On the Factorizations of Rational Matrix Functions with Applications to Integrable Systems and Discrete Painlevé Equations

Anton Dzhamay

School of Mathematical Sciences
University of Northern Colorado

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Goals and Motivation

We study the space of rational $m \times m$-matrix functions on the Riemann sphere with the fixed singularity data (i.e., asymptotic behavior at infinity and the zeroes and poles of the determinantal divisor). We are interested in finding good coordinates on this space, and also in describing ways of factorizing such matrix functions into products of elementary blocks.

The main reasons for the study are Krichever's project on understanding the Lagrangian description of discrete integrable systems from "basic principles" (e.g., the Lax representation); generalization of Moser-Veselov approach to integrability of discrete systems through refactorization transformations from polynomial to rational matrix functions; applications to discrete Painlevé equations.
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### Continuous Case

- **The Lax Pair Representations**
  \[ \dot{L} = [M, L] \]
  \[ \Delta L = RL\Delta R^{-1} \]

### Discrete Case

- **The Symplectic Form**
  \[ \omega = \sum \text{res} z_i \text{Tr}(\Psi^{-1} \delta L \wedge \delta \Psi) dz \]

\[ \omega = \sum \text{res} z_i, \zeta_\alpha \text{Tr}(\Psi^{-1} L^{-1} \delta L \wedge \delta \Psi) dz \]

where \( \Psi \) is the eigenfunction matrix,

\[ L\Psi = \Psi \Lambda \]

- **Hamiltonian Lagrangian**
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These symplectic forms are well-defined only on the "leaves" of some foliation of our phase space. In the discrete case, this corresponds to fixing the location of zeroes and poles of det $L(z)$. For $\tilde{L}(z) = R(z) L(z) R^{-1}(z)$ to be in the same space as $L(z), R(z)$, $L(z)$ has to be chosen in a special way, one way to do it is refactorization: $L(z) = L_1(z) L_2(z) \mapsto \tilde{L}(z) = L_2(z) L_1(z), R(z) = L_2(z)$.

Such transformations sometimes reduce to discrete Painlevé equations (same as in the Moser-Veselov description of the integrability mechanism of discrete systems).

Same type of transformations correspond to isomonodromic transformations of difference equations, only this time the determinant divisor "moves".
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Assumptions on the Singularity Structure

We consider rational matrix functions \( L(z) \) of rank \( m \) such that both \( L(z) \) and its inverse \( M(z) = L^{-1}(z) \) satisfy the following general conditions:

- \( L(z) \) is regular, diagonalizable (and diagonalized) at \( z = \infty \),
- \( L_0 = \lim_{z \to \infty} L(z) = \text{diag}\{\rho_1, \ldots, \rho_m\} \),
- \( M_0 = L^{-1}_0 \).
- \( L(z) \) has simple poles at the points \( z_1, \ldots, z_n \),
- \( M(z) \) has simple poles at the points \( \zeta_1, \ldots, \zeta_n \);
- The divisor \( (L(z)) = (\det L(z)) \) is simple as well,
- \( \det L(z) = \rho_1 \cdots \rho_m n \prod_{k=1}^{n} (z - \zeta_k) (z - z_k) \).
- This is the rank-one condition on the residues:
  - \( L_k = \text{res}_{z_k} L(z) = a_k b_k^\dagger = \alpha_k [a_k] [b_k^\dagger] \),
  - \( M_k = -\text{res}_{\zeta_k} M(z) = c_k d_k^\dagger = \beta_k [c_k] [d_k^\dagger] \).
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\]
Additive Representations

\[ L(z) = L_0 + \sum_{k=1}^{n} L_k z^{-k}, \]
\[ M(z) = M_0 - \sum_{k=1}^{n} M_k z^{-k}, \]

where

- \( L_0 = \text{diag}\{\rho_1, \ldots, \rho_m\} \)
- \( L_k = a_k b_k^\dagger \)
- \( M_0 = \text{diag}\{1/\rho_1, \ldots, 1/\rho_m\} \)
- \( M_k = c_k d_k^\dagger \)

Note that

- \( d_k^\dagger L(\zeta_k) = 0 \)
- \( L(\zeta_k) c_k = 0 \)
- \( b_k^\dagger M(z_k) = 0 \)
- \( M(z_k) a_k = 0 \).

The space of such \( L(z) \) with the fixed \( D = \sum_i z_i - \sum_i \zeta_i \) is \( M_D \mathcal{R} \).

Question: What are good coordinate systems on the space \( M_D \mathcal{R} \)?
Additive Representations

Additive Representations of $L(z)$ and $M(z)$

\[
L(z) = L_0 + \sum_{k=1}^{n} \frac{L_k}{z - z_k}, \quad \begin{align*}
L_0 &= \text{diag}\{\rho_1, \ldots, \rho_m\} \\
L_k &= a_k b_k^\dagger
\end{align*}
\]

\[
M(z) = M_0 - \sum_{k=1}^{n} \frac{M_k}{z - \zeta_k}, \quad \begin{align*}
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d_k^\dagger L(\zeta_k) = 0, \quad L(\zeta_k)c_k = 0, \quad b_k^\dagger M(z_k) = 0, \quad M(z_k)a_k = 0.
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The space of such $L(z)$ with the fixed $D = \sum_i z_i - \sum_i \zeta_i$ is $\mathcal{M}^D_r$. 

Anton Dzhmay (UNC)
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\]

The space of such $L(z)$ with the fixed $D = \sum_i z_i - \sum_i \zeta_i$ is $\mathcal{M}_r^D$.

Question

What are good coordinate systems on the space $\mathcal{M}_r^D$?
Lemma 1: Generically, the collection \( \{a_k, d^\dagger_k\} \) (or the collection \( \{c_k, b^\dagger_k\} \)) gives coordinates on the space \( M_{Dr} \).

Consider the equations

\[
M(z_k)a_k = 0 \quad \text{and} \quad d^\dagger_i L(\zeta_i) = 0.
\]

\[
M_0 a_k - n \sum_{i=1}^k c_i d^\dagger_i a_k z_k - \zeta_i = 0, \quad d^\dagger_i L_0 + n \sum_{k=1}^i d^\dagger_i a_k \zeta_i - z_k b^\dagger_k = 0.
\]

Then if the matrix \( [d^\dagger_i a_k z_k - \zeta_i] \) is invertible,

\[
c_i = L^{-1}_0 a_k [d^\dagger_i a_k z_k - \zeta_i] - 1, \quad b^\dagger_k = [d^\dagger_i a_k z_k - \zeta_i]^{-1} d^\dagger_i L_0.
\]

Unfortunately, Lagrangian coordinates seem to be a mix of vectors from these two collections.
Lemma

Generically, the collection \( \{a_k, d_k^\dagger\}_{k=1}^n \) (or the collection \( \{c_k, b_k^\dagger\}_{k=1}^n \)) gives coordinates on the space \( \mathcal{M}_D^r \).
Lemma

Generically, the collection \( \{a_k, d_k^\dagger\}_{k=1}^n \) (or the collection \( \{c_k, b_k^\dagger\}_{k=1}^n \)) gives coordinates on the space \( \mathcal{M}_D^r \).

Consider the equations \( M(z_k)a_k = 0 \) and \( d_i^\dagger L(\zeta_i) = 0 \):

\[
M_0a_k - \sum_{i=1}^n c_i \frac{d_i^\dagger a_k}{z_k - \zeta_i} = 0, \quad d_i^\dagger L_0 + \sum_{k=1}^n \frac{d_i^\dagger a_k}{\zeta_i - z_k} b_k^\dagger = 0.
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Then if the matrix

\[
\begin{bmatrix}
  d_i^\dagger a_k \\
  z_k - \zeta_i
\end{bmatrix}
\]

is invertible,
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Consider the equations

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M(z_k)a_k = 0 \quad \text{and} \quad d_i^\dagger L(\zeta_i) = 0:
\]

\[
M_0a_k - \sum_{i=1}^n c_i \frac{d_i^\dagger a_k}{z_k - \zeta_i} = 0, \quad d_i^\dagger L_0 + \sum_{k=1}^n \frac{d_i^\dagger a_k}{\zeta_i - z_k} b_k^\dagger = 0.
\]

Then if the matrix \( \begin{bmatrix} d_i^\dagger a_k \\ z_k - \zeta_i \end{bmatrix} \) is invertible,

\[
c_i = L_0^{-1} a_k \left( \frac{d_i^\dagger a_k}{z_k - \zeta_i} \right)^{-1}, \quad b_k^\dagger = \left( \frac{d_i^\dagger a_k}{z_k - \zeta_i} \right)^{-1} d_i^\dagger L_0.
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Lemma

Generically, the collection \( \{ a_k, d^\dagger_k \}_{k=1}^n \) (or the collection \( \{ c_k, b^\dagger_k \}_{k=1}^n \) ) gives coordinates on the space \( \mathcal{M}_r^D \).

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\]

Then if the matrix \( \left[ \frac{d_i^\dagger a_k}{z_k - \zeta_i} \right] \) is invertible,

\[
c_i = L_0^{-1} a_k \left[ \frac{d_i^\dagger a_k}{z_k - \zeta_i} \right]^{-1}, \quad b^\dagger_k = \left[ \frac{d_i^\dagger a_k}{z_k - \zeta_i} \right]^{-1} d_i^\dagger L_0.
\]

Unfortunately, Lagrangian coordinates seem to be a mix of vectors from these two collections.
Multiplicative Representation

Elementary Divisors (building blocks)

$$B(z) = I + z_0 - \zeta_0 z - \zeta_0 p q^\dagger q^\dagger p B(z) - 1 = I + \zeta_0 - z_0 z - \zeta_0 p q^\dagger q^\dagger p .$$

Properties of Elementary Divisors

1. $$\det B(z) = (z - \zeta_0) / (z - z_0);$$
2. $$B(z)p = (z - \zeta_0 z - \zeta_0) p, q^\dagger B(z) = (z - \zeta_0 z - \zeta_0) q^\dagger ;$$
3. $$B(z^* w) = v = \Rightarrow B(z) = I + 1/z - z_0 (z_0 - z^* w q^\dagger q^\dagger w + (z^* - \zeta_0) v q^\dagger q^\dagger v), p = \frac{\partial}{\partial q^\dagger} (\frac{z_0 - z^*}{\log(q^\dagger w)}) + (z^* - \zeta_0) \log(q^\dagger v) ;$$
4. $$w^\dagger B(z^*) = v^\dagger = \Rightarrow B(z) = I + 1/z - z_0 (z_0 - z^*) p w^\dagger w^\dagger p + (z^* - \zeta_0) v^\dagger v^\dagger p .$$
Multiplicative Representation

Elementary Divisors (building blocks)

\[ B(z) = I + \frac{z_0 - \zeta_0}{z - z_0} pq^\dagger \]

\[ B(z)^{-1} = I + \frac{\zeta_0 - z_0}{z - \zeta_0} pq^\dagger. \]
Multiplicative Representation

Elementary Divisors (building blocks)

\[ B(z) = I + \frac{z_0 - \zeta_0}{z - z_0} pp^\dagger \quad B(z)^{-1} = I + \frac{\zeta_0 - z_0}{z - \zeta_0} qq^\dagger. \]

Properties of Elementary Divisors
Multiplicative Representation

Elementary Divisors (building blocks)

\[ B(z) = I + \frac{z_0 - \zeta_0}{z - z_0} p q^\dagger \]

\[ B(z)^{-1} = I + \frac{\zeta_0 - z_0}{z - \zeta_0} q^\dagger p. \]

Properties of Elementary Divisors

(1) \( \det B(z) = (z - \zeta_0)/(z - z_0); \)
Multiplicative Representation

Elementary Divisors (building blocks)

\[ B(z) = I + \frac{z_0 - \zeta_0}{z - z_0} \frac{pq^\dagger}{q^\dagger p} \quad \text{and} \quad B(z)^{-1} = I + \frac{\zeta_0 - z_0}{z - \zeta_0} \frac{pq^\dagger}{q^\dagger p}. \]

Properties of Elementary Divisors

1. \( \det B(z) = \frac{(z - \zeta_0)}{(z - z_0)}; \)
2. \( B(z)p = \left( \frac{z - \zeta_0}{z - z_0} \right) p, \quad q^\dagger B(z) = \left( \frac{z - \zeta_0}{z - z_0} \right) q^\dagger; \)
Multiplicative Representation

Elementary Divisors (building blocks)

\[
B(z) = I + \frac{z_0 - \zeta_0}{z - z_0} pq^\dagger \quad B(z)^{-1} = I + \frac{\zeta_0 - z_0}{z - \zeta_0} q^\dagger p .
\]

Properties of Elementary Divisors

(1) \( \det B(z) = (z - \zeta_0)/(z - z_0) \);

(2) \( B(z)p = \left( \frac{z - \zeta_0}{z - z_0} \right) p , \quad q^\dagger B(z) = \left( \frac{z - \zeta_0}{z - z_0} \right) q^\dagger ; \)

(3) \( B(z^*)w = v \implies B(z) = I + \frac{1}{z - z_0} \left( (z_0 - z^*) \frac{wq^\dagger}{q^\dagger w} + (z^* - \zeta_0) \frac{vq^\dagger}{q^\dagger v} \right) \),
Multiplicative Representation

Elementary Divisors (building blocks)

\[ B(z) = I + \frac{z_0 - \zeta_0}{z - z_0} \mathbf{p} \mathbf{q}^\dagger \quad \text{and} \quad B(z)^{-1} = I + \frac{\zeta_0 - z_0}{z - \zeta_0} \mathbf{q}^\dagger \mathbf{p}. \]

Properties of Elementary Divisors

1. \( \det B(z) = (z - \zeta_0)/(z - z_0); \)
2. \( B(z)\mathbf{p} = \left( \frac{z - \zeta_0}{z - z_0} \right) \mathbf{p}, \quad \mathbf{q}^\dagger B(z) = \left( \frac{z - \zeta_0}{z - z_0} \right) \mathbf{q}^\dagger; \)
3. \( B(z^*)\mathbf{w} = \mathbf{v} \implies B(z) = I + \frac{1}{z - z_0} \left( (z_0 - z^*) \frac{\mathbf{w} \mathbf{q}^\dagger}{\mathbf{q}^\dagger \mathbf{w}} + (z^* - \zeta_0) \frac{\mathbf{v} \mathbf{q}^\dagger}{\mathbf{q}^\dagger \mathbf{v}} \right), \)

\[ \mathbf{p} = \frac{\partial}{\partial \mathbf{q}^\dagger} \left( (z_0 - z^*)(\log(\mathbf{q}^\dagger \mathbf{w}) + (z^* - \zeta_0)(\log(\mathbf{q}^\dagger \mathbf{v})) \right); \]
Multiplicative Representation

Elementary Divisors (building blocks)

\[ B(z) = I + \frac{z_0 - \zeta_0}{z - z_0} p q^\dagger \]
\[ B(z)^{-1} = I + \frac{\zeta_0 - z_0}{z - \zeta_0} p q^\dagger. \]

Properties of Elementary Divisors

(1) \( \det B(z) = (z - \zeta_0)/(z - z_0); \)
(2) \( B(z)p = \left(\frac{z - \zeta_0}{z - z_0}\right)p, \quad q^\dagger B(z) = \left(\frac{z - \zeta_0}{z - z_0}\right)q^\dagger; \)
(3) \( B(z^*)w = v \implies B(z) = I + \frac{1}{z - z_0} \left( (z_0 - z^*)\frac{w q^\dagger}{q^\dagger w} + (z^* - \zeta_0)\frac{v q^\dagger}{q^\dagger v} \right), \)
\[ p = \frac{\partial}{\partial q^\dagger} \left( (z_0 - z^*) \log(q^\dagger w) + (z^* - \zeta_0) \log(q^\dagger v) \right); \]
(4) \( w^\dagger B(z^*) = v^\dagger \implies B(z) = I + \frac{1}{z - z_0} \left( (z_0 - z^*)\frac{p w^\dagger}{w^\dagger p} + (z^* - \zeta_0)\frac{p v^\dagger}{v^\dagger p} \right). \)
Factors and Divisors

\[ L(z) = L_0 C_1(z) \cdots C_n(z), \]
\[ \det C_k(z) = z - \zeta_k z - \zeta_k \]
\[ L_r k(z) B_r k(z), \]
\[ \det B_r k(z) = z - \zeta_k z - \zeta_k \]
\[ B_l k(z) L_l k(z), \]
\[ \det B_l k(z) = z - \zeta_k z - \zeta_k \].

Definition

We call \( C_k(z) \) the factors of \( L(z) \) and \( B_r k(z) \) (resp. \( B_l k(z) \)) right (resp. left) divisors of \( L(z) \) (or \( M(z) \)).
Multiplicative Representation (continued)

Factors and Divisors

\[ L(z) = L_0 C_1(z) \cdots C_n(z), \quad \text{det } C_k(z) = \frac{z - \zeta_k}{z - z_k} \]

\[ = L^r_k(z) B^r_k(z), \quad \text{det } B^r_k(z) = \frac{z - \zeta_k}{z - z_k} \]

\[ = B^l_k(z) L^l_k(z), \quad \text{det } B^l_k(z) = \frac{z - \zeta_k}{z - z_k}. \]
Multiplicative Representation (continued)

Factors and Divisors

\[ L(z) = L_0 C_1(z) \cdot \ldots \cdot C_n(z), \quad \det C_k(z) = \frac{z - \zeta_k}{z - z_k} \]
\[ = L_k^r(z) B_k^r(z), \quad \det B_k^r(z) = \frac{z - \zeta_k}{z - z_k} \]
\[ = B_k^l(z) L_k^l(z), \quad \det B_k^l(z) = \frac{z - \zeta_k}{z - z_k}. \]

Definition

We call \( C_k(z) \) the factors of \( L(z) \) and \( B_k^r(z) \) (resp. \( B_k^l(z) \)) right (resp. left) divisors of \( L(z) \) (or \( M(z) \)).
Lemma

Let \( L_k = \text{res}_z L(z)|_{z = z_k} = a_k b_k^\dagger \) and \( M_k = -\text{res}_\zeta M(z)|_{\zeta = \zeta_k} = c_k d_k^\dagger \). Then

\[
B_r k(z) = I + z_k - \zeta_k z - z_k c_k b_k^\dagger b_k^\dagger c_k
\]

\[
B_l k(z) = I + z_k - \zeta_k z - z_k a_k d_k^\dagger d_k^\dagger a_k.
\]

Proof

Let \( L(z) = L_r k(z) B_r k(z) \). Taking the residue at \( z_k \) gives

\[
a_k b_k^\dagger = L_r k(z_k) p_r k(q_r k)^\dagger,
\]

and so \( (q_r k)^\dagger = b_k^\dagger \).

Similarly, taking the residue of \( M(z) = (B_r k(z))^{-1} (L_r k(z))^{-1} \) at \( \zeta_k \) gives

\[
p_r k = c_k.
\]
Lemma

Let \( L_k = \text{res}_{z_k} L(z) = a_k b_k^\dagger \) and \( M_k = - \text{res}_{\zeta_k} M(z) = c_k d_k^\dagger \). Then

\[
B^r_k(z) = I + \frac{z_k - \zeta_k}{z - z_k} \frac{c_k b_k^\dagger}{b_k^\dagger c_k}
\]

\[
B^l_k(z) = I + \frac{z_k - \zeta_k}{z - z_k} \frac{a_k d_k^\dagger}{d_k^\dagger a_k}.
\]
**Lemma**

Let $L_k = \text{res}_{z_k} L(z) = a_k b_k^\dagger$ and $M_k = - \text{res}_{\zeta_k} M(z) = c_k d_k^\dagger$. Then

$$B^r_k(z) = I + \frac{z_k - \zeta_k}{z - z_k} \frac{c_k b_k^\dagger}{b_k^\dagger c_k}$$

$$B^l_k(z) = I + \frac{z_k - \zeta_k}{z - z_k} \frac{a_k d_k^\dagger}{d_k^\dagger a_k}.$$

**Proof**

Let $L(z) = L_k^r(z) B^r_k(z)$. Taking the residue at $z_k$ gives

$$a_k b_k^\dagger = L_k^r(z_k) p_k^r (q_k^r)^\dagger,$$
and so $(q_k^r)^\dagger = b_k^\dagger$.

Similarly, taking the residue of $M(z) = (B^r_k(z))^{-1} (L_k^r(z))^{-1}$ at $\zeta_k$ gives $p_k^r = c_k$. 

Anton Dzhamay (UNC)  Refactorization dynamics  NMMP, July 19, 2009  11 / 29
Refactorization Transformations

From now on, restrict our attention to the quadratic ($n = 2$) case:

$L(z) = L_0 C_1(z) C_2(z) = B_2(z) L_0 \quad B_1(z) = B_1(z) L_0 B_2(z)$

Consider the map $L(z) \mapsto \tilde{L}(z) = R(z) L(z) R(z)^{-1}$ with $R(z) = B_1(z)$:

$L(z) = B_2(z) L_0 B_1(z) \mapsto \tilde{L}(z) = B_1(z) B_2(z) L_0 \tilde{B}_1(z) B_2(z)$.

Related isomonodromic transformation:

$L(z) = B_2(z) L_0 B_1(z) \mapsto \tilde{L}(z) = B_1(z) B_2(z) L_0 \tilde{B}_1(z) B_2(z)$.

Iterate it: this is our dynamics.

Question: is it possible to write down a Lagrangian generating this dynamics?
Refactorization Transformations

From now on, restrict our attention to the quadratic \( (n = 2) \) case:

\[
L(z) = L_0 C_1(z) C_2(z) = B'_2(z)L_0 B'_1(z) = B'_1(z)L_0 B'_2(z)
\]
Refactorization Transformations

From now on, restrict our attention to the quadratic ($n = 2$) case:

$$L(z) = L_0 C_1(z) C_2(z) = B_2^l(z) L_0 B_1^r(z) = B_1^l(z) L_0 B_2^r(z)$$

Consider the map $L(z) \mapsto \tilde{L}(z) = R(z) L(z) R(z)^{-1}$ with $R(z) = B_1^r(z)$:

$$L(z) = B_2^l(z) L_0 B_1^r(z) \mapsto \tilde{L}(z) = B_1^r(z) B_2^l(z) L_0 = \tilde{B}_2^l(z) L_0 \tilde{B}_1^r(z).$$

Related isomonodromic transformation:

$$L(z) = B_2^l(z) L_0 B_1^r(z) \mapsto \tilde{L}(z) = B_1^r(z+1) B_2^l(z) L_0 = \tilde{B}_2^l(z) L_0 \tilde{B}_1^r(z).$$

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Question: is it possible to write down a Lagrangian generating this dynamics?
Refactorization Transformations

From now on, restrict our attention to the quadratic \((n = 2)\) case:

\[
L(z) = L_0 C_1(z) C_2(z) = B_2^l(z) L_0 B_1^r(z) = B_1^l(z) L_0 B_2^r(z)
\]

Consider the map \(L(z) \mapsto \tilde{L}(z) = R(z) L(z) R(z)^{-1}\) with \(R(z) = B_1^r(z)\):

\[
\tilde{L}(z) = B_1^r(z) B_2^l(z) L_0 = B_2^l(z) L_0 B_1^r(z).
\]

Related isomonodromic transformation:

\[
L(z) = B_2^l(z) L_0 B_1^r(z) \mapsto \tilde{L}(z) = B_1^r(z + 1) B_2^l(z) L_0 = B_2^l(z) L_0 B_1^r(z).
\]
Refactorization Transformations

From now on, restrict our attention to the quadratic \((n = 2)\) case:

\[
L(z) = L_0 C_1(z) C_2(z) = B_2^l(z) L_0 B_1^r(z) = B_1^l(z) L_0 B_2^r(z)
\]

Consider the map \(L(z) \mapsto \tilde{L}(z) = R(z) L(z) R(z)^{-1}\) with \(R(z) = B_1^r(z)\):

\[
L(z) = B_2^l(z) L_0 B_1^r(z) \mapsto \tilde{L}(z) = B_1^r(z) B_2^l(z) L_0 = \tilde{B}_2^l(z) L_0 \tilde{B}_1^r(z).
\]

Related isomonodromic transformation:

\[
L(z) = B_2^l(z) L_0 B_1^r(z) \mapsto \tilde{L}(z) = B_1^r(z + 1) B_2^l(z) L_0 = \tilde{B}_2^l(z) L_0 \tilde{B}_1^r(z).
\]

Iterate it: this is our dynamics.
Refactorization Transformations

From now on, restrict our attention to the quadratic ($n = 2$) case:

$$L(z) = L_0 C_1(z) C_2(z) = B_2^l(z) L_0 B_1^r(z) = B_1^r(z) L_0 B_2^r(z)$$

Consider the map $L(z) \mapsto \tilde{L}(z) = R(z) L(z) R(z)^{-1}$ with $R(z) = B_1^r(z)$:

$$L(z) = B_2^l(z) L_0 B_1^r(z) \mapsto \tilde{L}(z) = B_1^r(z) B_2^l(z) L_0 = \tilde{B}_2^l(z) L_0 \tilde{B}_1^r(z).$$

Related isomonodromic transformation:

$$L(z) = B_2^l(z) L_0 B_1^r(z) \mapsto \tilde{L}(z) = B_1^r(z + 1) B_2^l(z) L_0 = \tilde{B}_2^l(z) L_0 \tilde{B}_1^r(z).$$

Iterate it: this is our dynamics.

**Question**: is it possible to write down a Lagrangian generating this dynamics?
Discrete Euler-Lagrange equations

Let $Q$ be the configuration space of our discrete dynamical system.

Continuous Case

$\mathcal{L} = \mathcal{L}(Q, \dot{Q}) \in \mathcal{F}(TQ)$

$\Rightarrow \mathcal{L} = \mathcal{L}(Q, \tilde{Q}) \in \mathcal{F}(Q \times Q)$

Discrete Case

$S(\gamma) = \int_{\gamma} \mathcal{L}(Q, \dot{Q}) dt$

$\Rightarrow S(\{Q_k\}) = \sum_k \mathcal{L}(Q_k, Q_k + 1)$

Euler-Lagrange Equations (from $\delta S = 0$)

$\frac{\partial \mathcal{L}}{\partial Q} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{Q}} = 0$

$\Rightarrow \frac{\partial \mathcal{L}}{\partial Y}(Q, \tilde{Q}) + \frac{\partial \mathcal{L}}{\partial X}(Q, \tilde{Q}) = 0$
Discrete Euler-Lagrange equations

Let \( Q \) be the configuration space of our discrete dynamical system.

**Continuous Case**

- The Lagrangian
  \[
  \mathcal{L} = \mathcal{L}(Q, \dot{Q}) \in \mathcal{F}(TQ)
  \]
- Action
  \[
  S(\gamma) = \int_\gamma \mathcal{L}(Q, \dot{Q}) \, dt
  \]
- Euler–Lagrange Equations (from \( \delta S = 0 \))
  \[
  \frac{\partial \mathcal{L}}{\partial Q} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{Q}} = 0
  \]
Discrete Euler-Lagrange equations

Let $Q$ be the configuration space of our discrete dynamical system.

Continuous Case

- The Lagrangian
  \[ \mathcal{L} = \mathcal{L}(Q, \dot{Q}) \in \mathcal{F}(TQ) \implies \mathcal{L} = \mathcal{L}(Q, \tilde{Q}) \in \mathcal{F}(Q \times Q) \]

- Action
  \[ S(\gamma) = \int_{\gamma} \mathcal{L}(Q, \dot{Q}) \, dt \implies S(\{Q_k\}) = \sum_k \mathcal{L}(Q_k, Q_{k+1}) \]

- Euler-Lagrange Equations (from $\delta S = 0$)
  \[ \frac{\partial \mathcal{L}}{\partial Q} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{Q}} = 0 \implies \frac{\partial \mathcal{L}}{\partial Y}(Q, Q) + \frac{\partial \mathcal{L}}{\partial X}(Q, \tilde{Q}) = 0 \]

Discrete Case
Discrete Lagrangian Integrable Systems

\[ \text{Find a map } \eta : \mathbb{Q} \times \mathbb{Q} \to \mathbb{P}, \text{ where } \mathbb{P} \text{ is a space of matrix polynomials} \]

The isospectral re-factorization map \( R : \mathbb{L}(z) \to \tilde{\mathbb{L}}(z) = \mathbb{L}^2(z) \mathbb{L}(z) \mathbb{L}^{-1}(z) \) is the discrete analogue of the Lax-pair representation.

This is exactly our setup, the ordering of the poles determines the order of the factors.
Discrete Lagrangian Integrable Systems

Moser-Veselov Approach to Integrability of Discrete Systems

Discrete system is a map \( \tilde{Q} = \phi(Q, \tilde{Q}) \) or \( (Q, \tilde{Q}) = \Phi(Q, \tilde{Q}) \).

Find a map \( \eta : Q \times Q \rightarrow P \), where \( P \) is a space of matrix polynomials in a spectral variable \( z \), such that we have the following diagram:

The isospectral re-factorization map \( R : L(z) \rightarrow \tilde{L}(z) = L_2(z) \cdot L(z) \cdot L^{-1/2}(z) \) is the discrete analogue of the Lax-pair representation.

This is exactly our setup, the ordering of the poles determines the order of the factors.
Discrete Lagrangian Integrable Systems

Moser-Veselov Approach to Integrability of Discrete Systems

- Discrete system is a map $\tilde{Q} = \phi(Q, \bar{Q})$ or $(Q, \tilde{Q}) = \Phi(Q, \bar{Q})$. 
Discrete Lagrangian Integrable Systems

Moser-Veselov Approach to Integrability of Discrete Systems

- Discrete system is a map \( \tilde{Q} = \phi(Q, \tilde{Q}) \) or \( (Q, \tilde{Q}) = \Phi(Q, Q) \).
- Find a map \( \eta : \mathcal{Q} \times \mathcal{Q} \rightarrow \mathcal{P} \), where \( \mathcal{P} \) is a space of matrix polynomials in a spectral variable \( z \), such that we have the following diagram:
Discrete Lagrangian Integrable Systems

Moser-Veselov Approach to Integrability of Discrete Systems

- Discrete system is a map $\tilde{Q} = \phi(Q, Q)$ or $(Q, \tilde{Q}) = \Phi(Q, Q)$.
- Find a map $\eta : Q \times Q \to P$, where $P$ is a space of matrix polynomials in a spectral variable $z$, such that we have the following diagram:

\[ \cdots \xrightarrow{\Phi} (Q, Q) \xrightarrow{\Phi} (Q, \tilde{Q}) \xrightarrow{\Phi} \cdots \]
Discrete Lagrangian Integrable Systems

Moser-Veselov Approach to Integrability of Discrete Systems

- Discrete system is a map $\tilde{Q} = \phi(\tilde{Q}, Q)$ or $(Q, \tilde{Q}) = \Phi(Q, Q)$.
- Find a map $\eta : Q \times Q \rightarrow \mathcal{P}$, where $\mathcal{P}$ is a space of matrix polynomials in a spectral variable $z$, such that we have the following diagram:

\[
\begin{array}{c}
\cdots \xrightarrow{\Phi} (Q, Q) \xrightarrow{\Phi} (Q, \tilde{Q}) \xrightarrow{\Phi} \cdots \\
\downarrow{\eta} \quad \quad \quad \downarrow{\eta} \\
\cdots \xrightarrow{R} L(z) = L_2(z)L_1(z) = L_1(z)L_2(z) \xrightarrow{R} \tilde{L}(z) = L_2(z)L_1(z) = \tilde{L}_1(z)\tilde{L}_2(z) \xrightarrow{R} \cdots
\end{array}
\]
Discrete Lagrangian Integrable Systems

Moser-Veselov Approach to Integrability of Discrete Systems

- Discrete system is a map \( \tilde{Q} = \phi(Q, Q) \) or \((Q, \tilde{Q}) = \Phi(Q, Q)\).
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\[
\cdots \xrightarrow{\Phi} (Q, Q) \xrightarrow{\Phi} (Q, \tilde{Q}) \xrightarrow{\Phi} \cdots \xrightarrow{\eta} \cdots
\]

\[
\cdots \xrightarrow{R} L(z) = L_2(z)L_1(z) = L_1(z)L_2(z) \xrightarrow{R} \tilde{L}(z) = L_2(z)L_1(z) = \tilde{L}_1(z)\tilde{L}_2(z) \xrightarrow{R} \cdots
\]

- the isospectral re-factorization map

\[
R : L(z) \rightarrow \tilde{L}(z) = L_2(z)L(z)L_2^{-1}(z)
\]

is the discrete analogue of the Lax-pair representation.
Moser-Veselov Approach to Integrability of Discrete Systems

- Discrete system is a map \( \tilde{Q} = \phi(Q, Q) \) or \( (Q, \tilde{Q}) = \Phi(Q, Q) \).
- Find a map \( \eta : Q \times Q \rightarrow P \), where \( P \) is a space of matrix polynomials in a spectral variable \( z \), such that we have the following diagram:

\[
\begin{array}{ccccccc}
\cdots & \xrightarrow{\Phi} & (Q, Q) & \xrightarrow{\Phi} & (Q, \tilde{Q}) & \xrightarrow{\Phi} & \cdots \\
\downarrow{\eta} & & & & \downarrow{\eta} & & \\
\cdots & \xrightarrow{R} & L(z) = L_2(z)L_1(z) = L_1(z)L_2(z) & \xrightarrow{R} & \tilde{L}(z) = L_2(z)\tilde{L}_1(z) = \tilde{L}_1(z)\tilde{L}_2(z) & \xrightarrow{R} & \cdots \\
\end{array}
\]

- the isospectral re-factorization map

\[
R : L(z) \rightarrow \tilde{L}(z) = L_2(z)L(z)L_2^{-1}(z)
\]

is the discrete analogue of the Lax-pair representation.

This is exactly our setup, the ordering of the poles determines the order of the factors.
Isospectral Dynamics

- Identify the configuration space \( Q \).
- Find the parametrization map \( \eta : Q \times Q \rightarrow M \).
- Find the equations of motion in terms of these coordinates.
- Find the Lagrangian function \( L = L(\mathbf{X}, \mathbf{Y}) \in \mathbb{F}(Q \times Q) \).

Consider the diagram...
Isospectral Dynamics

We need to do the following:

1. Identify the configuration space $Q$.
2. Find the parametrization map $\eta: Q \times Q \rightarrow M$.
3. Find the equations of motion in terms of these coordinates.
4. Find the Lagrangian function $L = L(X, Y) \in F(Q \times Q)$.
Isospectral Dynamics

We need to do the following:

- Identify the configuration space $Q$. 
Isospectral Dynamics

We need to do the following:

- Identify the configuration space $Q$.
- Find the parametrization map $\eta : Q \times Q \rightarrow \mathcal{M}_D^r$. 

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Isospectral Dynamics

We need to do the following:

- Identify the configuration space $\mathcal{Q}$.
- Find the parametrization map $\eta : \mathcal{Q} \times \mathcal{Q} \to \mathcal{M}_D^r$.
- Find the equations of motion in terms of these coordinates.
Isospectral Dynamics

We need to do the following:

- Identify the configuration space $Q$.
- Find the parametrization map $\eta : Q \times Q \to M^r_D$.
- Find the equations of motion in terms of these coordinates.
- Find the Lagrangian function $\mathcal{L} = \mathcal{L}(X, Y) \in \mathcal{F}(Q \times Q)$. 
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Consider the diagram
Isospectral Dynamics

We need to do the following:

- Identify the configuration space $\mathcal{Q}$.
- Find the parametrization map $\eta : \mathcal{Q} \times \mathcal{Q} \rightarrow \mathcal{M}_D^r$.
- Find the equations of motion in terms of these coordinates.
- Find the Lagrangian function $\mathcal{L} = \mathcal{L}(X, Y) \in \mathcal{F}(\mathcal{Q} \times \mathcal{Q})$.

Consider the diagram
Isospectral Dynamics

We need to do the following:

- Identify the configuration space $Q$.
- Find the parametrization map $\eta : Q \times Q \rightarrow M_D^r$.
- Find the equations of motion in terms of these coordinates.
- Find the Lagrangian function $L = L(X, Y) \in F(Q \times Q)$.

Consider the diagram

\[
\begin{align*}
L(z) &= B_2^l(z)L_0B_1^r(z) \\
\tilde{L}(z) &= B_1^r(z)B_2^l(z)L_0 = \tilde{B}_2^l(z)L_0\tilde{B}_1^r(z) \\
(p_2^l, (q_2^l)^\dagger, p_1^r, (q_1^r)^\dagger) &\quad \downarrow \\
(\tilde{p}_2^l, (\tilde{q}_2^l)^\dagger, \tilde{p}_1^r, (\tilde{q}_1^r)^\dagger) &\quad \downarrow \\
(Q, \tilde{Q}) &\quad \eta &\quad \eta &\quad \eta &\quad \eta \\
\end{align*}
\]
The Coordinates

\[
\eta \quad \downarrow \quad \downarrow \quad \uparrow \quad \uparrow
\]

\[
L(z) = B_{l_2}(z) \quad L_0 \quad B_{r_1}(z) = e B_{l_2}(z) \quad L_0 e B_{r_2}(z)
\]

\[
(p_{l_2}, (q_{l_2})^\dagger, p_{r_1}, (q_{r_1})^\dagger) \quad \downarrow \quad \downarrow \quad \uparrow \quad \uparrow
\]

\[
(\tilde{p}_{l_2}, (\tilde{q}_{l_2})^\dagger, \tilde{p}_{r_1}, (\tilde{q}_{r_1})^\dagger)
\]

We want:

• \[ p_{r_1} = c_1 = c_1(a_2, b_1^\dagger, \tilde{a}_2, \tilde{b}_1^\dagger) \]

• \[ \tilde{p}_{r_1} = \tilde{c}_1 = \tilde{c}_1(a_2, b_1^\dagger, \tilde{a}_2, \tilde{b}_1^\dagger) \]

• \[ (q_{l_2})^\dagger = d_2^\dagger = d_2^\dagger(a_2, b_1^\dagger, \tilde{a}_2, \tilde{b}_1^\dagger) \]

• \[ (\tilde{q}_{l_2})^\dagger = \tilde{d}_2^\dagger = \tilde{d}_2^\dagger(a_2, b_1^\dagger, \tilde{a}_2, \tilde{b}_1^\dagger) \]
The Coordinates

\[
\begin{align*}
L(z) &= B_2^r(z)L_0B_1^r(z) \\
\tilde{L}(z) &= B_2^r(z)B_2^r(z)L_0 = B_2^r(z)L_0B_1^r(z) = B_1^r(z)L_0B_2^r(z)
\end{align*}
\]

\[
\begin{align*}
(p_2^l, (q_2^l)^\dagger, p_1^r, (q_1^r)^\dagger) \\
(\tilde{p}_2^l, (\tilde{q}_2^l)^\dagger, \tilde{p}_1^r, (\tilde{q}_1^r)^\dagger)
\end{align*}
\]
The Coordinates

Let \( Q = (p_2^l = a_2, (q_1^r)^\dagger = b_1^\dagger) \in Q = \mathbb{C}^m \times \mathbb{C}^m \)
The Coordinates

\[ L(z) = B_2^l(z)L_0B_1^r(z) \]

\[ \tilde{L}(z) = B_1^r(z)B_2^l(z)L_0 = B_2^l(z)L_0B_1^r(z) = B_1^r(z)L_0B_2^l(z) \]

Let \( Q = (p_2^l, a_2, q_1^r, b_1^\dagger) \in Q = \mathbb{P}^{m-1} \times \mathbb{P}^{m-1} \)
The Coordinates

\[ \mathbf{L}(z) = B_2^r(z)L_0 B_1^r(z) \]

\[ \tilde{\mathbf{L}}(z) = B_1^r(z)B_2^r(z)L_0 = \tilde{B}_2^r(z)L_0 \tilde{B}_1^r(z) = \tilde{B}_1^r(z)L_0 \tilde{B}_2^r(z) \]

Let \( \mathbf{Q} = (p_2^l, (q_2^r)^\dagger, p_1^r, (q_1^r)^\dagger) \in \mathbb{P}^{m-1} \times \mathbb{P}^{m-1} \)

Also, \( \tilde{\mathbf{Q}} = (\tilde{a}_2, \tilde{b}_1^\dagger) \).

The Coordinates

Let $Q = \left( p_2^l, (q_2^r)^\dagger, p_1^r, (q_1^r)^\dagger \right) \in Q = \mathbb{P}^{m-1} \times \mathbb{P}^{m-1}$

Also, $\tilde{Q} = (\tilde{a}_2, \tilde{b}_1^\dagger)$.

We want:

- $p_1^r = c_1 = c_1(a_2, b_1^\dagger, \tilde{a}_2, \tilde{b}_1^\dagger)$
- $\tilde{p}_1^r = \tilde{c}_1 = \tilde{c}_1(a_2, b_1^\dagger, \tilde{a}_2, \tilde{b}_1^\dagger)$
- $(q_2^l)^\dagger = d_2^\dagger = d_2^\dagger(a_2, b_1^\dagger, \tilde{a}_2, \tilde{b}_1^\dagger)$
- $(\tilde{q}_2^l)^\dagger = \tilde{d}_2^\dagger = \tilde{d}_2^\dagger(a_2, b_1^\dagger, \tilde{a}_2, \tilde{b}_1^\dagger)$
\[ -\frac{\partial L}{\partial x} ^1 (Q, \tilde{Q}) = c_1 = \tilde{c} \]
\[ \frac{\partial L}{\partial y} ^1 (Q, \tilde{Q}) \]
\[ -\frac{\partial L}{\partial x} ^2 (Q, \tilde{Q}) = d^2 = \tilde{d} \]
\[ -\frac{\partial L}{\partial y} ^2 (Q, \tilde{Q}) \]
Equations of Motion

\[
\begin{align*}
    c_1 &= (z_1 - z_2) a_2 b_1^* + (z_2 - \zeta_1) \tilde{a}_2 \tilde{b}_1^* \tilde{c}_1 \\
    \tilde{c}_1 &= (z_1 - \zeta_2) L_{-1} a_2 \tilde{b}_1^* + (\zeta_2 - \zeta_1) \tilde{L}_{-1} \tilde{a}_2 \tilde{b}_1^* \\
    d_2^* &= (z_2 - z_1) b_1^* b_1^* a_2 + (z_1 - \zeta_2) \tilde{b}_1^* \tilde{L}_{-1} \tilde{a}_2 \\
    \tilde{d}_2^* &= (z_2 - \zeta_1) b_1^* b_1^* \tilde{a}_2 + (\zeta_1 - \zeta_2) \tilde{b}_1^* \tilde{L}_{-1} \tilde{a}_2
\end{align*}
\]

\[\text{Discrete Euler-Lagrange Equations}\]
\[\partial L / \partial x_1^*(Q, \tilde{Q}) = c_1 = \tilde{c}_1 \]
\[\partial L / \partial y_1^*(Q, \tilde{Q}) = \partial L / \partial x_2^*(Q, \tilde{Q}) = d_2^* = \tilde{d}_2^* = -\partial L / \partial y_2^*(Q, \tilde{Q})\]
Equations of Motion

\[ c_1 = (z_1 - z_2) \frac{a_2}{b_1 a_2} + (z_2 - \zeta_1) \frac{\tilde{a}_2}{b_1 \tilde{a}_2} \]
Equations of Motion

\[ c_1 = (z_1 - z_2) \frac{a_2}{b_1^\dagger a_2} + (z_2 - \zeta_1) \frac{\tilde{a}_2}{b_1^\dagger \tilde{a}_2} \]

\[ \tilde{c}_1 = (z_1 - \zeta_2) \frac{L_0^{-1} a_2}{\tilde{b}_1^\dagger L_0^{-1} a_2} + (\zeta_2 - \zeta_1) \frac{L_0^{-1} \tilde{a}_2}{\tilde{b}_1^\dagger L_0^{-1} \tilde{a}_2} \]
Equations of Motion

\[ \mathbf{c}_1 = (z_1 - z_2) \frac{\mathbf{a}_2}{\mathbf{b}_1^\dagger \mathbf{a}_2} + (z_2 - \zeta_1) \frac{\tilde{\mathbf{a}}_2}{\mathbf{b}_1^\dagger \tilde{\mathbf{a}}_2} \]

\[ \tilde{\mathbf{c}}_1 = (z_1 - \zeta_2) \frac{\mathbf{L}_0^{-1} \mathbf{a}_2}{\tilde{\mathbf{b}}_1^\dagger \mathbf{L}_0^{-1} \mathbf{a}_2} + (\zeta_2 - \zeta_1) \frac{\mathbf{L}_0^{-1} \tilde{\mathbf{a}}_2}{\tilde{\mathbf{b}}_1^\dagger \mathbf{L}_0^{-1} \tilde{\mathbf{a}}_2} \]

\[ \mathbf{d}_2^\dagger = (z_2 - z_1) \frac{\tilde{\mathbf{b}}_1^\dagger}{\mathbf{b}_1^\dagger \mathbf{a}_2} + (z_1 - \zeta_2) \frac{\tilde{\mathbf{b}}_1^\dagger \mathbf{L}_0^{-1}}{\mathbf{b}_1^\dagger \mathbf{L}_0^{-1} \mathbf{a}_2} \]
Equations of Motion

- $c_1 = (z_1 - z_2) \frac{a_2}{b_1 a_2} + (z_2 - \zeta_1) \frac{\tilde{a}_2}{b_1 \tilde{a}_2}$

- $\tilde{c}_1 = (z_1 - \zeta_2) \frac{L_0^{-1} a_2}{\tilde{b}_1 L_0^{-1} a_2} + (\zeta_2 - \zeta_1) \frac{L_0^{-1} \tilde{a}_2}{\tilde{b}_1 L_0^{-1} \tilde{a}_2}$

- $d_2^\dagger = (z_2 - z_1) \frac{b_1^\dagger}{b_1 a_2} + (z_1 - \zeta_2) \frac{b_1 L_0^{-1}}{\tilde{b}_1 L_0^{-1} a_2}$

- $\tilde{d}_2^\dagger = (z_2 - \zeta_1) \frac{b_1^\dagger}{b_1 \tilde{a}_2} + (\zeta_1 - \zeta_2) \frac{b_1 L_0^{-1}}{\tilde{b}_1 L_0^{-1} \tilde{a}_2}$
Equations of Motion

\[ c_1 = (z_1 - z_2) \frac{a_2}{b^\dagger_1 a_2} + (z_2 - \zeta_1) \frac{\tilde{a}_2}{b^\dagger_1 \tilde{a}_2} = - \frac{\partial L}{\partial x^\dagger_1} (Q, \tilde{Q}) \]

\[ \tilde{c}_1 = (z_1 - \zeta_2) \frac{L^{-1}_0 a_2}{\tilde{b}^\dagger_1 L^{-1}_0 a_2} + (\zeta_2 - \zeta_1) \frac{L^{-1}_0 \tilde{a}_2}{\tilde{b}^\dagger_1 L^{-1}_0 \tilde{a}_2} = \frac{\partial L}{\partial y^\dagger_1} (Q, \tilde{Q}) \]

\[ d^\dagger_2 = (z_2 - z_1) \frac{b^\dagger_1}{b^\dagger_1 a_2} + (z_1 - \zeta_2) \frac{\tilde{b}^\dagger_1 L^{-1}_0}{\tilde{b}^\dagger_1 L^{-1}_0 a_2} = \frac{\partial L}{\partial x^\dagger_2} (Q, \tilde{Q}) \]

\[ \tilde{d}^\dagger_2 = (z_2 - \zeta_1) \frac{b^\dagger_1}{b^\dagger_1 \tilde{a}_2} + (\zeta_1 - \zeta_2) \frac{\tilde{b}^\dagger_1 L^{-1}_0}{\tilde{b}^\dagger_1 L^{-1}_0 \tilde{a}_2} = - \frac{\partial L}{\partial y^\dagger_2} (Q, \tilde{Q}) \]
Equations of Motion

\[ c_1 = (z_1 - z_2) \frac{a_2}{b_1 a_2} + (z_2 - \zeta_1) \frac{\tilde{a}_2}{b_1 \tilde{a}_2} = - \frac{\partial L}{\partial x_1^\dagger} (Q, \tilde{Q}) \]

\[ \tilde{c}_1 = (z_1 - \zeta_2) \frac{L_0^{-1} a_2}{\tilde{b}_1 L_0^{-1} a_2} + (\zeta_2 - \zeta_1) \frac{L_0^{-1} \tilde{a}_2}{\tilde{b}_1 L_0^{-1} \tilde{a}_2} = \frac{\partial L}{\partial y_1^\dagger} (Q, \tilde{Q}) \]

\[ d_2^\dagger = (z_2 - z_1) \frac{b_1^\dagger}{b_1 a_2} + (z_1 - \zeta_2) \frac{\tilde{b}_1 L_0^{-1}}{\tilde{b}_1 L_0^{-1} \tilde{a}_2} = \frac{\partial L}{\partial x_2} (Q, \tilde{Q}) \]

\[ \tilde{d}_2^\dagger = (z_2 - \zeta_1) \frac{b_1^\dagger}{b_1 \tilde{a}_2} + (\zeta_1 - \zeta_2) \frac{\tilde{b}_1 L_0^{-1}}{\tilde{b}_1 L_0^{-1} \tilde{a}_2} = - \frac{\partial L}{\partial y_2} (Q, \tilde{Q}) \]

Discrete Euler-Lagrange Equations

\[- \frac{\partial L}{\partial x_1^\dagger} (Q, \tilde{Q}) = c_1 = \tilde{c}_1 = \frac{\partial L}{\partial y_1^\dagger} (Q, Q) \]

\[ \frac{\partial L}{\partial x_2} (Q, \tilde{Q}) = d_2^\dagger = \tilde{d}_2^\dagger = - \frac{\partial L}{\partial y_2} (Q, Q) \]
The Lagrangian

Thus, we have the following

Theorem

The equations of both the isospectral and isomonodromic dynamic can be written in the Lagrangian form with

\[ L(X, Y, t) = (z_2(t) - z_1(t)) \log(x_1^\dagger x_2) + (\zeta_2(t) - \zeta_1(t)) \log(y_1^\dagger L^{-1} y_2) + (\zeta_1(t) - z_2(t)) \log(x_1^\dagger y_2), \]

where

\[ X = (x_1, x_1^\dagger), \quad Y = (y_1, y_1^\dagger), \]

in the isomonodromic case

\[ z_1(t) = z_1 - t, \quad \zeta_1(t) = \zeta_1 - t, \]

and in the isospectral case

\[ z_1(t) = z_1, \quad \zeta_1(t) = \zeta_1. \]

\[ L(X, Y) \]

is time-independent.
The Lagrangian

Thus, we have the following

**Theorem**

The equations of both the isospectral and isomonodromic dynamic can be written in the Lagrangian form with

\[
\mathcal{L}(X, Y, t) = (z_2 - z_1(t)) \log(x_1^\dagger x_2) + (z_1(t) - \zeta_2) \log(y_1^\dagger L_0^{-1} x_2) \\
+ (\zeta_2 - \zeta_1(t)) \log(y_1^\dagger L_0^{-1} y_2) + (\zeta_1(t) - z_2) \log(x_1^\dagger y_2),
\]

where \( X = (x_1, x_2^\dagger) \) and \( Y = (y_1, y_2^\dagger) \), in the isomonodromic case \( z_1(t) = z_1 - t, \zeta_1(t) = \zeta_1 - t \), and in the isospectral case \( z_1(t) = z_1, \zeta_1(t) = \zeta_1 \) and \( \mathcal{L}(X, Y) \) is time-independent.
Sketch of the proof

Consider $\tilde{L}(z) = B_r(z) B_l(z) L_0 = \tilde{B}_l(z) L_0 \tilde{B}_r(z)$.

Consider $\tilde{M}(z) = L^{-1} - B_l(z) - B_r(z) = \tilde{B}_r(z) - L^{-1} \tilde{B}_l(z)$.

$\tilde{c}_1 = (z_1 - z_2) a_2^t 1 a_2 + (z_2 - \zeta_1) \tilde{a}_2^t 1 \tilde{a}_2$.

$\tilde{c}_1 = (z_1 - \zeta_2) L^{-1} 0 a_2^t \tilde{b}_1 L^{-1} 0 a_2 + (\zeta_2 - \zeta_1) L^{-1} 0 \tilde{a}_2^t \tilde{b}_1 L^{-1} 0 \tilde{a}_2$. 
Sketch of the proof

Consider \( \tilde{L}(z) = B_1(z)B_2(z)L_0 = \tilde{B}_2(z)L_0\tilde{B}_1(z). \)
Sketch of the proof

Consider $\tilde{L}(z) = B_1^r(z)B_2^l(z)L_0 = \tilde{B}_2^l(z)L_0\tilde{B}_1^r(z)$.

- $\text{res}_{z_2}: \quad B_1^r(z_2)a_2d_2^\dagger L_0 = \tilde{a}_2\tilde{d}_2^\dagger L_0\tilde{B}_1^r(z_2)$
Sketch of the proof

Consider \( \tilde{L}(z) = B'_1(z)B'_2(z)L_0 = \tilde{B}'_2(z)L_0\tilde{B}'_1(z) \).

- \( \text{res}_{z_2} : \quad B'_1(z_2)a_2\tilde{d}_2^\dagger L_0 = \tilde{a}_2\tilde{d}_2^\dagger L_0\tilde{B}'_1(z_2) \)
Sketch of the proof

Consider \( \tilde{L}(z) = B_1^r(z)B_2^l(z)L_0 = \tilde{B}_2^l(z)L_0\tilde{B}_1^r(z) \).

- \( \text{res}_{z_2} : B_1^r(z_2)a_2^d d_2^\dagger L_0 = \tilde{a}_2^d d_2^\dagger L_0 \tilde{B}_1^r(z_2) \)

\[
c_1 = (z_1 - z_2) \frac{a_2}{b_1 a_2} + (z_2 - \zeta_1) \frac{\tilde{a}_2}{b_1 \tilde{a}_2}
\]
Sketch of the proof

Consider $\tilde{L}(z) = B_1^r(z)B_2^l(z)L_0 = \tilde{B}_2^l(z)L_0\tilde{B}_1^r(z)$.

- $\text{res}_{z_2} : \quad B_1^r(z_2)a_2d_2^\dagger L_0 = \tilde{a}_2d_2^\dagger L_0\tilde{B}_1^r(z_2)$

\[
c_1 = (z_1 - z_2)\frac{a_2}{b_1^\dagger a_2} + (z_2 - \zeta_1)\frac{\tilde{a}_2}{b_1^\dagger \tilde{a}_2}
\]

Consider $\tilde{M}(z) = L_0^{-1}B_2^l(z)^{-1}B_1^r(z)^{-1} = \tilde{B}_1^r(z)^{-1}L_0^{-1}\tilde{B}_2^l(z)^{-1}$. 
Sketch of the proof

Consider $\widetilde{L}(z) = B_1^r(z)B_2^l(z)L_0 = \widetilde{B}_2^l(z)L_0\widetilde{B}_1^r(z)$.

- $\text{res}_{z_2}: \quad B_1^r(z_2)a_2d_2^\dagger L_0 = \tilde{a}_2\tilde{d}_2^\dagger L_0\tilde{B}_1^r(z_2)$

$$c_1 = (z_1 - z_2)\frac{a_2}{b_1^\dagger a_2} + (z_2 - \zeta_1)\frac{\tilde{a}_2}{b_1^\dagger \tilde{a}_2}$$

Consider $\widetilde{M}(z) = L_0^{-1}B_2^l(z)^{-1}B_1^r(z)^{-1} = \tilde{B}_1^r(z)^{-1}L_0^{-1}\tilde{B}_2^l(z)^{-1}$.

- $\text{res}_{\zeta_2}: \quad L_0^{-1}a_2d_2^\dagger B_1^r(\zeta_2)^{-1} = \tilde{B}_1^r(\zeta_2)^{-1}L_0^{-1}\tilde{a}_2\tilde{d}_2^\dagger$
Sketch of the proof

Consider \( \tilde{L}(z) = B_1^r(z)B_2^l(z)L_0 = B_2^l(z)L_0B_1^r(z) \).

- \( \text{res}_{z_2} : \quad B_1^r(z_2)a_2d_2^\dagger L_0 = \tilde{a}_2d_2^\dagger L_0B_1^r(z_2) \)

\[
\tilde{c}_1 = (z_1 - z_2) \frac{a_2}{b_1^\dagger a_2} + (z_2 - \zeta_1) \frac{\tilde{a}_2}{b_1^\dagger \tilde{a}_2}
\]

Consider \( \tilde{M}(z) = L_0^{-1}B_2^l(z)^{-1}B_1^r(z)^{-1} = \tilde{B}_1^r(z)^{-1}L_0^{-1}\tilde{B}_2^l(z)^{-1} \).

- \( \text{res}_{\zeta_2} : \quad L_0^{-1}a_2d_2^\dagger B_1^r(\zeta_2)^{-1} = \tilde{B}_1^r(\zeta_2)^{-1}L_0^{-1}\tilde{a}_2d_2^\dagger \)
Sketch of the proof

Consider $\tilde{L}(z) = B_1^f(z)B_2^l(z)L_0 = B_2^l(z)L_0B_1^f(z)$.

- \( \text{res}_{z_2}: \quad B_1^f(z_2)a_2\tilde{d}_2L_0 = \tilde{a}_2\tilde{d}_2L_0B_1^f(z_2) \)

\[
\begin{align*}
c_1 &= (z_1 - z_2)\frac{a_2}{b_1a_2} + (z_2 - \zeta_1)\frac{\tilde{a}_2}{b_1\tilde{a}_2}
\end{align*}
\]

Consider $\tilde{M}(z) = L_0^{-1}B_2^l(z)^{-1}B_1^f(z)^{-1} = \tilde{B}_1^f(z)^{-1}L_0^{-1}\tilde{B}_2^l(z)^{-1}$.

- \( \text{res}_{\zeta_2}: \quad L_0^{-1}a_2\tilde{d}_2^\dagger B_1^f(\zeta_2)^{-1} = \tilde{B}_1^f(\zeta_2)^{-1}L_0^{-1}\tilde{a}_2\tilde{d}_2^\dagger \)

\[
\begin{align*}
\tilde{c}_1 &= (z_1 - \zeta_2)\frac{L_0^{-1}a_2}{\tilde{b}_1L_0^{-1}a_2} + (\zeta_2 - \zeta_1)\frac{L_0^{-1}\tilde{a}_2}{\tilde{b}_1L_0^{-1}\tilde{a}_2}
\end{align*}
\]
Coordinates on $\mathcal{M}^r_D$

Since $B_r^1(z) \sim B_l^2(z) \circ L_0 = B_l^1(z) L_0 B_r^2(z)$,
$c_2 = L^{-1} a \sim_2$, and $d_1^\dagger = b \sim_1^\dagger$,
and we have:

Theorem

The vectors $(c_2, d_1^\dagger; a_2, b_1^\dagger)$, considered up to rescaling, are coordinates on
the space $\mathcal{M}^r_D$.

To recover $L_{\pm 1}(z)$, consider the function

$L((x_2, x_1^\dagger), (y_2, y_1^\dagger)) = (z_2 - z_1) \log(x_1^\dagger L_0 x_2) + (z_1 - \zeta_2) \log(y_1^\dagger x_2) + (\zeta_2 - \zeta_1) \log(y_1^\dagger L^{-1}_0 y_2) + (\zeta_1 - z_2) \log(x_1^\dagger y_2)$.

Then $a_1 = -\frac{\partial L}{\partial x_1^\dagger}((c_2, d_1^\dagger), (a_2, b_1^\dagger))$,
$b_1^\dagger_2 = \frac{\partial L}{\partial x_2}((c_2, d_1^\dagger), (a_2, b_1^\dagger))$,
$c_1 = \frac{\partial L}{\partial y_1^\dagger}((c_2, d_1^\dagger), (a_2, b_1^\dagger))$,
$d_1^\dagger_2 = -\frac{\partial L}{\partial y_2}((c_2, d_1^\dagger), (a_2, b_1^\dagger))$. 
Coordinates on $\mathcal{M}_D^r$

Since $\underline{B}_1(z) \underline{B}_2(z) L_0 = \underline{B}_1(z) L_0 \underline{B}_2(z)$, $c_2 = L_0^{-1} a_2$, $d_1^\dagger = b_1^\dagger$, and we have:

**Theorem**

The vectors $(c_2, d_1^\dagger; a_2, b_1^\dagger)$, considered up to rescaling, are coordinates on the space $\mathcal{M}_D^r$. To recover $L^{\pm 1}(z)$, consider the function

$$L((x_2, x_1^\dagger), (y_2, y_1^\dagger)) = (z_2 - z_1) \log(x_1^\dagger L_0 x_2) + (z_1 - \zeta_2) \log(y_1^\dagger x_2)$$
$$+ (\zeta_2 - \zeta_1) \log(y_1^\dagger L_0^{-1} y_2) + (\zeta_1 - z_2) \log(x_1^\dagger y_2).$$

Then

$$a_1 = - \frac{\partial L}{\partial x_1^\dagger} ((c_2, d_1^\dagger); (a_2, b_1^\dagger)); \quad b_2^\dagger = \frac{\partial L}{\partial x_2} ((c_2, d_1^\dagger); (a_2, b_1^\dagger));$$
$$c_1 = \frac{\partial L}{\partial y_1^\dagger} ((c_2, d_1^\dagger); (a_2, b_1^\dagger)); \quad d_2^\dagger = - \frac{\partial L}{\partial y_2} ((c_2, d_1^\dagger); (a_2, b_1^\dagger)).$$
Relation to discrete Painlevé equations
Relation to discrete Painlevé equations

In the rank $r = 2$ case, the isomonodromic dynamics, written down in the so-called spectral coordinates, is described by the discrete Painlevé equations.
Relation to discrete Painlevé equations

In the rank \( r = 2 \) case, the isomonodromic dynamics, written down in the so-called spectral coordinates, is described by the discrete Painlevé equations.

References
Relation to discrete Painlevé equations

In the rank $r = 2$ case, the isomonodromic dynamics, written down in the so-called spectral coordinates, is described by the discrete Painlevé equations.

References

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Relation to discrete Painlevé equations

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Main new feature of our approach is the use of rational functions, which sometimes gives computational advances, emphasis on the re-factorization, and the relationship to the Lagrangian I mentioned earlier.
Rank 2 case: general remarks

We consider

\[ L(z) = L_0 + L_1 z - z_1 + L_2 z - z_2 \]

and

\[ M(z) = L(z) - 1 = M_0 - M_1 z - \zeta_1 - M_2 z - \zeta_2, \]

\[ L_0 = \text{diag}\{\rho_1, \rho_2\}, \]

\[ M_0 = \text{diag}\{1/\rho_1, 1/\rho_2\}, \]

\[ L_i = \alpha_i \begin{bmatrix} \alpha_i \beta_i & \gamma_i \beta_i \end{bmatrix}, \]

\[ M_i = \beta_i \begin{bmatrix} \beta_i \gamma_i & \gamma_i \gamma_i \end{bmatrix}, \]

\[ (i = 1, 2), \]

\[ \det L(z) = \rho_1 \rho_2 (z - \zeta_1)(z - \zeta_2)(z - z_1)(z - z_2). \]

Then

\[ L(z) = \begin{bmatrix} \rho_1 + \alpha_1 a_1 z - z_1 + \alpha_2 a_2 z - z_2 & \alpha_1 b_1 z - z_1 + \alpha_2 b_2 z - z_2 \\ \alpha_1 z - z_1 + \alpha_2 z - z_2 & \rho_2 + \alpha_1 b_1 z - z_1 + \alpha_2 b_2 z - z_2 \end{bmatrix} \]
Rank 2 case: general remarks

We consider

\[ L(z) = L_0 + \frac{L_1}{z-z_1} + \frac{L_2}{z-z_2} \quad \text{and} \quad M(z) = L(z)^{-1} = M_0 - \frac{M_1}{z-\zeta_1} - \frac{M_2}{z-\zeta_2}, \]
Rank 2 case: general remarks

We consider

- \( L(z) = L_0 + \frac{L_1}{z-z_1} + \frac{L_2}{z-z_2} \) and \( M(z) = L(z)^{-1} = M_0 - \frac{M_1}{z-\zeta_1} - \frac{M_2}{z-\zeta_2} \),
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- \( L_i = \alpha_i \begin{bmatrix} a_i \\ 1 \end{bmatrix} \begin{bmatrix} 1 & b_i \end{bmatrix} \), \( M_i = \beta_i \begin{bmatrix} c_i \\ 1 \end{bmatrix} \begin{bmatrix} 1 & d_i \end{bmatrix} \), \( i = 1, 2 \),
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- \( \det L(z) = \rho_1 \rho_2 \frac{(z-\zeta_1)(z-\zeta_2)}{(z-z_1)(z-z_2)}. \)
We consider

- \( \mathbf{L}(z) = \mathbf{L}_0 + \frac{\mathbf{L}_1}{z-z_1} + \frac{\mathbf{L}_2}{z-z_2} \) and \( \mathbf{M}(z) = \mathbf{L}(z)^{-1} = \mathbf{M}_0 - \frac{\mathbf{M}_1}{z-\zeta_1} - \frac{\mathbf{M}_2}{z-\zeta_2} \),
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- \( \det \mathbf{L}(z) = \rho_1 \rho_2 \frac{(z-\zeta_1)(z-\zeta_2)}{(z-z_1)(z-z_2)} \).
- Then

\[
\mathbf{L}(z) = \begin{bmatrix}
\rho_1 + \frac{\alpha_1 a_1}{z-z_1} + \frac{\alpha_2 a_2}{z-z_2} & \frac{\alpha_1 a_1 b_1}{z-z_1} + \frac{\alpha_2 a_2 b_2}{z-z_2} \\
\frac{\alpha_1}{z-z_1} + \frac{\alpha_2}{z-z_2} & \rho_2 + \frac{\alpha_1 b_1}{z-z_1} + \frac{\alpha_2 b_2}{z-z_2}
\end{bmatrix}
\]
 ranked 2 case and difference Painlevé

**Spectral Coordinates**

\[
\mu(z) := L(z) = \alpha_1 z - z_1 + \alpha_2 z - z_2 = \hat{\mu}(z - \gamma)(z - z_1)(z - z_2).
\]

Then \(\alpha_i = \text{res}_{z_i} \mu(z) = \hat{\mu}(\gamma - z_i)(z_j - z_i),\) \(i, j = 1, 2\) and \(i \neq j\).

We defined \(\gamma\) by the condition \(\mu(\gamma) = L(z_2) = 0\). Also, \(L(\gamma) = \begin{pmatrix} \rho_1 & \pi_1 \ast 0 & \rho_2 \\ \pi_2 \end{pmatrix}\), \(\pi_1 \pi_2 = (z - \zeta_1)(z - \zeta_2)(z - z_1)(z - z_2)\).

Define \(\pi\) by \(\pi_1 = (\gamma - \zeta_2)(\gamma - z_1)\pi_2\) (and so \(\pi_2 = (\gamma - \zeta_1)(\gamma - z_2)\pi_1\)).

The pair \((\gamma, \pi)\) is called the spectral coordinates of \(L(z)\).
Spectral Coordinates

Put

$$\mu(z) := L(z)_{21} = \frac{\alpha_1}{z - z_1} + \frac{\alpha_2}{z - z_2} = \frac{\hat{\mu}(z - \gamma)}{(z - z_1)(z - z_2)}.$$
Spectral Coordinates

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- Then \( \alpha_i = \text{res}_{z_i} \mu(z) = \frac{\hat{\mu}(\gamma - z_i)}{(z_j - z_i)}, i, j = 1, 2 \) and \( i \neq j \).
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- We defined \( \gamma \) by the condition \( \mu(\gamma) = L_{21}(\gamma) = 0 \). Also,

\[ L(\gamma) = \begin{bmatrix} \rho_1 \pi_1 & \ast \\ 0 & \rho_2 \pi_2 \end{bmatrix}, \quad \pi_1 \pi_2 = \frac{(z - \zeta_1)(z - \zeta_2)}{(z - z_1)(z - z_2)}. \]
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- Then \( \alpha_i = \text{res}_{z_i} \mu(z) = \frac{\hat{\mu}(\gamma - z_i)}{(z_j - z_i)}, \) \( i, j = 1, 2 \) and \( i \neq j. \)

- We defined \( \gamma \) by the condition \( \mu(\gamma) = L_{21}(\gamma) = 0. \) Also,
  \[ L(\gamma) = \begin{bmatrix} \rho_1 \pi_1 & * \\ 0 & \rho_2 \pi_2 \end{bmatrix}, \quad \pi_1 \pi_2 = \frac{(z - \zeta_1)(z - \zeta_2)}{(z - z_1)(z - z_2)}. \]

- Define \( \pi \) by \( \pi_1 = \frac{(\gamma - \zeta_2)}{(\gamma - z_1)} \pi \) (and so \( \pi_2 = \frac{(\gamma - \zeta_1)}{(\gamma - z_2)} \pi \)).
Spectral Coordinates

- Put
  \[ \mu(z) := L(z)_{21} = \frac{\alpha_1}{z - z_1} + \frac{\alpha_2}{z - z_2} = \frac{\hat{\mu}(z - \gamma)}{(z - z_1)(z - z_2)}. \]

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The Spectral Coordinates

The pair \( (\gamma, \pi) \) is called the spectral coordinates of \( L(z) \).
We need the following notation:

\[ \phi_i(a,b) := \pi_i(\gamma - a) - (\gamma - b) \]

Then \( L(z) \) in spectral coordinates:

\[
\begin{align*}
L(z)_{11} &= \rho_1 \phi_1(z^2, z) z - z^2 + \mu(z) a_1 \\
L(z)_{22} &= \rho_2 \phi_2(z^2, z) z - z^2 + \mu(z) b_2
\end{align*}
\]
Spectral Coordinates (continued)

- We need the following Notation

\[ \varphi_i(a, b) := \pi_i(\gamma - a) - (\gamma - b). \]
Spectral Coordinates (continued)

- We need the following

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Spectral Coordinates (continued)

- We need the following

**Notation**

\[ \phi_i(a, b) := \pi_i(\gamma - a) - (\gamma - b). \]

- Then

**L(z) in spectral coordinates**

\[
\begin{align*}
L(z)_{11} &= \frac{\rho_1 \varphi_1(z_2, z)}{z - z_2} + \mu(z)a_1 = \frac{\rho_1 \varphi_1(z_1, z)}{z - z_1} + \mu(z)a_2, \\
L(z)_{22} &= \frac{\rho_2 \varphi_2(z_2, z)}{z - z_2} + \mu(z)b_1 = \frac{\rho_2 \varphi_2(z_1, z)}{z - z_1} + \mu(z)b_2.
\end{align*}
\]
Normalization (difference case)

Our normalization condition is

\[ L_\infty = -r_\infty L(z) = L_1 + L_2 = \begin{pmatrix} \rho_1 k_1 \mu \rho_2 k_2 \end{pmatrix}. \]

Then \( \hat{\mu} = \mu, \rho_1 k_1 = \rho_1 \phi(z_2) + \mu a_1 g \) gives

\[ a_1 = \rho_1 \mu(k_1 - \phi(z_2)), \]

and so on.

The inverse matrix

\[ M(z) = M_0 - M_1 z - \mu_1 \zeta_2 - M_2 z - \zeta_2 \]

has similar parameters:

- \( L(z) \) and \( M(z) \):
  - \( z_1 z_2 \zeta_1 \zeta_2 \rho_1 \rho_2 k_1 k_2 \mu \pi \)
  - \( M(z) \):
    - \( \zeta_1 \zeta_2 z_1 z_2 \rho_1 \rho_2 - k_1 - k_2 - \mu \rho_1 \rho_2 \pi_1 \pi_2 \)

This follows from

\[ M_\infty = -L^{-1} L_0 L^{-1} \]

and

\[ M(\gamma) = \begin{pmatrix} 1 & \rho_1 \pi_1^* & 0 & \rho_2 \pi_2 \end{pmatrix}. \]
Normalization (difference case)

Our normalization condition is

\[ L_{\infty} = - \text{res}_{\infty} L(z) = L_1 + L_2 := \begin{bmatrix} \rho_1 k_1 & * \\ \mu & \rho_2 k_2 \end{bmatrix} \].
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Then \( \hat{\mu} = \mu \),
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Then \( \hat{\mu} = \mu, \rho_1 k_1 = \rho_1 \varphi_1(z_2, z_2) + \mu a_1 \) gives \( a_1 = \frac{\rho_1}{\mu} (k_1 - \varphi_1(z_2, z_2)) \), and so on.
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Then \( \hat{\mu} = \mu \), \( \rho_1 k_1 = \rho_1 \varphi_1(z_2, z_2) + \mu a_1 \) gives \( a_1 = \frac{\rho_1}{\mu} (k_1 - \varphi_1(z_2, z_2)) \), and so on.

The inverse matrix \( M(z) = M_0 - \frac{M_1}{z - \zeta_1} - \frac{M_2}{z - \zeta_2} \) has similar parameters:
Normalization (difference case)

Our normalization condition is

$$L_{\infty} = - \text{res}_{\infty} L(z) = L_1 + L_2 := \begin{bmatrix} \rho_1 k_1 & \ast \\ \mu & \rho_2 k_2 \end{bmatrix}.$$ 

Then $\hat{\mu} = \mu, \rho_1 k_1 = \rho_1 \varphi_1(z_2, z_2) + \mu a_1$ gives $a_1 = \frac{\rho_1}{\mu} (k_1 - \varphi_1(z_2, z_2)),$ and so on.

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Normalization (difference case)

Our normalization condition is

\[ L_{\infty} = - \text{res}_{\infty} L(z) = L_1 + L_2 := \begin{pmatrix} \rho_1 k_1 & * \\ \mu & \rho_2 k_2 \end{pmatrix}. \]

Then \( \hat{\mu} = \mu, \rho_1 k_1 = \rho_1 \varphi_1(z_2, z_2) + \mu a_1 \) gives \( a_1 = \frac{\rho_1}{\mu} (k_1 - \varphi_1(z_2, z_2)) \), and so on.

The inverse matrix \( M(z) = M_0 - \frac{M_1}{z - \zeta_1} - \frac{M_2}{z - \zeta_2} \) has similar parameters:

**Types of** \( L(z) \) **and** \( M(z) \)

| \( L(z) \)  | \( z_1 \) | \( z_2 \) | \( \zeta_1 \) | \( \zeta_2 \) | \( \rho_1 \) | \( \rho_2 \) | \( k_1 \) | \( k_2 \) | \( \frac{\mu}{\rho_1 \rho_2} \) | \( \gamma \) | \( \frac{1}{\rho_1} \) | \( \frac{1}{\rho_2} \) |
| \( M(z) \)  | \( \zeta_1 \) | \( \zeta_2 \) | \( z_1 \) | \( z_2 \) | \( \frac{1}{\rho_1} \) | \( \frac{1}{\rho_2} \) | \( -k_1 \) | \( -k_2 \) | \( -\frac{\mu}{\rho_1 \rho_2} \) | \( \gamma \) | \( \frac{1}{\rho_1} \) | \( \frac{1}{\rho_2} \) |

This follows from \( M_{\infty} = -L_0^{-1} L_{\infty} L_0^{-1} \) and \( M(\gamma) = \begin{pmatrix} \frac{1}{\rho_1} & \frac{1}{\pi_1} & * \\ 0 & \frac{1}{\rho_2} & \frac{1}{\pi_2} \end{pmatrix} \).
$L(z)$ in spectral coordinates

Additive form of $L(z)$ in spectral coordinates

$$a_1 = \rho_1 \mu(k_1 - \phi_1(z_2, z_2))$$
$$a_2 = \rho_1 \mu(k_1 - \phi_1(z_1, z_1))$$

$$b_1 = \rho_2 \mu(k_2 - \phi_2(z_2, z_2)/\pi_1)$$
$$b_2 = \rho_2 \mu(k_2 - \phi_2(z_1, z_1)/\pi_1)$$

Additive form of $M(z)$ in spectral coordinates

$$c_1 = \rho_2 \mu(k_1 - \phi_1(\zeta_2, \zeta_2)/\pi_1)$$
$$c_2 = \rho_2 \mu(k_1 - \phi_1(\zeta_1, \zeta_1)/\pi_1)$$

$$d_1 = \rho_1 \mu(k_2 - \phi_2(\zeta_2, \zeta_2)/\pi_2)$$
$$d_2 = \rho_1 \mu(k_2 - \phi_2(\zeta_1, \zeta_1)/\pi_2)$$

Together they completely describe left and right divisors of $L(z)$ and $M(z)$.\ICENSE{Anton Dzhamay (UNC) Refactorization dynamics NMMP, July 19, 2009} 26 / 29
$L(z)$ in spectral coordinates

Additive form of $L(z)$ in spectral coordinates

\[
\begin{align*}
    a_1 &= \frac{\rho_1}{\mu} (k_1 - \varphi_1(z_2, z_2)) \\
    b_1 &= \frac{\rho_2}{\mu} (k_2 - \varphi_2(z_2, z_2)) \\
    a_2 &= \frac{\rho_1}{\mu} (k_1 - \varphi_1(z_1, z_1)) \\
    b_2 &= \frac{\rho_2}{\mu} (k_2 - \varphi_2(z_1, z_1)).
\end{align*}
\]
**L(z) in spectral coordinates**

Additive form of \( L(z) \) in spectral coordinates

\[
a_1 = \frac{\rho_1}{\mu} (k_1 - \varphi_1(z_2, z_2)) \quad \quad \quad \quad \quad a_2 = \frac{\rho_1}{\mu} (k_1 - \varphi_1(z_1, z_1))
\]
\[
b_1 = \frac{\rho_2}{\mu} (k_2 - \varphi_2(z_2, z_2)) \quad \quad \quad \quad \quad b_2 = \frac{\rho_2}{\mu} (k_2 - \varphi_2(z_1, z_1)).
\]

Additive form of \( M(z) \) in spectral coordinates

\[
c_1 = \frac{\rho_2}{\mu} (k_1 - \varphi_1(\zeta_2, \zeta_2)/\pi_1) \quad \quad \quad \quad \quad c_2 = \frac{\rho_2}{\mu} (k_1 - \varphi_1(\zeta_1, \zeta_1)/\pi_1)
\]
\[
d_1 = \frac{\rho_1}{\mu} (k_2 - \varphi_2(\zeta_2, \zeta_2)/\pi_2) \quad \quad \quad \quad \quad b_2 = \frac{\rho_1}{\mu} (k_2 - \varphi_2(\zeta_1, \zeta_1)/\pi_2).
\]
\( L(z) \) in spectral coordinates

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a_1 &= \frac{\rho_1}{\mu} (k_1 - \varphi_1(z_2, z_2)) \\
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b_2 &= \frac{\rho_2}{\mu} (k_2 - \varphi_2(z_1, z_1)).
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Together they completely describe left and right divisors of \( L(z) \) and \( M(z) \).
Isomonodromy and dP-V
Isomonodromy and dP-V

We consider the isomonodromy transformation

\[ \tilde{L}(z) = R(z + 1)L(z)R^{-1}(z) \]
Isomonodromy and dP-V

We consider the isomonodromy transformation

$$\tilde{L}(z) = R(z + 1)L(z)R^{-1}(z)$$

for the linear difference system

$$\Psi(z + 1) = L(z)\Psi(z)$$
Isomonodromy and dP-V

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$$\tilde{L}(z) = R(z + 1)L(z)R^{-1}(z)$$

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$$\Psi(z + 1) = L(z)\Psi(z)$$

with $R(z) = B_1^r(z)$, where $L(z) = B_2^l(z)L_0B_1^r(z)$:
Isomonodromy and dP-V

We consider the isomonodromy transformation

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for the linear difference system

\[ \psi(z + 1) = L(z)\psi(z) \]

with \( R(z) = B^r_1(z) \), where \( L(z) = B^l_2(z)L_0B^r_1(z) \):

\[ \tilde{L}(z) = B^r_1(z + 1)B^l_2(z)L_0 = \tilde{B}^l_1(z)L_0\tilde{B}^r_2(z) = \tilde{B}^l_2(z)L_0\tilde{B}^r_1(z). \]
Isomonodromy and dP-V

We consider the isomonodromy transformation

\[
\tilde{L}(z) = R(z + 1) L(z) R^{-1}(z)
\]

for the linear difference system

\[
\Psi(z + 1) = L(z) \Psi(z)
\]

with \( R(z) = B_1^r(z) \), where \( L(z) = B_2^l(z) L_0 B_1^r(z) \):

\[
\tilde{L}(z) = B_1^r(z + 1) B_2^l(z) L_0 = \tilde{B}_1^l(z) L_0 \tilde{B}_2^r(z) = \tilde{B}_2^l(z) L_0 \tilde{B}_1^r(z).
\]

Types of \( L(z) \) and \( \tilde{L}(z) \)

| \( L(z) \):  | \( z_1 \) | \( z_2 \) | \( \zeta_1 \) | \( \zeta_2 \) | \( \rho_1 \) | \( \rho_2 \) | \( k_1 \) | \( k_2 \) | \( \mu \) | \( \gamma \) | \( \pi \) |
| \( \tilde{L}(z) \): | \( \tilde{z}_1 = z_1 - 1 \) | \( \tilde{z}_2 = z_2 \) | \( \tilde{\zeta}_1 = \zeta_1 - 1 \) | \( \tilde{\zeta}_2 = \zeta_2 \) | \( \rho_1 \) | \( \rho_2 \) | \( k_1 \) | \( k_2 \) | \( \tilde{\mu} \) | \( \tilde{\gamma} \) | \( \tilde{\pi} \) |
The parameters $\tilde{\mu}$, $\tilde{\gamma}$, $\tilde{\pi}$ and $\mu, \gamma, \pi$ are related by the following equations:

\[
\begin{align*}
\tilde{\mu} &= \mu \rho_1 \left( \pi_1 - z_1 \right) - \rho_2 \left( \gamma - \zeta_2 \right) \\
\rho_2 \left( \pi_1 - z_1 \right) - \rho_2 \left( \gamma - \zeta_2 \right) &= \pi_1 \tilde{\pi}_1 \\
\rho_1 \left( \gamma - z_1 \right) \left( \tilde{\gamma} - \tilde{\zeta}_1 \right) &= \rho_2 \left( \gamma - \zeta_2 \right) \left( \tilde{\gamma} - \tilde{\zeta}_2 \right) \\
\end{align*}
\]

This explains the normalization $\pi = \pi_1 \left( \gamma - z_1 \right) \left( \gamma - \zeta_2 \right)$.

\[
\begin{align*}
\pi - 1 &= \rho_2 \left( k_2 - z_1 + \zeta_2 + 1 \right) \\
\end{align*}
\]

(dPV (a))

\[
\begin{align*}
\tilde{\gamma} + \gamma &= z_2 + \zeta_2 + k_1 + \zeta_2 - z_1 \\
\end{align*}
\]

(dPV (b))
Difference Painlevé V

The parameters $\tilde{\mu}, \tilde{\gamma}, \tilde{\pi}$ and $\mu, \gamma, \pi$ are related by the following equations:
Difference Painlevé V

The parameters $\tilde{\mu}, \tilde{\gamma}, \tilde{\pi}$ and $\mu, \gamma, \pi$ are related by the following equations:

$$
\tilde{\mu} = \mu \frac{\rho_1(\pi_1 - z_1) - \rho_2(\gamma - \zeta_2)}{\rho_2(\pi_1 - z_1) - \rho_2(\gamma - \zeta_2)}
$$
Difference Painlevé V

The parameters $\tilde{\mu}, \tilde{\gamma}, \tilde{\pi}$ and $\mu, \gamma, \pi$ are related by the following equations:

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\]

\[
\pi_1 \tilde{\pi}_1 = \frac{\rho_2(\gamma - \zeta_2)(\tilde{\gamma} - \tilde{\zeta}_1)}{\rho_1(\gamma - z_1)(\tilde{\gamma} - \tilde{\zeta}_2)}
\]
Difference Painlevé V

The parameters $\tilde{\mu}, \tilde{\gamma}, \tilde{\pi}$ and $\mu, \gamma, \pi$ are related by the following equations:

$$\tilde{\mu} = \mu \frac{\rho_1(\pi_1 - z_1) - \rho_2(\gamma - \zeta_2)}{\rho_2(\pi_1 - z_1) - \rho_2(\gamma - \zeta_2)}$$

$$\pi_1 \tilde{\pi}_1 = \frac{\rho_2(\gamma - \zeta_2)(\tilde{\gamma} - \tilde{\zeta}_1)}{\rho_1(\gamma - z_1)(\tilde{\gamma} - \tilde{\zeta}_2)}$$

This explains the normalization $\pi = \pi_1 \frac{(\gamma - z_1)}{(\gamma - \zeta_2)}$:

$$\pi \tilde{\pi} = \frac{\rho_2(\tilde{\gamma} - \tilde{\zeta}_1)(\tilde{\gamma} - \tilde{\zeta}_1)}{\rho_1(\tilde{\gamma} - \tilde{\zeta}_2)(\tilde{\gamma} - \tilde{\zeta}_2)}$$

$$\tilde{\gamma} + \gamma = z_2 + \zeta_2 + \frac{k_1 + \zeta_2 - z_1}{\pi - 1} + \frac{\rho_2(k_2 - z_1 + \zeta_2 + 1)}{\rho_1\pi - \rho_2}$$

(dPV (a))

(dPV (b))
Writing $B_s(z) = 1 + \frac{G_s}{z - z_s}$, we see
Writing $B_s(z) = 1 + \frac{G_s}{z-z_s}$, we see

$$L_\infty = G_2 L_0 + L_0 G_1', \quad \tilde{L}_\infty = (G_1' + G_2')L_0$$
Writing $B_s(z) = 1 + \frac{G_s}{z-z_s}$, we see

$$L_\infty = G_2' L_0 + L_0 G_1', \quad \tilde{L}_\infty = (G_1' + G_2') L_0$$

Thus,

$$\tilde{\mu} = (\tilde{L}_\infty)_{21} = \mu + [G_1', L_0]_{21} = \mu + (\rho_1 - \rho_2)(G_1'_{21})$$
Writing $B_s(z) = 1 + \frac{G_s}{z - z_s}$, we see

\[ L_\infty = G_2^l L_0 + L_0 G_1^r, \quad \tilde{L}_\infty = (G_1^r + G_2^l) L_0 \]

Thus,

\[ \tilde{\mu} = (\tilde{L}_\infty)_{21} = \mu + [G_1^r, L_0]_{21} = \mu + (\rho_1 - \rho_2)(G_1^r)_{21} \]

Also, uniqueness of the left divisors gives $B_1^r(z + 1) = B_1^l(z)$ and so $G_1^r = \tilde{G}_1^l$. 
Writing $B_s(z) = 1 + \frac{G_s}{z - z_s}$, we see

$$L_\infty = G'_2 L_0 + L_0 G'_1, \quad \tilde{L}_\infty = (G'_1 + G'_2) L_0$$

Thus,

$$\tilde{\mu} = (\tilde{L}_\infty)_{21} = \mu + [G'_1, L_0]_{21} = \mu + (\rho_1 - \rho_2)(G'_1)_{21}$$

Also, uniqueness of the left divisors gives $B'_1(z + 1) = \tilde{B}'_1(z)$ and so $G'_1 = \tilde{G}'_1$. Since

$$G'_1 = \frac{z_1 - \zeta_1}{b^\dagger_1 c_1} \begin{bmatrix} \frac{\rho_2}{\mu} (k_1 - \varphi_1(\zeta_2, \zeta_2)/\pi_1) \\ 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{\rho_2}{\mu} (k_2 - \varphi_2(z_2, z_2)) \end{bmatrix}$$

$$\tilde{G}'_1 = \frac{z_1 - \zeta_1}{\tilde{d}^\dagger_1 \tilde{a}_1} \begin{bmatrix} \frac{\rho_1}{\tilde{\mu}} (k_1 - \tilde{\varphi}_1(\tilde{z}_2, \tilde{z}_2)) \\ 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{\rho_1}{\tilde{\mu}} (k_2 - \tilde{\varphi}_2(\tilde{z}_2, \tilde{z}_2)/\tilde{\pi}_2) \end{bmatrix},$$

the rest is a simple direct computation.