Localized objects in the Faddeev-Skyrme model

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Introduction

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In this talk: Review of work on *Hopfions in Faddeev's model*, done in collaboration with P. Salo and J. Jäykkä.

Topology in $\mathbb{R}^3$: Hopfions

- Carrier field: 3D unit vector field $\mathbf{n}$ in $\mathbb{R}^3$, locally smooth.
- 3D-unit vectors can be represented by points on the surface of the sphere S^2.
- Asymptotically trivial: $\mathbf{n}(\mathbf{r}) \to \mathbf{n}_\infty$, when $|\mathbf{r}| \to \infty$
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Therefore

$$\mathbf{n} : S^3 \to S^2.$$

Such functions are characterized by the Hopf charge, i.e., by the homotopy class $\pi_3(S^2) = \mathbb{Z}$.

A concrete Hopfion

Example of vortex ring with Hopf charge 1:

$$\mathbf{n} = \left(\frac{4(2xz - y(r^2-1))}{(1+r^2)^2}, \frac{4(2yz + x(r^2-1))}{(1+r^2)^2}, 1 - \frac{8(r^2-z^2)}{(1+r^2)^2} \right).$$

where $r^2 = x^2 + y^2 + z^2$.

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Note that

- $\mathbf{n} = (0,0,1)$ at infinity (in any direction).
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Computing the Hopf charge:

Given $\mathbf{n} : \mathbb{R}^3 \to S^2$ define $F_{ij} = \epsilon_{abc} n^a \partial_i n^b \partial_j n^c$.

Given F_{ij} construct A_j so that $F_{ij} = \partial_i A_j - \partial_j A_i$, then

$$Q = \frac{1}{16\pi^2} \int \epsilon^{ijk} A_i F_{jk} \, d^3x.$$

Possible physical realization

VOLUME 75, NUMBER 18 PHYSICAL REVIEW LETTERS 30 OCTOBER 1995

Phase Diagram of Vortices in Superfluid ^{3}He-A

Ü. Parts, J. M. Karimäki, J. H. Koivuniemi, M. Krusius, V. M. H. Ruutu, E. V. Thuneberg, and G. E. Volovik*
Low Temperature Laboratory, Helsinki University of Technology, 02150 Espoo, Finland
(Received 6 June 1995)

Four alternative but topologically different structures of vorticity exist in rotating ^{3}He-A. As a function of magnetic field (H) and rotation velocity (Ω), we identify with NMR the type of vortex which is nucleated during cooling from the normal to the superfluid phase. The measurements are compared to the calculated equilibrium phase diagram of vortices in the H-Ω plane at temperatures $T \lesssim T_c$. Slow transitions are found to reproduce the calculated equilibrium state.

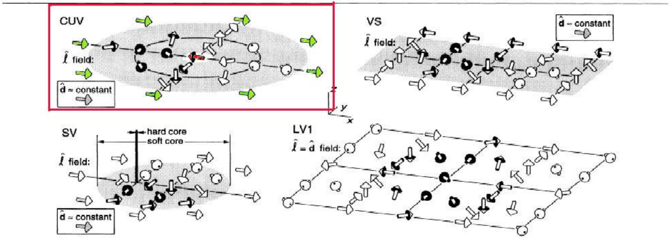

FIG. 1. Four vortex structures of rotating ^{3}He-A: continuous unlocked vortex (CUV), vortex sheet (VS), singular vortex (SV), and locked vortex (LV1). The arrows denote the orientation of $\hat{\mathbf{l}}$ in the x-y plane. The rotation axis $\mathbf{\Omega}$ is parallel to z. The shaded area marks the "soft core" of the unlocked vortices (CUV, VS, and SV) where $\hat{\mathbf{d}}$ and $\hat{\mathbf{l}}$ deviate from each other. In the LV, $\hat{\mathbf{d}}$ and $\hat{\mathbf{l}}$ follow each other everywhere. The $\hat{\mathbf{l}}$ field is continuous with the exception of the SV, where $\hat{\mathbf{l}}$ is not defined in the "hard core." In all cases the vorticity has periodicity in the x-y plane, but the complete periodic unit is depicted for the LV1 only. For the VS one full periodic unit in the x direction is shown; by stacking these units one after another, its soft core becomes a continuous sheet. The CUV is equivalent to one period of the VS, when it is bent and closed to a cylinder. The length scales are 0.01 and 10 μm for the hard and soft cores, respectively, and 200 μm (at Ω = 1 rads/s) for the unit cell.

Faddeev's model

In 1975 Faddeev proposed the Lagrangian (energy)

$$E = \int \left[(\partial_i \mathbf{n})^2 + g\, F_{ij}^2 \right] d^3x, \quad F_{ij} := \mathbf{n} \cdot \partial_\mathbf{i} \mathbf{n} \times \partial_\mathbf{j} \mathbf{n}.$$

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Vakulenko and Kapitanskii (1979): a lower limit for the energy,

$$E \geq c\,|Q|^{\frac{3}{4}},$$

where c is some constant, and Q the Hopf charge.

Similar upper bound has been derived recently by Lin and Yang.

Numerical studies of Faddeev's model

What is the minimum energy state for a given Hopf charge?

Studied in 1997-2004 by Gladikowski and Hellmund, Faddeev and Niemi, Battye and Sutcliffe, and Hietarinta and Salo.

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Full 3D minimization using dissipative dynamics:
$\mathbf{n}_{new} = \mathbf{n}_{old} - \delta \nabla_{\mathbf{n}(\mathbf{r})} L$.
No assumptions on symmetry, on the contrary:
Linked unknots of various charges.

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J. Hietarinta and P. Salo: *Faddeev-Hopf knots: dynamics of linked unknots*, Phys. Lett. B 451, 60-67 (1999).

J. Hietarinta and P. Salo: *Ground state in the Faddeev-Skyrme model*, Phys. Rev. D 62, 081701(R) (2000).

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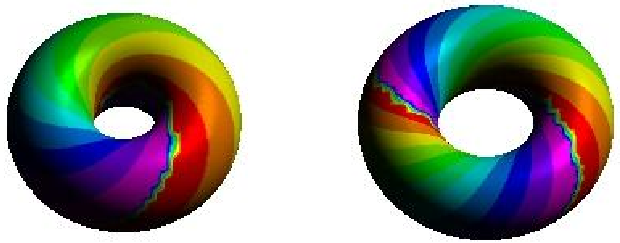
Longitudes are represented by colors on the equilatitude surface (under a global gauge rotation only colors change): we paint the surfaces using longitudes.

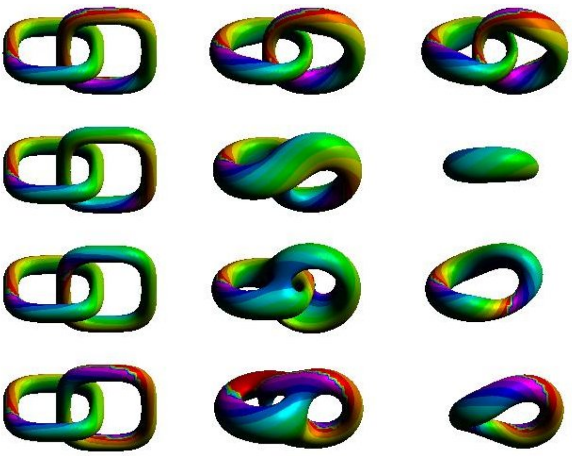
Isosurface $n_3 = 0$ (equator) for $|Q| = 1, 2$



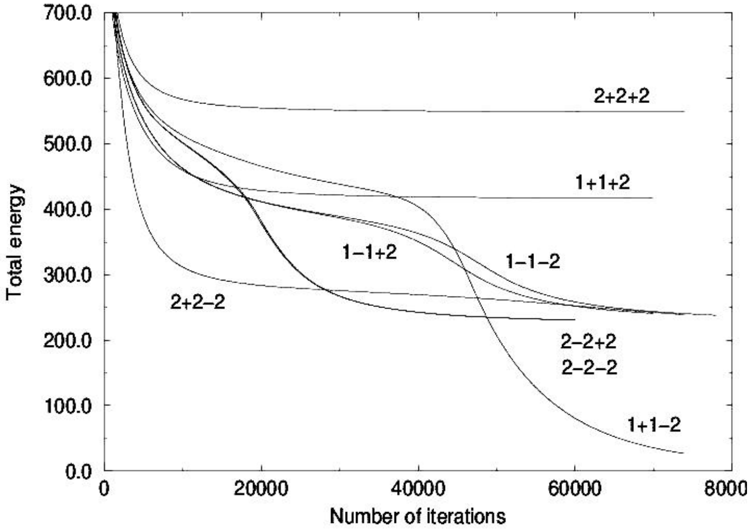
Color order and handedness of twist determine Hopf charge.
Inside the torus is the core, where $n_3 = -1$.

Evolution of linked unknots

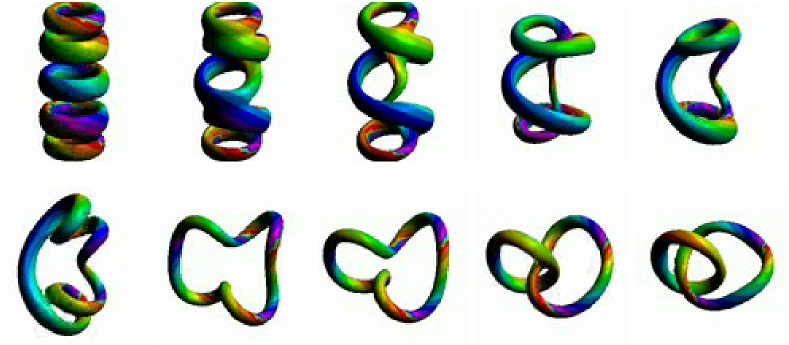
Total charge = sum of individual charges + linking number



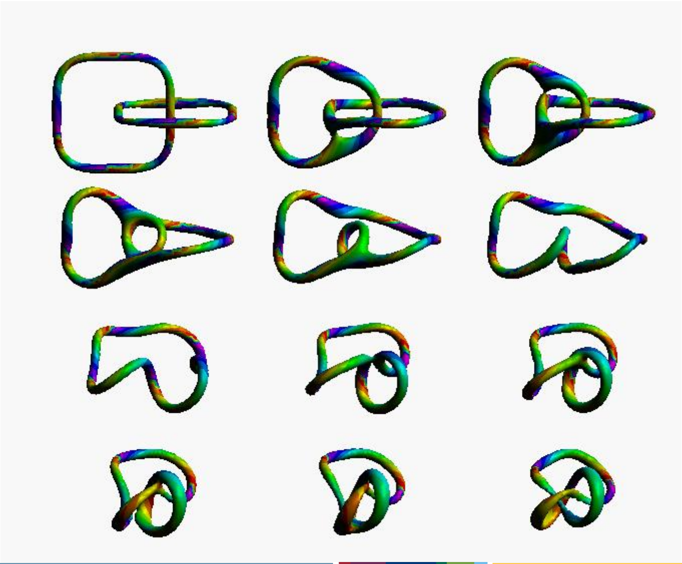
Energy evolution in minimization



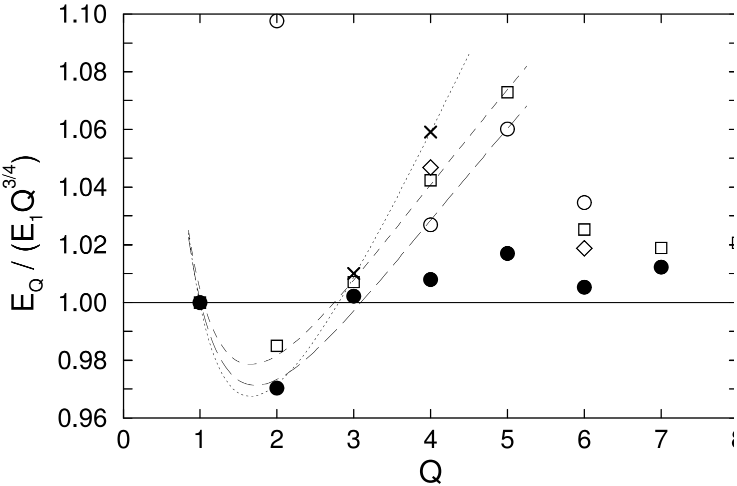
Evolution $(1,5) \to 1+2+2$



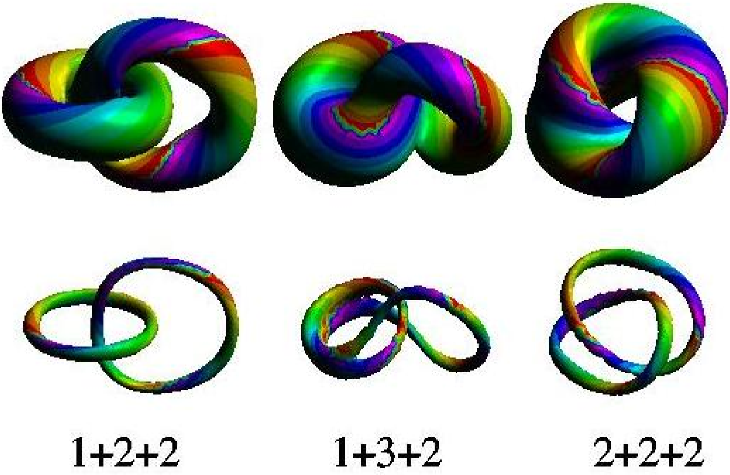
Evolution $5 + 4 - 2 \rightarrow$ trefoil



Vakulenko bound



Different and improved final states



Framed links and ribbon knots

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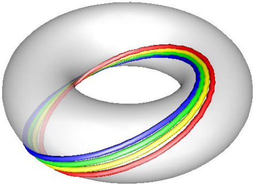
One way to describe framed links is to use directed ribbons, which are preimages of line segments.



Computing the charge

For a ribbon define:

- **twist** = linking number of the ribbon core with a ribbon boundary, locally.
- **writhe** = signed crossover number of the ribbon core with itself.
- **linking number** = $\frac{1}{2}$(sum of signed crossings)

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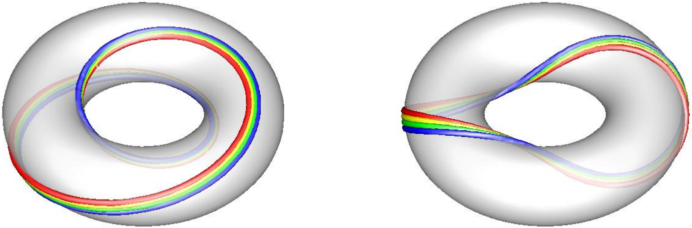
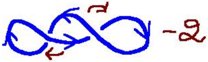
The Hopf charge can be determined either by
twist + writhe
or
linking number of the two ribbon boundaries,
or
linking number of the preimages of any pair of regular points.

Ribbon view, $Q = -2$

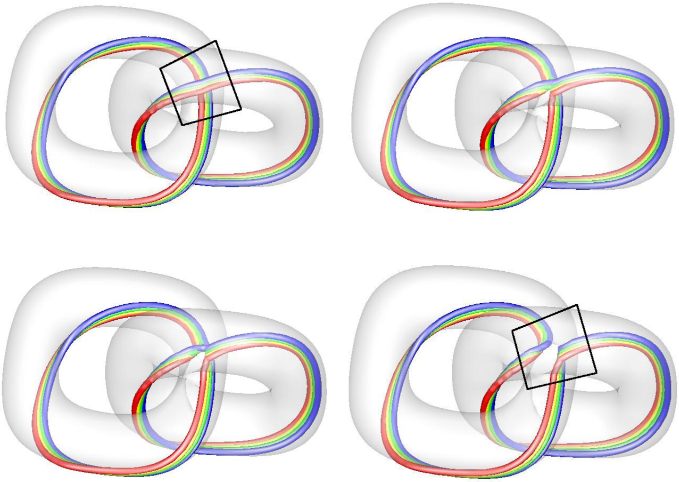


Two ways to get charge -2: twice around small vs. large circle.
The first one has twist $= -1$, writhe $= -1$,
the second twist $= -2$, writhe $= 0$.

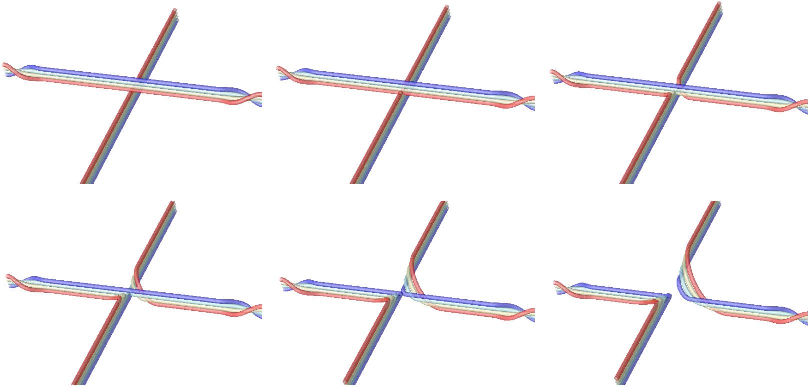
Both have boundary linking number $= -2$.



Example of ribbon deformation during minimization

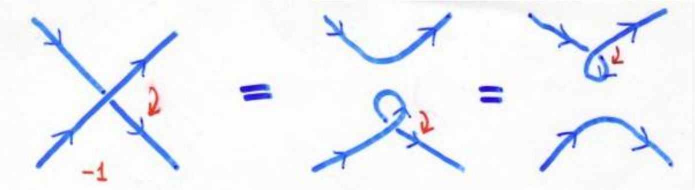


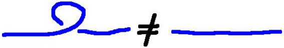
Close-up of the deformation process



Diagrammatic rule for deformations

Knot deformations correspond to ribbon deformations, e.g., crossing and breaking, but the Hopf charge will be conserved.



Note that when considering equivalence of ribbon diagrams type I Reidemeister move is not valid: 

What is different with vortices

- Vortices do not allow 1-point compactification of $\mathbb{R}^3 \to S^3$.
- Instead we have $R^2 \times T$ if periodic in z-direction or T^3 if periodic in all directions.

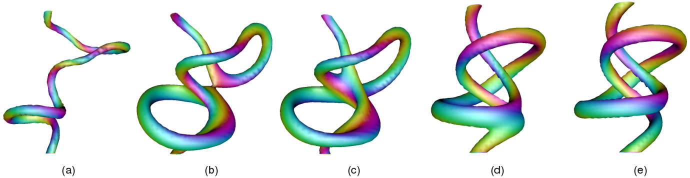
Topological conserved quantities studied by Pontrjagin in 1941, They are mainly related to vortex punctures of the periodic box.

Single twisted vortex

J. Hietarinta, J. Jäykkä and P. Salo: *Dynamics of vortices and knots in Faddeev's model*, JHEP Proc.: PrHEP unesp2002/17
`http://pos.sissa.it//archive/conferences/008/017/un`

J. Hietarinta, J. Jäykkä and P. Salo: *Relaxation of twisted vortices in the Faddeev-Skyrme model*, Phys. Lett. A 321, 324-329 (2004).

Knotting as usual if tightly wound:



Vortex bunches and unwinding

If the vortices are close enough there can be bunching or in the fully periodic case, Hopfion unwinding.

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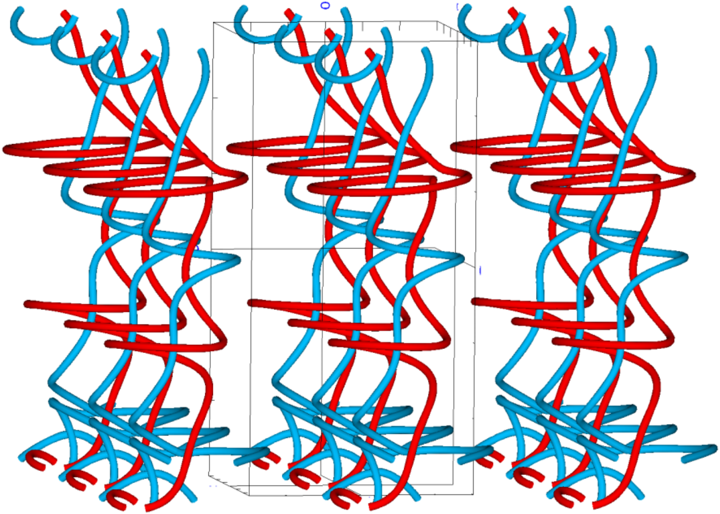
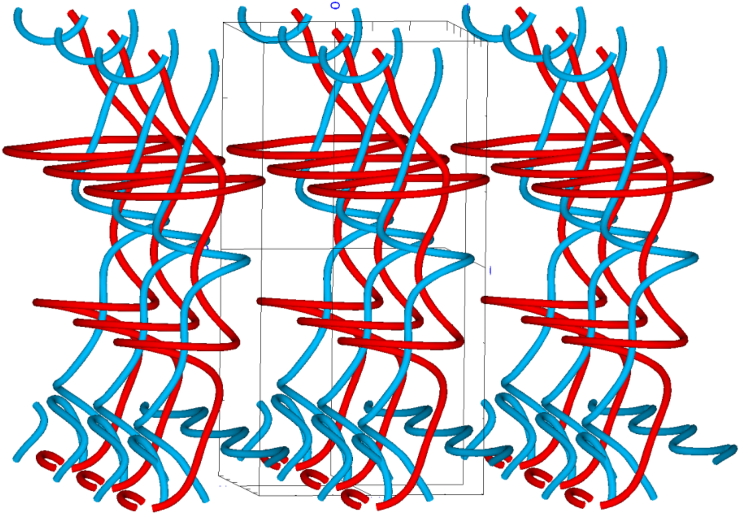
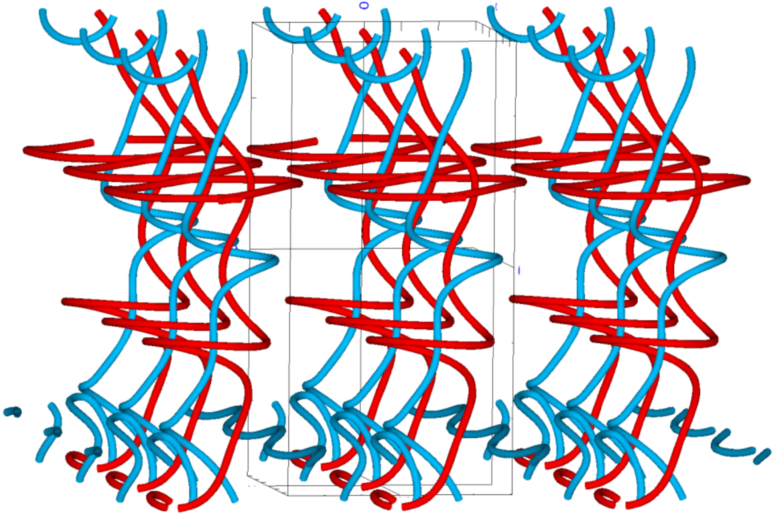
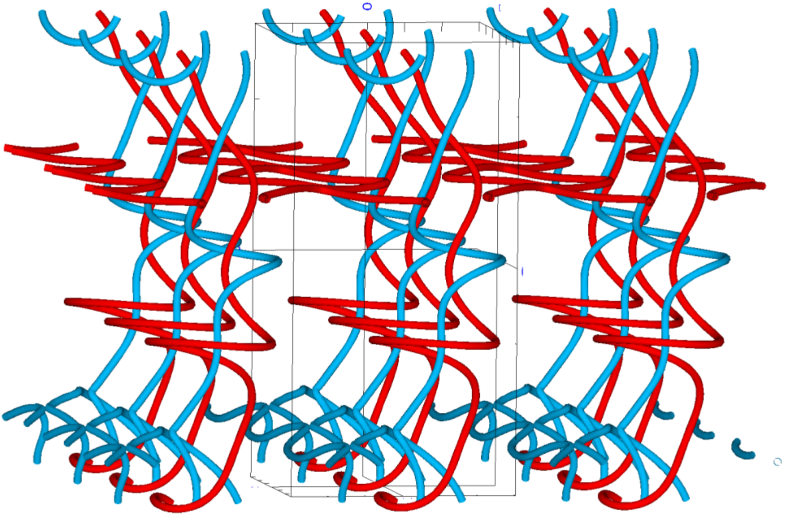
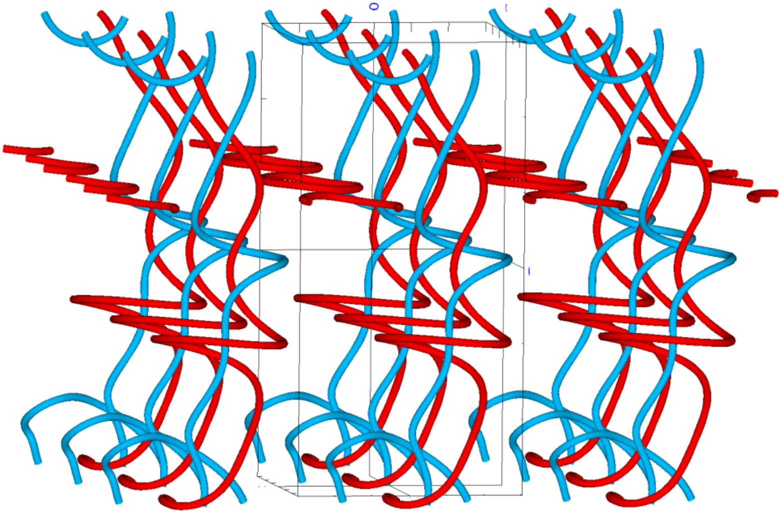
First a 3×3 section of the fully periodic case.

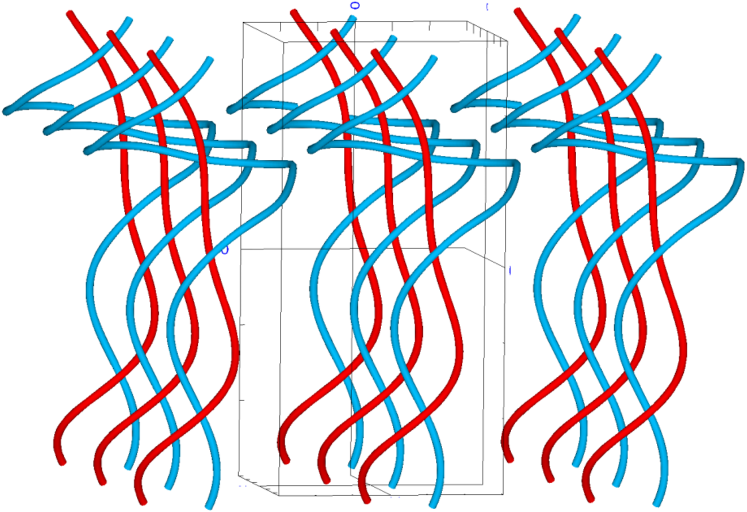


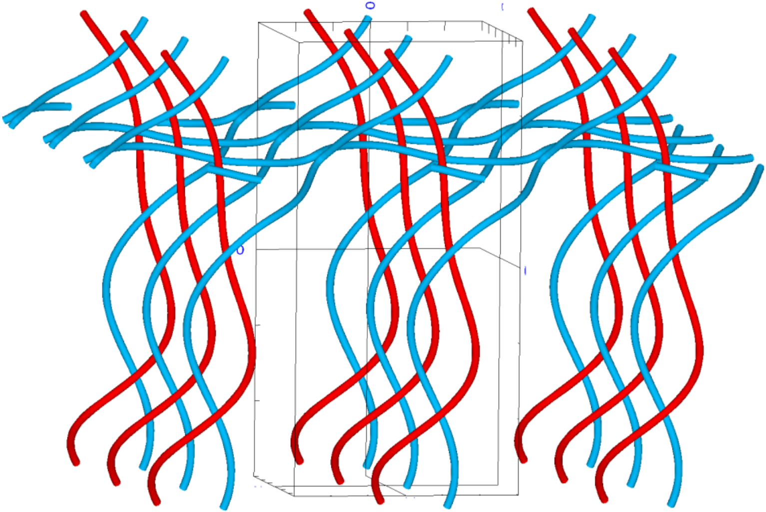


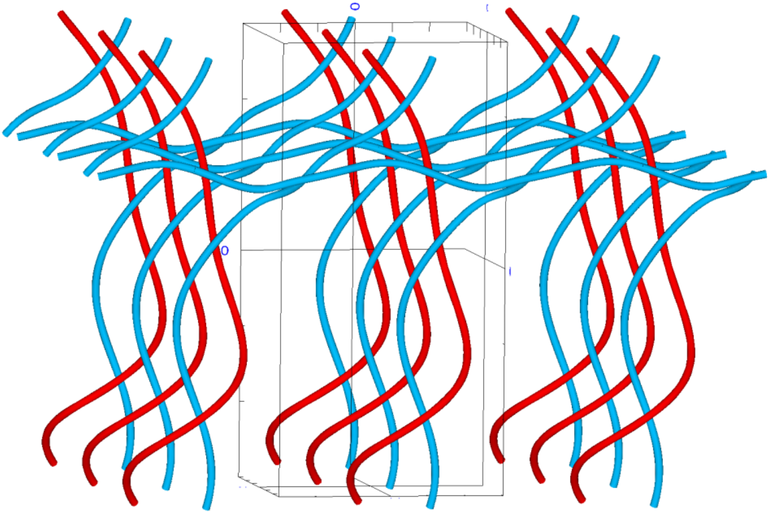








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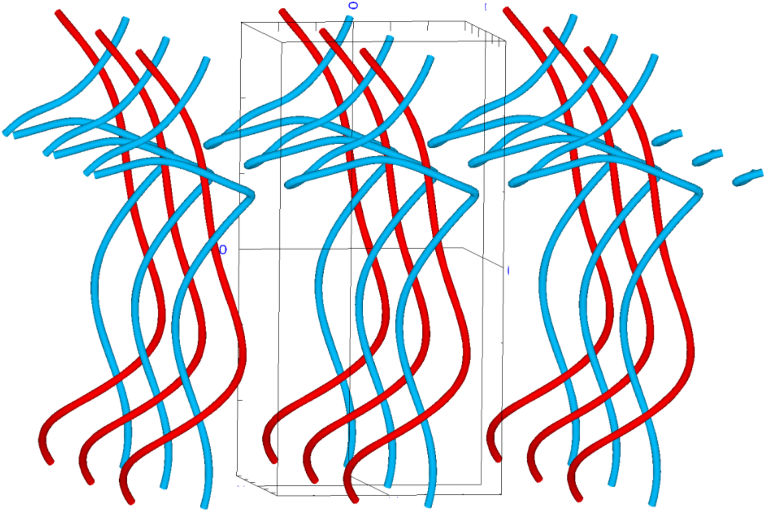
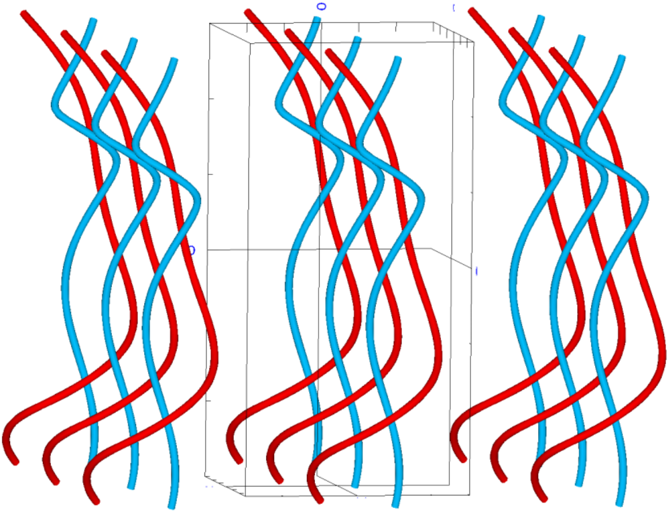
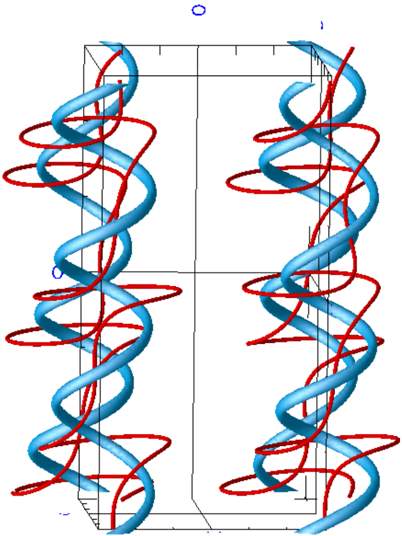




4×4



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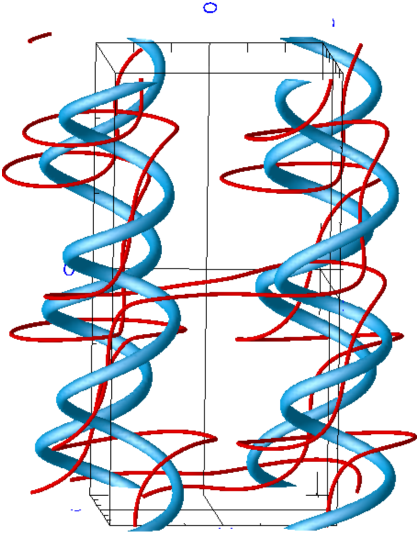

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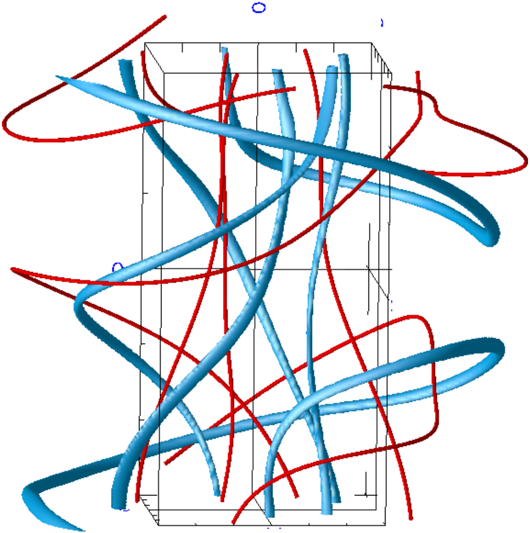

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See also:
`http://users.utu.fi/hietarin/knots/index.html`