_x + F_{u_x} u_{xx} + g F_{u_{xx}}, \\ B &= A_x + A_u u_x + A_{u_x} u_{xx} + g A_{u_{xx}}. \end{aligned}$$

Remark: The CLBSs are associated with the geometric properties of nonlinear parabolic equations such as the B-concave, B-convex property and Harnack inequalities.

♠ Quasi-local symmetries

Definition 1.5. If a symmetry of (1) depend on a nonlocal variable, then it is said to be a *quasi-local symmetry* .

• Nonlinear diffusion equation (Ibragimov et al 1987, 1991; Bluman et al 1988)

$$u_t = [D(u)u_x]_x.$$

• Nonlinear Newton fluid equation with convection term (Chou, Qu, 1999; Popovych, Ivanova, 2005)

$$u_t = [D(u)(u_x)^n + P(u)]_x.$$

- Variable-coefficient nonlinear equations with convection term (Popovych, Ivanova, Sophocleous, 2009)

$$f(x)u_t = [g(x)D(u)u_x + h(x)P(u)]_x.$$

- Systems of nonlinear parabolic equations (Qu, 2007; Qu and Kang, 2008; Ivanova, Sophocleous, 2008)

$$u_t = [f(u,v)u_x + g(u,v)v_x + r(u,v)]_x,$$
$$v_t = [p(u,v)v_x + q(u,v)u_x + s(u,v)]_x.$$

1.2. The third-order evolution equations

♠ Lie point symmetries

- The KdV equation (Kostin, 1969)

$$u_t + u_{xxx} + 6uu_x = 0.$$

- Generalized KdV-Burgers equation (Zakharov and Korobeinikov, 1980; Korobeinikov, 1983)

$$u_t + \beta u_{xxx} + \gamma u^m u_x - \mu u_{xx} + \alpha \frac{u}{t} = 0.$$

- Variable-coefficient KdV equation (Gazeau and Winternitz, 1992)

$$u_t + g(x,t)u_{xxx} + f(x,t)uu_x = 0.$$

- The general KdV equation (Gungor, Lahno and Zhdanov, 2004)

$$u_t = u_{xxx} + f(t, x, u, u_x, u_{xx}).$$

♠ Lie Bäcklund symmetries

- The KdV equation (GGKM, 1974) has the recursion operator

$$\Re = D_x^2 + 4u + 2u_x D_x^{-1}.$$

- The potential mKdV equation (Ibragimov and Shabat, 1979) has the recursion operator

$$\Re = D_x^2 + \frac{2}{3}w_x - \frac{1}{3}D^{-1}w_{xx}.$$

- The modified KdV equation (Olver, 1977)

$$u_t = u_{xxx} + \frac{3}{2}u^2 u_x$$

has the recursion operator

$$\Re = D_x^2 + u^2 + u_x D_x^{-1} u.$$

- The Burgers equation

$$u_t = u_{xxx} + 3(uu_x)_x + 3u^2 u_x$$

has the recursion operator

$$\Re = D_x + u + u_x D_x^{-1}.$$

- The quasi-linear third-order evolution equation (Clarkson, Ablowits, Fokas, Svinolupov, Sokolov, Yamilov et al, 1987-2005)

$$u_t = g(u) u_{xxx} + f(u, u_x, u_{xx}),$$

- System of third-order evolution equations (Mikhailov, Shabat, Sokolov, Yamilov, Olver, Wang JP, Svinolupov, Anco, Wolf et al, 1989-2005)

$$\vec{u}_t = g(\vec{u}) \vec{u}_{xxx} + \vec{f}(\vec{u}, \vec{u}_x, \vec{u}_{xx}).$$

♠ Nonlocal symmetries

• The KdV equation (Krasilshchik, Vinogradov, 1984; Gutherie, 1993, 1994, Lou, 1991-1995; Dodd, 1994).

• The KdV-type equation (Qu, Song, 2009).

1.3. The fourth-order evolution equations

$$u_t = F(t, x, u, u_x, u_{xx}, u_{xxx})u_{xxxx} + G(t, x, u, u_x, u_{xx}, u_{xxx}). \quad (7)$$

Examples:

- The Cahn-Hilliard equation

$$u_t = -u_{xxxx} - (u^2 + u)_{xx}.$$

- The Kuramoto-Sivashinsky equation

$$u_t = -u_{xxxx} - u_{xx} - \frac{1}{2}u_x^2.$$

- The Fisher-Kolmogorov equation

$$u_t = -u_{xxxx} + u_{xx} - u^3 + u.$$

- The Swift-Hohenberg equation

$$u_t = -u_{xxxx} - 2u_{xx} - u^3 + (k-1)u, \quad k \in R.$$

- The surface tension driven thin film flows

$$u_t = -(u^3 u_{xxx} + f(u, u_x, u_{xx}))_x.$$

When $f = 0$, it is the capillary-driven flow.

- The evolution of rupture of a liquid surface

$$u_t = -[M(u)(R(u)u_{xx} + Q(u))_x]_x.$$

- The curve evolution flows

$$\gamma_t = f(\kappa, \kappa_s, \kappa_{ss})\vec{n}.$$

Remark: The case F=Constant has been discussed by Q. Huang, V. Lahno, CZ Qu, R. Zhdanov (JMP, 2009)

2. Classification of Lie point symmetries

Lemma 1.1. (Levi-Matsev) Let L be a finite-dimensional Lie algebra over the field R or C. Let N be its radical. Then there exists a semi-simple Lie subalgebra S of L such that

$$L = S \oplus N.$$

It reduces to do the following tasks

1). The classification of all semi-simple Lie algebra.

2). The classification of all solvable Lie algebra.

3). The classification of all algebras which are semi-direct sums of semi-simple Lie algebra and solvable Lie algebra. (Mubarakzyanov, 1963).

2.1. Preliminary group analysis of (7)

Prop. 2.1. The most general symmetry group of (7) is generated by the infinitesimal operators

$$V = \tau(t)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u,$$

where τ, ξ and η are real-valued functions satisfying equations

$$\eta^t - [\tau F_t + \xi F_x + \eta F_u + \eta^x F_{u_x} + \eta^{xx} F_{u_2} + \eta^{xxx} F_{u_3}]u_4 - \eta^{xxxx} F$$
$$-[\tau G_t + \xi F_x + \eta G_u + \eta^x G_{u_x} + \eta^{xx} G_{u_2} + \eta^{xxx} G_{u_3}]|_{u_t = F u_4 + G} = 0,$$

where η^t, η^x, $\cdots$, η^{xxxx} are the prolongations of η.

Theorem 2.1. There are two inequivalent equations (7) invariant under one-parameter symmetry groups:

$$A_1^1 = \langle \partial_t \rangle : \; u_t = F(x, u, u_x, u_2, u_3)u_4 + G(x, u, u_x, u_2, u_3),$$
$$A_1^2 = \langle \partial_x \rangle : \; u_t = F(t, u, u_x, u_2, u_3)u_4 + G(t, u, u_x, u_2, u_3).$$

Here F and G are arbitrary smooth functions. Furthermore, the associated symmetry algebra is maximal in Lie's sense.

2.2. Classification invariant under semi-simple Lie algebras of (7)

The lowest order real semi-simple Lie algebras are isomorphic to one of the following two three-dimensional algebras:

$$so(3):\quad [V_1, V_2] = V_3,\ [V_1, V_3] = -V_2,\ [V_2, V_3] = V_1;$$
$$sl(2, R):\ [V_1, V_2] = 2V_2,\ [V_1, V_3] = -2V_3,\ [V_2, V_3] = V_1.$$

Theorem 2.2. There are no *so*(3)-invariant equations of the form (7).

Theorem 2.3. There are six inequivalent realizations of $sl(2, R)$, which are admitted by (7).

Theorem 2.4. The invariant equations listed in Theorem 2.3 exhaust the list of all possible inequivalent PDEs (7), whose invariant algebras are semi-simple.

2.3. Classification invariant under solvable Lie algebras

• Equations with two-dimensional Lie algebras

There are two non-isomorphic two-dimensional Lie algebras

$$A_{2.1}: \quad [V_1, V_2] = 0, \qquad A_{2.2}: \quad [V_1, V_2] = V_2.$$

Theorem 2.5. There exist three Abelian and four non-Abelian two-dimensional symmetry algebras which are admitted by (7).

• Equations admitting three-dimensional solvable Lie algebras

Three-dimensional solvable Lie algebras are separated into two classes. One contains the algebras which are direct sums of lower dimensional algebras, the other includes the remaining algebras. We consider the decomposable and non-decomposable Lie algebras separately.

Theorem 2.6. There exist fourteen decomposable and fourty two non-decomposable three-dimensional symmetry algebras which are admitted by (7).

• Equations admitting four-dimensional solvable Lie algebras

⋆ Equations with four-dimensional decomposable algebras

The list of non-isomorphic four-dimensional decomposable Lie algebras contains the following ten algebras:

$$\begin{aligned} &A_{2.2} \oplus A_{2.2} = 2A_{2.2},\\ &A_{3.1} \oplus A_1 = 4A_1,\\ &A_{3.2} \oplus A_1 = A_{2.2} \oplus 2A_1,\\ &A_{3.i} \oplus A_1 \ (i = 3, 4, \cdots, 9). \end{aligned}$$

⋆ Equations with four-dimensional non-decomposable algebras

There exist ten non-isomorphic four-dimensional non-decomposable Lie algebras, $A_{4.i}$ $(i = 1, 2, \cdots, 10)$:

$A_{4.1}:$ $[X_2, X_4] = X_1, [X_3, X_4] = X_2;$

$A_{4.2}:$ $[X_1, X_4] = qX_1, [X_2, X_4] = X_2, [X_3, X_4] = X_2 + X_3, \ q \neq 0;$

$A_{4.3}:$ $[X_1, X_4] = X_1, [X_3, X_4] = X_2;$

$A_{4.4}:$ $[X_1, X_4] = X_1, [X_2, X_4] = X_1 + X_2, [X_3, X_4] = X_2 + X_3;$

$A_{4.5}:$ $[X_1, X_4] = X_1, [X_2, X_4] = qX_2, [X_3, X_4] = pX_3,$
$-1 \leqslant p \leqslant q \leqslant 1, \ pq \neq 0;$

$A_{4.6}:$ $[X_1, X_4] = qX_1, [X_2, X_4] = pX_2 - X_3, [X_3, X_4] = X_2 + pX_3,$
$q \neq 0, \ p \geqslant 0;$

$$
\begin{aligned}
A_{4.7}: \quad & [X_2, X_3] = X_1, [X_1, X_4] = 2X_1, [X_2, X_4] = X_2, \\
& [X_3, X_4] = X_2 + X_3; \\
A_{4.8}: \quad & [X_2, X_3] = X_1, [X_1, X_4] = (1+q)X_1, [X_2, X_4] = X_2, \\
& [X_3, X_4] = qX_3, \ |q| \leqslant 1; \\
A_{4.9}: \quad & [X_2, X_3] = X_1, [X_1, X_4] = 2qX_1, [X_2, X_4] = qX_2 - X_3, \\
& [X_3, X_4] = X_2 + qX_3, \ q \geqslant 0; \\
A_{4.10}: \quad & [X_1, X_3] = X_1, [X_2, X_3] = X_2, [X_4, X_1] = X_2, [X_2, X_4] = X_1.
\end{aligned}
$$

Theorem 2.7. There exist eighty five decomposable and seventy four non-decomposable four-dimensional symmetry algebras which are admitted by (7).

3. Classification of quasi-local symmetries

Note that the most general infinitesimal operator, V, admitted by evolution equation (7) reads as

$$V = \tau(t)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u$$

and the maximal equivalence group of Eq. (7) takes the form

$$\bar{t} = T(t), \quad \bar{x} = X(t,x,u), \quad \bar{u} = U(t,x,u),$$

where $\dot{T} \neq 0$ and $D(X,U)/D(x,u) \neq 0$.
If $\tau = 0$, we have $V = \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u$ and there is an equivalence transformation $(t,x,u) \longrightarrow (\bar{t},\bar{x},\bar{u})$ that reduces V to the canonical form ∂_u (we drop the bars). Eq. (7) now becomes

$$u_t = F(t,x,u_x,u_{xx},u_{xxx})u_{xxxx} + G(t,x,u_x,u_{xx},u_{xxx}). \qquad (8)$$

Differentiating Eq. (8) with respect to x yields

$$u_{tx} = F u_{xxxxx} + (F_x + F_{u_x} u_{xx} + F_{u_{xx}} u_{xxx} + F_{u_{xxx}} u_{xxxx}) u_{xxxx}$$
$$+ G_x + G_{u_x} u_{xx} + G_{u_{xx}} u_{xxx} + G_{u_{xxx}} u_{xxxx}.$$

Making the nonlocal change of variables

$$\bar{t} = t, \quad \bar{x} = x, \quad \bar{u} = u_x \tag{9}$$

and dropping the bars, we finally get

$$u_t = F u_{xxxx} + (F_x + F_u u_x + F_{u_x} u_{xx} + F_{u_{xx}} u_{xxx}) u_{xxx}$$
$$+ G_x + G_u u_x + G_{u_x} u_{xx} + G_{u_{xx}} u_{xxx}. \tag{10}$$

where $F = F(t, x, u, u_x, u_{xx})$ and $G = G(t, x, u, u_x, u_{xx})$.
Thus the nonlocal transformation (9) preserves the differential structure of the class of evolution equation (8).

Assume that Eq. (8) admits r-parameter Lie transformation group

$$t' = T(t, \vec{\theta}), \quad x' = X(t, x, u, \vec{\theta}), \quad u' = U(t, x, u, \vec{\theta}) \tag{11}$$

with the vector of group parameters $\vec{\theta} = (\theta_1, \ldots, \theta_r)$ and $r \geq 2$. It leads to

$$u'_{x'} = \frac{U_u u_x + U_x}{X_u u_x + X_x}.$$

The transformation group (11) now reads as

$$t' = T(t, \vec{\theta}), \quad x' = X(t, x, v, \vec{\theta}), \quad u' = \frac{U_v u + U_x}{X_v u + X_x}. \tag{12}$$

with $v = \partial_x^{-1} u$ and $U = U(t, x, v, \vec{\theta})$, where $\partial_x^{-1}\partial_x \equiv \partial_x \partial_x^{-1} \equiv 1$.

Consequently, Eq. (10) admits symmetry group (12). Note that (12) depend explicitly on v, then (12) is a quasi-local symmetry of Eq. (10). Evidently, Eq. (10) admits nonlocal symmetry iff transformation (12) satisfies one of the constraints

$$X_v \neq 0 \quad \text{or} \quad \frac{\partial}{\partial \mathrm{v}}\left(\frac{\mathrm{U_v u + U_x}}{\mathrm{X_v u + X_x}}\right) \neq 0,$$

or, equivalently,

$$X_v \neq 0 \quad \text{or} \quad \mathrm{X_v} = 0, \quad \mathrm{U_{xv}^2 + U_{vv}^2} \neq 0.$$

Theorem 3.1 Evolution equation (8) can be transformed to Eq. (10) admitting a quasi-local symmetry if (8) admits a Lie symmetry whose infinitesimal generator satisfies one of the following conditions:

$$\xi_u \neq 0,$$
$$\text{or} \quad \xi_u = 0, \quad \eta_{xu}^2 + \eta_{uu}^2 \neq 0.$$

By making a hodograph transformation

$$\bar{t} = t, \quad \bar{x} = u, \quad \bar{u} = x,$$

and dropping the bars, the evolution equation

$$u_t = F(t, u, u_x, u_{xx}, u_{xxx})u_{xxxx} + G(t, u, u_x, u_{xx}, u_{xxx}) \qquad (13)$$

can be transformed to an equation of the form (8).

Corollary 3.1. Eq. (7) can be reduced to an equation possessing a quasi-local symmetry if Eq. (13) admits a Lie symmetry satisfying one of the inequalities

$$\eta_x \neq 0, \quad \text{or} \quad \eta_x = 0, \quad \xi_{xu}^2 + \xi_{xx}^2 \neq 0.$$

- The algorithm for constructing evolution equations (1) admitting quasi-local symmetries:

1. Select all invariant equations, whose invariance algebras contain at least one operator of the form $V = \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u$.

2. For each of these equations, make a suitable equivalence transformation reducing V to the canonical form ∂_u, the original equations being transformed to evolution equations of the form (8).

3. For each Lie algebra admitted by (8) check whether its infinitesimal generator satisfies one of conditions in Theorem 3.1. This analysis yields the list of evolution equations (7) that can be reduced to those admitting nonlocal symmetries.

4. Performing the nonlocal change of variables (9) transforms Eq. (8) to (10) which has quasi-local symmetries (12).

Theorem 3.2. There exist four three-dimensional semi-simple, twelve three-dimensional solvable, sixty five four-dimensional solvable quasi-local symmetry algebras which are admitted by (7).

Example . Consider the Lie algebra

$$2A_{2.2}^2 = \langle -t\partial_t - x\partial_x, \partial_t, \partial_u, \mathrm{e}^{\mathrm{u}}\partial_{\mathrm{u}} \rangle$$

and its corresponding invariant equation

$$u_t = x^3F(\omega_1,\omega_2)u_{xxxx} - x^3(u_x^4 - 6u_x^2u_{xx} + 4u_xu_{xxx} + 3u_{xx}^2)F(\omega_1,\omega_2) + u_xG(\omega_1,\omega_2),$$

where $\omega_1 = x(u_x^2 - u_{xx})u_x^{-1}$, $\omega_2 = x^2(u_x^3 - 3u_xu_{xx} + u_{xxx})u_x^{-1}$ and F, G are arbitrary smooth functions. Differentiating the above equation with respect to x and replacing u_x with u yield the evolution equation

$$\begin{aligned} u_t = \; & x^3Fu_{xxxx} + 3x^2Fu_{xxx} + [x^3u_{xxx} - x^3(u_x^4 - 6u_x^2u_{xx} + 4u_xu_{xxx} + 3u_{xx}^2)][F_{\omega_1}\sigma_1 \\ & + F_{\omega_2}\sigma_2] - x^2[3u^4 + 4xu^3u_x - 6(xu_{xx} + 3u_x)u^2 + (12u_{xx} - 12xu_x^2 + 4xu_{xxx})u \\ & + 9u_x^2 + 10xu_xu_{xx}]F + u_xG + u\sigma_1G_{\omega_1} + u\sigma_2G_{\omega_2}, \end{aligned}$$

with

$$\omega_1 = \frac{x(u^2 - u_x)}{u}, \ \omega_2 = \frac{x^2(u^3 - 3uu_x + u_{xx})}{u},$$

and

$$\sigma_1 = \frac{u^3 + xu_x u^2 - (xu_{xx} + u_x)u + xu_x^2}{u^2},$$
$$\sigma_2 = \frac{x[2u^4 + 2xu_x u^3 - 3(xu_{xx} + 2u_x)u^2 + (xu_{xxx} + 2u_{xx})u - xu_x u_{xx}]}{u^2}.$$

The new equation admits the nonlocal symmetry

$$t' = t, \quad x' = x, \quad u' = \frac{u}{1 - \theta \mathrm{e}^{\mathrm{v}}},$$

with symmetry generator $e^v u \partial_u$. Here $v = \partial_x^{-1} u$.

4. Invariant sets of nonlinear evolution equations

Consider the equation (J.R. King, 2001)

$$u_t = -f(u)u_{xxxx} - g(u)u_x u_{xxx} - h(u)u_{xx}^2 \\ -d(u)u_x^2 u_{xx} + p(u)u_{xx} + q(u)u_x^2, \tag{14}$$

where f, g, d, h, p and q are smooth functions of u.

The method of invariant set is available to study (14).

Theorem 4.1. (Qu, 2007) There exist certain equations (4) which admit the invariant set $u_x = F(x, u)$ for some F.

For example, $F = (a/x)u + bu^n$, when $b = 0$, it describes the scaling invariance.

Thank you!