Chords and Solitons: C & G of the KP Equation

Yuji Kodama
Ohio State University

Joint work with
S. Chakravarty, C-Y. Kao, M. Oikawa and H. Tsuji

International Workshop on Nonlinear and Modern Mathematical Physics
Beijing Xiedao Group, Beijing, China, July 15-21, 2009
The KP Equation

The KP equation:

\[
\frac{\partial}{\partial x} \left( 4 \frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + 6u \frac{\partial u}{\partial x} \right) + 3 \frac{\partial^2 u}{\partial y^2} = 0.
\]

The solution \( u(x, y, t) \) in terms of the \( \tau \)-function:

\[ u(x, y, t) = 2 \frac{\partial^2}{\partial x^2} \ln \tau(x, y, t). \]

The \( \tau \)-function given by the Wronskian determinant:

\[ \tau_N = \text{Wr}(f_1, \ldots, f_N). \]
The KP Equation

The linearly independent set \( \{ f_i(x, y, t) : i = 1, \ldots, N \} \):

\[
\begin{align*}
\frac{\partial f_i}{\partial y} &= \frac{\partial^2 f_i}{\partial x^2}, \\
\frac{\partial f_i}{\partial t} &= -\frac{\partial^3 f_i}{\partial x^3}.
\end{align*}
\]

Heat equation

Finite dimensional solutions:

\[
\begin{align*}
f_i(x, y, t) &= \sum_{j=1}^{M} a_{ij} E_j(x, y, t), & i = 1, \ldots, N < M, \\
E_j(x, y, t) &= \exp(k_j x + k_j^2 y - k_j^3 t), & j = 1, \ldots, M.
\end{align*}
\]

with the ordering \( k_1 < k_2 < \cdots < k_M \).
The KP Equation

Note that we have a Grassmannian picture, i.e. $\text{Gr}(N, M)$:

- $\text{Span}_\mathbb{R}\{E_j : j = 1, \ldots, M\} \cong \mathbb{R}^M$.
- $\text{Span}_\mathbb{R}\{f_i : i = 1, \ldots, N\}$ forms an $N$-dimensional subspace in $\mathbb{R}^M$,

$$(f_1, \ldots, f_N) = (E_1, \ldots, E_M) A^T,$$

where $A$-matrix is defined by

$$A = \begin{pmatrix}
a_{11} & \cdots & \cdots & a_{1M} \\
\vdots & \ddots & \ddots & \vdots \\
a_{N1} & \cdots & \cdots & a_{NM}
\end{pmatrix} \in M_{N \times M}(\mathbb{R}).$$

Each solution can be parametrized by the $A$-matrix.
The KP Equation

- For $\forall H \in GL_N(\mathbb{R})$, $(g_1, \ldots, g_N) = (f_1, \ldots, f_N)H$ gives the same solution, i.e. $\tau(g) = |H|\tau(f)$. This implies that the $\tau$-function is identified as a point on the Grassmannian $Gr(N, M)$, i.e.

$$Gr(N, M) \cong GL_N(\mathbb{R}) \backslash M_{N \times M}(\mathbb{R}),$$

with dim $Gr(N, M) = NM - N^2 = N(M - N)$.

- $H \in GL_N(\mathbb{R})$ gives a row reduction of the $A$-matrix. For example, a generic $A$ can be written in the form $(RREF)$,

$$A = \begin{pmatrix}
1 & \cdots & 0 & * & \cdots & * \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 1 & * & \cdots & *
\end{pmatrix}$$
The KP Equation

- $\text{Gr}(N, M)$ has a Schubert decomposition,

$$\text{Gr}(N, M) = \bigsqcup_{1 \leq j_1 < \ldots < j_N \leq M} W(j_1, \ldots, j_N),$$

where $(j_1, \ldots, j_N)$ is a Schubert symbol representing the pivot indices.

- The set of the Schubert symbols forms a partially ordered set (POSET) with a weak Bruhat order, i.e.

$$(j_1, \ldots, j_N) \iff \exists! \sigma \in S_M/P_N,$$

where $S_M$ is the permutation group of order $M$, and $P_N$ is a parabolic subgroup generated by the simple reflections (transposition) $s_k = (k, k + 1)$ without $s_{M-N}$. 
The KP Equation

Example:

- \( \text{Gr}(1, 2) = W(1) \sqcup W(2) \) where \( W(1) = \{(1, *)\} \) and \( W(2) = \{(0, 1)\} \). In terms of the permutation \( S_2 \), we have

\[
(1 \ 2) \overset{s_1^{-1}}{\longrightarrow} (2 \ 1).
\]

- \( \text{Gr}(1, 3) = W(1) \sqcup W(2) \sqcup W(3) \), and in terms of the permutation \( S_3/P_1 \) with \( P_1 = \langle s_2 \rangle \),

\[
(1 \ 2 \ 3) \overset{s_2^{-1}}{\longrightarrow} (1 \ 3 \ 2) \overset{s_1^{-1}}{\longrightarrow} (2 \ 3 \ 1).
\]

- \( \text{Gr}(2, 4) = W(1, 2) \sqcup W(1, 3) \sqcup \cdots \sqcup W(3, 4) \), and

\[
(1 \ 2 \ 3 \ 4) \overset{s_2^{-1}}{\longrightarrow} (1 \ 3 \ 2 \ 4) \cdots (2 \ 4 \ 1 \ 3) \overset{s_2^{-1}}{\longrightarrow} (3 \ 4 \ 1 \ 2).
\]
Example: For \(N = 1\), the function \(w = \frac{\partial}{\partial x} \ln \tau_1\) satisfies

\[
\frac{\partial w}{\partial y} = 2w \frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2} \quad \text{(The Burgers equation)}.
\]

A shock solution is given by \(\tau_1 = f_1 = E_1 + aE_2, (A = (1, a))\),

\[
w = \frac{1}{2}(k_1 + k_2) + \frac{1}{2}(k_2 - k_1) \tanh \frac{1}{2}(\theta_2 - \theta_1 + \ln a),
\]

where \(\theta_j = k_j x + k_j^2 y - k_j^3 t\). Notice that for \(k_1 < k_2\),

\[
w \rightarrow \begin{cases} 
  k_1 & x \rightarrow -\infty \\
  k_2 & x \rightarrow \infty
\end{cases}
\]
The KP Equation

Example 1: One line-soliton solution with $\tau = E_1 + aE_2$.

3D figure of $u = 2\frac{\partial w}{\partial x}$, and the contour plot. The numbers (i) represent the dominant exponential term in the $\tau$-function. We denote this $[1, 2]$-soliton.
The KP Equation

One-soliton solution is given by a balance between two exponential terms, and in general it is expressed with the parameters \( \{k_i, k_j\} \),

\[
u = A_{[i,j]} \text{sech}^2 \Theta_{[i,j]},
\]

where the amplitude \( A_{[i,j]} \) and the phase \( \Theta_{[i,j]} \) are

\[
A_{[i,j]} = \frac{1}{2} (k_i - k_j)^2, \quad \Theta_{[i,j]} = \frac{1}{2} (\theta_j - \theta_i).
\]

The slope of the soliton in the \( xy \)-plane is given by

\[
\tan \Psi_{[i,j]} = \frac{K^y_{[i,j]}}{K^x_{[i,j]}} = k_i + k_j.
\]
Example 2: Y-type solution with \( \tau_1 = f_1 = E_1 + aE_2 + bE_3 \),

In each region, one of the exponential terms is dominant. Each line-soliton is given by the balance between two exponential terms, \( E_i \) and \( E_j \), denoted as \((i)\) and \((j)\).
The KP Equation

Chord diagrams: One can express each soliton solution in a chord diagram (= permutation). For example, Y-type soliton with the parameters \( \{ k_1, k_2, k_3 \} \) is given by

\[
\pi = \begin{pmatrix}
1 & 2 & 3 \\
3 & 1 & 2
\end{pmatrix}
\]

- The upper part represents \([1, 3]\)-soliton in \( y > 0 \).
- The lower part represents \([1, 2]\)- and \([2, 3]\)-solitons in \( y < 0 \).
The $\tau$-function is given by

$$
\tau(x, y, t) = \det \left( K D(x, y, t) A^T \right),
$$

where $D = \text{diag}(E_1, \ldots, E_M)$ with $E_j = \exp(k_j x + k_j^2 y + k_j^3 t)$, and $K$ is the $N \times M$ matrix given by

$$
K = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
k_1 & k_2 & \cdots & k_M \\
\vdots & \vdots & \ddots & \vdots \\
k_1^{N-1} & k_2^{N-1} & \cdots & k_M^{N-1}
\end{pmatrix}
$$

Recall that $\tilde{A} = HA$ with $H \in GL_N(\mathbb{R})$ gives the same solution, i.e. $A$ can be written in RREF.
Lemma: (Binet-Cauchy) The $\tau$-function can be expanded as

$$
\tau_N = \sum_{1 \leq i_1 < \cdots < i_N \leq M} \xi(i_1, \ldots, i_N) E(i_1, \ldots, i_N),
$$

where $\xi(i_1, \ldots, i_N)$ is the $N \times N$ minor of the $A$-matrix, and $E(i_1, \ldots, i_N)$ is given by

$$
E(i_1, \ldots, i_N) = \left( \prod_{1 \leq j < l \leq N} (k_{i_j} - k_{i_l}) \right) E_{i_1} \cdots E_{i_N} > 0,
$$

Note that if all the $N \times N$ minors of the $A$-matrix are non-negative (i.e. $A$ is totally non-negative), then the $\tau$-function is positive definite. Namely, $u$ is non-singular.
We say that the $A$-matrix is irreducible, if
- in each column, there is at least one nonzero element,
- in each raw, there is at least one more nonzero element in addition to the pivot.

Example: For $N = 2$ and $M = 4$, there are only two types of irreducible $A$-matrices in RREF:

\[
\begin{pmatrix}
1 & 0 & * & * \\
0 & 1 & * & *
\end{pmatrix}, \quad \begin{pmatrix}
1 & * & 0 & * \\
0 & 0 & 1 & *
\end{pmatrix}.
\]

Note that other cases can be expressed by a smaller matrix of $N' \times M'$ with either $N' < N$ or $M' < M$.
Classification Theorem

Example: For $N = 2$, $M = 4$, there are seven types of the $A$-matrices in RREF which are both irreducible and totally non-negative:

$$
\begin{bmatrix}
1 & 0 & -c & -d \\
0 & 1 & a & b
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 0 & -b & -c \\
0 & 1 & a & 0
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 0 & 0 & -c \\
0 & 1 & a & b
\end{bmatrix}
$$

$$
\begin{bmatrix}
1 & 0 & 0 & -b \\
0 & 1 & a & 0
\end{bmatrix}, \quad
\begin{bmatrix}
1 & a & 0 & -c \\
0 & 0 & 1 & b
\end{bmatrix}, \quad
\begin{bmatrix}
1 & a & 0 & 0 \\
0 & 0 & 1 & b
\end{bmatrix}
$$

Here $a, b, c$ and $d$ are positive numbers, and for the first one, either $ad - cb > 0$ or $= 0$. The total number of nonzero minors is at least four, and the maximal number is six.
Theorem 1: Let \( \{e_1, \ldots, e_N\} \) be the pivot indices, and let \( \{g_1, \ldots, g_{M-N}\} \) be the non-pivot indices for an irreducible and totally non-negative \( A \)-matrix. Then the soliton solution associated with the \( A \)-matrix has

(a) \( N \) line-solitons of \([e_n, j_n]\)-type for \( n = 1, \ldots, N \) as \( y \to \infty \),

(b) \( M - N \) line-solitons of \([i_m, g_m]\)-type for \( m = 1, \ldots, M - N \) as \( y \to -\infty \).
Classification Theorem

Theorem 2: The set of those solitons \([e_n, j_n]\) and \([i_m, g_m]\) are expressed by a unique chord diagram which corresponds a derangement of the permutation group \(S_M\), i.e.

\[
\begin{pmatrix}
e_1 & \cdots & e_N & g_1 & \cdots & g_{M-N} \\
\dot{j}_1 & \cdots & \dot{j}_N & i_1 & \cdots & i_{M-N}
\end{pmatrix}
\]

Theorem 3: Conversely, for each chord diagram associated with the derangement, one can construct an \(A\)-matrix, and the corresponding \(\tau\)-function gives the solution of the KP equation having line-solitons expressed by the chord diagram. The entries of the \(A\)-matrix give the scattering data, i.e. the locations of those line-solitons and their interaction properties.
Example: $N = 2, M = 4$. We have seven different types of $(2, 2)$-soliton solution, which are parametrized by the permutation group $S_4$:

The 4-tuples of the diagrams represent the permutation,

$$
\pi = \begin{pmatrix}
1 & 2 & 3 & 4 \\
\pi(1) & \pi(2) & \pi(3) & \pi(4)
\end{pmatrix} = (\pi(1), \ldots, \pi(4)).
$$
Exact Solutions

Example 1: O-type soliton solution.

\[ \pi = (2143) \]

\[ A_{[1,2]} + A_{[3,4]} < u_{\text{center}} < (\sqrt{A_{[1,2]}} + \sqrt{A_{[3,4]}})^2. \]
Exact Solutions

Example 2: P-type soliton solution.

\[ \pi = (4321) \]

\[ \left( \sqrt{A_{[1,4]}} - \sqrt{A_{[2,3]}} \right)^2 < u_{\text{center}} < A_{[1,4]} - A_{[2,3]} \]
Example 3: \((3142)\)-type soliton solution.

Note that \([1, 4]\) gives the maximum amplitude \(A = \frac{1}{2}(k_4 - k_1)^2\).
Exact Solutions

Example 4: T-type soliton solution (i.e. (3412)-type).

Notice that the front half is the same as (3142)-type.
Exact Solutions

Example: $N = 3, M = 6$ (7-dimensional solution).

- $t = -30$
- $t = 0$
- $t = 30$

\[
A = \begin{pmatrix}
1 & 0 & -a & -b & 0 & c \\
0 & 1 & d & e & 0 & -f \\
0 & 0 & 0 & 0 & 1 & g
\end{pmatrix}
\]

$\pi = (451263)$
Numerical Simulations

The initial wave profile:

\[
A_{[i,j]} = \frac{1}{2} (k_i - k_j)^2
\]

\[
\tan \Psi_{[i,j]} = k_i + k_j
\]

\[
A_{[i,j]} = A_0
\]

\[
A_{[m,n]} = 2
\]

\[
\Psi_{[m,n]} = -\Psi_{[i,j]} = \Psi_0
\]
Numerical Simulations

Physical example: The Mach reflection with a rigid wall:

Here $\Psi_0 < \Psi_c$. The right figure shows the equivalent system.
Numerical Simulations

V-shape initial data of O-type:

T=0

T=0.9
Numerical Simulations

(3142)-type:

T=0

T=6

Chords and Solitons: C & G of the KP Equation – p. 5/9
Numerical Simulations

(3142)-type with a cut:
Numerical Simulations

P-type:

T=0

T=9.6623

Chords and Solitons: C & G of the KP Equation – p. 5/9
Summary for V-shape IWs

Chord diagrams for V-shape initial waves: $A=2$
Example 1 (O-type): $A_0 = 1, \Psi_0 \approx \pm 72^\circ$.

(a): $(k_1, k_2, k_3, k_4) = \frac{1}{4}(-3 + 2\sqrt{2}, -3 - 2\sqrt{2}, 2, 10)$.

(b): $(k_1, k_2, k_3, k_4) = \frac{1}{4}(-10, -2, 3 - 2\sqrt{2}, 3 + 2\sqrt{2})$. 
Summary for V-shape IWs

Example 2 ((3142) and dual): $A_0 = 3$, $\Psi_0 = \pm 45^\circ$.

(a): $(k_1, k_2, k_3, k_4) = \frac{1}{2}(-\sqrt{6} + 1, -1, \sqrt{6} - 1, 3)$.

(b): $(k_1, k_2, k_3, k_4) = \frac{1}{2}(-3, -(-\sqrt{6} - 1), 1, \sqrt{6} + 1)$. 
Summary for V-shape IWs

Example 2 Exact: \( A_0 = 3, \ \Psi_0 = \pm 45^\circ. \)

(a): \((k_1, k_2, k_3, k_4) = \frac{1}{2}(-\sqrt{6} + 1, \ -1, \ \sqrt{6} - 1, \ 3).\)

(b): \((k_1, k_2, k_3, k_4) = \frac{1}{2}(-3, \ - (\sqrt{6} - 1), \ 1, \ \sqrt{6} + 1).\)
Summary for V-shape IWs

Example 3 ((1,3)- and dual): $\Psi_0 = 0^\circ$.

(a): $(k_1, k_2, k_3, k_4) = \frac{1}{2}(-2, -\sqrt{2}, \sqrt{2}, 2) \Rightarrow A_0 = 1.$

(b): $(k_1, k_2, k_3, k_4) = (-\sqrt{3}, -1, 1, \sqrt{3}) \Rightarrow A_0 = 6.$
Summary for V-shape IWs

Example 3 Exact ((1,3)- and dual): $\Psi_0 = 0^\circ$.

(a): $(k_1, k_2, k_3, k_4) = \frac{1}{2}(-2, -\sqrt{2}, \sqrt{2}, 2) \Rightarrow A_0 = 1.$

(b): $(k_1, k_2, k_3, k_4) = (-\sqrt{3}, -1, 1, \sqrt{3}) \Rightarrow A_0 = 6.$
Summary for X-shape IWs

Chord diagrams for X-shape initial waves:

\[ \tan \Psi_0 \]

\[ 0 \quad 1 \quad \sqrt{2A_0} \]

\[ k_1 = k_2 \]
\[ k_2 = k_3 \]
\[ k_3 = k_4 \]
Summary for X-shape IWs

T-type solution: $A_0 = 8$, $\Psi_0 = 45^\circ$ ($\Psi_c \approx 63.4^\circ$).

(a): Simulation for sum of two line-solitons with $A_0 = 2$.
(b): Exact solution with $(k_1, k_2, k_3, k_4) = \frac{1}{2}(-3, -1, 1, 3)$. 
Example of 3 half-waves

(415362)-type solution (one of (3,3)-type solitons):

(a) Initial wave with $[1, 4], [3, 5]$ for $y > 0$ and $[2, 6]$ for $y < 0$.
(b) Exact solution with $(k_1, \ldots, k_6) = \frac{1}{2}(-4, -3, -1, 0, 1, 4)$. 

References


References

- S. Chakravarty and Y. K., *Combinatorics and geometry of soliton solutions of the KP equation*, coming soon.


