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An Eight Component Integrable Hamiltonian Hierarchy from a Reduced Seventh-Order Matrix Spectral Problem

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Abstract

We present an eight component integrable Hamiltonian hierarchy, based on a reduced seventh order matrix spectral problem, with the aim of aiding the study and classification of multicomponent integrable models and their underlying mathematical structures. The zero-curvature formulation is the tool to construct a recursion operator from the spatial matrix problem. The second and third set of integrable equations present integrable nonlinear Schrödinger and modified Korteweg-de Vries type equations, respectively. The trace identity is used to construct Hamiltonian structures, and the first three Hamiltonian functionals so generated are computed.

Keywords

Matrix Spectral Problem, Zero Curvature Equation, Lax Pair, Integrable Hierarchy, NLS Equations, mKdV Equations, Hamiltonian Structure, Lie Bracke

1. Introduction

Multi-component integrable models are one class of nonlinear integrable PDE models which possess diversity and difficulty, with many physical applications. We aim to exploit an integrable Hamiltonian hierarchy with eight components, motivated by a recent reduced matrix spectral problem [1], which provide novel integrable nonlinear Schrodinger models and modified Korteweg-De Vries models.

Consider a linear and non-linear partial differential equation (PDE):

$$u_t = \xi\left(u, u_x, \cdots, u_{N_x}\right) \tag{1.1}$$

where $u(x,t)$ is a real or complex valued function. A PDE is called Lax-integrable if there exists a Lax pair, *i.e.*, a pair of two linear ODEs,

$$\psi_x = U\psi, \quad \psi_t = V\psi.$$

where U and V are $n \times n$ matrices depending on dependent and independent variables, such that the compatibility of the mixed derivatives $\partial_t\psi_x = \partial_x\psi_t$ holds. When a PDE possesses a Lax pair, it is quite common that it possesses an infinite number of conserved quantities and an infinite number of symmetries.

Based on recursion operators associated with Lax pairs, integrable hierarchies could be presented, which consist of infinitely many Lax-integrable PDEs. In an integrable hierarchy, every element is characterized by an infinite number of conserved functionals that mutually commute when subjected to the corresponding Poisson bracket [2] and infinitely many symmetries commuting under the Lie bracket of vector fields.

In this work, Lax pairs associated with a reduced [1] seventh order spectral problem, are used to generate an eight component integrable hierarchy. Within the related matrix loop algebra, a pseudoregular element is selected to construct a spectral matrix. The characteristics of this pseudoregular element guarantee the existence of a Laurent series solution $W = \sum_{s\geq 0}\lambda^{-s}W^{[s]}$ to the stationary zero curvature equation $W_x = i[U,W]$ and will lead to the formation of an integrable hierarchy via zero curvature equations:

$$U_{t_g} - V_x^{[g]} + i\left[U, V^{[g]}\right] = 0, \quad g \geq 0. \tag{1.2}$$

The zero curvature equations are the compatibility conditions of the spatial and temporal matrix spectral problems:

$$-i\psi_x = U\psi, -i\psi_{t_g} = V^{[g]}\psi, \quad g \geq 0, \tag{1.3}$$

with ψ being an eigenfunction.

The paper is structured as follows. Section 2 computes an eight component integrable hierarchy from a reduced seventh order matrix spectral problem. Section 3 presents Hamiltonian structures for the eight component integrable hierarchy. These structures are constructed using the trace identity [3] since the corresponding Lie algebra is semisimple. In particular, we compute the first three Hamiltonian functionals. The Hamiltonian structures establish a connection between symmetries and conserved quantities. The last section is devoted to a conclusion.

2. An Eight Component Integrable Hierarchy

Consider the following potential vector, with λ being the spectral parameter:

$$u = u(x,t) = \left(p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4\right)^{\mathrm{T}}. \tag{2.1}$$

We first present the spatial matrix spectral problem as follows:

$$-i\psi_x = U\psi = U(u,\lambda), U = \begin{bmatrix} \lambda & p_1 & p_2 & p_3 & p_4 & p_4 & 0 \\ q_1 & 0 & 0 & 0 & 0 & 0 & p_1 \\ q_2 & 0 & 0 & 0 & 0 & 0 & p_2 \\ q_3 & 0 & 0 & 0 & 0 & 0 & p_3 \\ q_4 & 0 & 0 & 0 & 0 & 0 & p_4 \\ q_4 & 0 & 0 & 0 & 0 & 0 & p_4 \\ 0 & q_1 & q_2 & q_3 & q_4 & q_4 & -\lambda \end{bmatrix}, \tag{2.2}$$

We initially address the stationary zero curvature equation

$$W_x = i[U,W] \tag{2.3}$$

by seeking a Laurent series solution:

$$W = \begin{bmatrix} a & b_1 & b_2 & b_3 & b_4 & b_4 & 0 \\ c_1 & 0 & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,4} & b_1 \\ c_2 & -d_{1,2} & 0 & d_{2,3} & d_{2,4} & d_{2,4} & b_2 \\ c_3 & -d_{1,3} & -d_{2,3} & 0 & d_{3,4} & d_{3,4} & b_3 \\ c_4 & -d_{1,4} & -d_{2,4} & -d_{3,4} & 0 & 0 & b_4 \\ c_4 & -d_{1,4} & -d_{2,4} & -d_{3,4} & 0 & 0 & b_4 \\ 0 & c_1 & c_2 & c_3 & c_4 & c_4 & -a \end{bmatrix} = \sum_{s\geq 0} \lambda^{-s} W^{[s]} \tag{2.4}$$

$$W^{[s]} = \begin{bmatrix} a^{[s]} & b_1^{[s]} & b_2^{[s]} & b_3^{[s]} & b_4^{[s]} & b_4^{[s]} & 0 \\ c_1^{[s]} & 0 & d_{1,2}^{[s]} & d_{1,3}^{[s]} & d_{1,4}^{[s]} & d_{1,4}^{[s]} & b_1^{[s]} \\ c_2^{[s]} & -d_{1,2}^{[s]} & 0 & d_{2,3}^{[s]} & d_{2,4}^{[s]} & d_{2,4}^{[s]} & b_2^{[s]} \\ c_3^{[s]} & -d_{1,3}^{[s]} & -d_{2,3}^{[s]} & 0 & d_{3,4}^{[s]} & d_{3,4}^{[s]} & b_3^{[s]} \\ c_4^{[s]} & -d_{1,4}^{[s]} & -d_{2,4}^{[s]} & -d_{3,4}^{[s]} & 0 & 0 & b_4^{[s]} \\ c_4^{[s]} & -d_{1,4}^{[s]} & -d_{2,4}^{[s]} & -d_{3,4}^{[s]} & 0 & 0 & b_4^{[s]} \\ 0 & c_1^{[s]} & c_2^{[s]} & c_3^{[s]} & c_4^{[s]} & c_4^{[s]} & -a^{[s]} \end{bmatrix}. \tag{2.5}$$

Since

$$[U,W] = \left([U,W]_{jk}\right)_{7\times 7}, \tag{2.6}$$

where

$$[U,W]_{11} = -b_1q_1 - b_2q_2 - b_3q_3 - 2b_4q_4 + c_1p_1 + c_2p_2 + c_3p_3 + 2c_4p_4,$$

$$[U,W]_{12} = -ap_1 + \lambda b_1 - d_{1,2}p_2 - d_{1,3}p_3 - 2d_{1,4}p_4,$$

$$[U,W]_{13} = -ap_2 + \lambda b_2 + d_{1,2}p_1 - d_{2,3}p_3 - 2d_{2,4}p_4,$$

$$[U,W]_{14} = -ap_3 + \lambda b_3 + d_{1,3}p_1 + d_{2,3}p_2 - 2d_{3,4}p_4,$$

$$[U,W]_{15} = [U,W]_{16} - ap_4 + \lambda b_4 + d_{1,4}p_1 + d_{2,4}p_2 + d_{3,4}p_3, [U,W]_{17} = 0,$$

$$[U,W]_{21} = aq_1 - \lambda c_1 - d_{1,2}q_2 - d_{1,3}q_3 - 2d_{1,4}q_4,$$

$$[U,W]_{22} = 0, [U,W]_{23} = -b_1q_2 + b_2q_1c_1p_2 + c_2p_1,$$

$$
\begin{aligned}
&[U,W]_{24} = -b_1q_3 + b_3q_1 - c_1p_3 + c_3p_1,\\
&[U,W]_{25} = [U,W]_{26} = -b_1q_4 + b_4q_1 - c_1p_4 + c_4p_1,\\
&[U,W]_{27} = -ap_1 + \lambda b_1 - d_{1,2}p_2 - d_{1,3}p_3 - 2d_{1,4}p_4,\\
&[U,W]_{31} = aq_2 - \lambda c_2 + d_{1,2}q_1 - d_{2,3}q_3 - 2d_{2,4}q_4,\\
&[U,W]_{32} = b_1q_2 - b_2q_1 + c_1p_2 - c_2p_1,\\
&[U,W]_{33} = 0, [U,W]_{34} = -b_2q_3 + b_3q_2 - c_2p_3 + c_3p_2,\\
&[U,W]_{35} = -[U,W]_{36} = -b_2q_4 + b_4q_2 - c_2p_4 + c_4p_2,\\
&[U,W]_{37} = -ap_2 + \lambda b_2 + d_{1,2}p_1 - d_{2,3}p_3 - 2d_{2,4}p_4,\\
&[U,W]_{41} = aq_3 - \lambda c_3 + d_{1,3}q_1 + d_{2,3}q_2 - 2d_{3,4}q_4,\\
&[U,W]_{42} = b_1q_3 - b_3q_1 + c_1p_3 - c_3p_1,\\
&[U,W]_{43} = b_2q_3 - b_3q_2 + c_2p_3 - c_3p_2, [U,W]_{44} = -0,\\
&[U,W]_{45} = [U,W]_{46} = -b_3q_4 + b_4q_3 - c_3p_4 + c_4p_3,\\
&[U,W]_{47} = -ap_3 + \lambda b_3 + d_{1,3}p_1 + d_{2,3}p_2 - 2d_{3,4}p_4,\\
&[U,W]_{51} = [U,W]_{61} = aq_4 - \lambda c_4 + d_{1,4}q_1 + d_{2,4}q_2 + d_{3,4}q_3,\\
&[U,W]_{52} = [U,W]_{62} = b_1q_4 - b_4q_1 + c_1p_4 - c_4p_1,\\
&[U,W]_{53} = [U,W]_{63} = b_2q_4 - b_4q_2 + c_2p_4 - c_4p_2,\\
&[U,W]_{54} = -[U,W]_{64} = b_3q_4 - b_4q_3 + c_3p_4 - c_4p_3,\\
&[U,W]_{55} = [U,W]_{56} = 0,\\
&[U,W]_{57} = [U,W]_{67} = -ap_4 + \lambda b_4 + d_{1,4}p_1 + d_{2,4}p_2 + d_{3,4}p_3,\\
&[U,W]_{71} = 0, [U,W]_{72} = aq_1 - \lambda c_1 - d_{1,2}q_2 - d_{1,3}q_3 - 2d_{1,4}q_4,\\
&[U,W]_{73} = aq_2 - \lambda c_2 + d_{1,2}q_1 - d_{2,3}q_3 - 2d_{2,4}q_4,\\
&[U,W]_{74} = aq_3 - \lambda c_3 + d_{1,3}q_1 + d_{2,3}q_2 - 2d_{3,4}q_4,\\
&[U,W]_{75} = [U,W]_{76} = aq_4 - \lambda c_4 + d_{1,4}q_1 + d_{2,4}q_2 + d_{3,4}q_3,\\
&[U,W]_{77} = -ap_4 + \lambda b_4 + d_{1,4}p_1 + d_{2,4}p_2 + d_{3,4}p_3,
\end{aligned}
\tag{2.7}
$$

The stationary zero curvature equation yields the ensuing recursion relations and initial conditions:

$$
\begin{cases}
a_x^{[0]} = 0, b_j^{[0]} = 0, c_j^{[0]} = 0, 1 \le j \le 4,\\
d_{m,n,x}^{[0]} = 0; m,n = \{(1,2)(1,3)(1,4)(2,3)(2,4)(3,4)\}
\end{cases}
\tag{2.8}
$$

$$\begin{cases} b_1^{[s+1]} = -ib_{1,x}^{[s]} + a^{[s]}p_1 + d_{1,2}^{[s]}p_2 + d_{1,3}^{[s]}p_3 + 2d_{1,4}^{[s]}p_4 \\ b_2^{[s+1]} = -ib_{2,x}^{[s]} + a^{[s]}p_2 - d_{1,2}^{[s]}p_1 + d_{2,3}^{[s]}p_3 + 2d_{2,4}^{[s]}p_4 \\ b_3^{[s+1]} = -ib_{3,x}^{[s]} + a^{[s]}p_3 - d_{1,3}^{[s]}p_1 - d_{2,3}^{[s]}p_2 + 2d_{3,4}^{[s]}p_4 \\ b_4^{[s+1]} = -ib_{4,x}^{[s]} + a^{[s]}p_4 - d_{1,4}^{[s]}p_1 - d_{2,4}^{[s]}p_2 - d_{3,4}^{[s]}p_3 \end{cases} \tag{2.9}$$

$$\begin{cases} c_1^{[s+1]} = ic_{l,x}^{[s]} + a^{[s]}q_1 - d_{1,2}^{[s]}q_2 - d_{1,3}^{[s]}q_3 - 2d_{1,4}^{[s]}q_4 \\ c_2^{[s+1]} = ic_{2,x}^{[s]} + a^{[s]}q_2 + d_{1,2}^{[s]}q_1 - d_{2,3}^{[s]}q_3 - 2d_{2,4}^{[s]}q_4 \\ c_3^{[s+1]} = ic_{3,x}^{[3]} + a^{[s]}q_3 + d_{1,3}^{[s]}q_1 + d_{2,3}^{[s]}q_2 - 2d_{3,4}^{[s]}q_4 \\ c_4^{[s+1]} = ic_{4,x}^{[s]} + a^{[s]}q_4 + d_{1,4}^{[s]}q_1 + d_{2,4}^{[s]}q_2 + d_{3,4}^{[s]}q_3 \end{cases} \tag{2.10}$$

$$\begin{cases} d_{m,n,x}^{[s+1]} = i\left[-b_m^{[s+1]}q_n + b_n^{[s+1]}q_m - c_m^{[s+1]}p_n + c_n^{[s+1]}p_m\right] \\ a_x^{[s+1]} = i\left[-b_1^{[s+1]}q_1 - b_2^{[s+1]}q_2 - b_3^{[s+1]}q_3 - 2b_4^{[s+1]}q_4 - c_1^{[s+1]}p_1 - c_2^{[s+1]}p_2 - c_3^{[s+1]}p_3 - 2c_4^{[s+1]}p_4\right], \end{cases} \tag{2.11}$$

with $s \geq 0$.

Subsequently, taking the following initial values,

$$a^{[0]} = 1, d_{m,n}^{[0]} = 0, \tag{2.12}$$

and setting the constants of integration to be zero,

$$a^{[s]}\Big|_{u=0} = 0, d_{m,n}^{[s]}\Big|_{u=0} = 0, s \geq 1,$$

we determine a unique solution W. Consequently, we can compute the first four sets of $a^{[s]}, b_j^{[s]}, c_j^{[s]}$ and $d_{m,n}^{[s]}$ as follows:

$$\begin{cases} b_j^{[1]} = p_j, \ c_j^{[1]} = q_j \\ d_{m,n}^{[1]} = 0, \ a^{[1]} = 0 \end{cases}$$

$$\begin{cases} b_j^{[2]} = -ip_{j,x}, \ \ c_j^{[2]} = iq_{j,x}, \\ d_{m,n}^{[2]} = -p_m q_n + p_n q_m, \ \ a^{[2]} = -p_1q_1 - p_2q_2 - p_3q_3 - 2p_4q_4 \end{cases}$$

$$\begin{cases} b_1^{[3]} = -p_{1,xx} - p_1^2q_1 + p_2^2q_1 + p_3^2q_1 + 2p_4^2q_1 - 2p_1p_2q_2 - 2p_1p_3q_3 - 4p_1p_4q_4 \\ b_2^{[3]} = -p_{2,xx} + p_1^2q_2 - p_2^2q_2 + p_3^2q_2 + 2p_4^2q_2 - 2p_2p_1q_1 - 2p_2p_3q_3 - 4p_2p_4q_4 \\ b_3^{[3]} = -p_{3,xx} + p_1^2q_3 + p_2^2q_3 - p_3^2q_3 + 2p_4^2q_3 - 2p_3p_1q_1 - 2p_3p_2q_2 - 4p_3p_4q_4 \\ b_4^{[3]} = -p_{4,xx} + p_1^2q_4 + p_2^2q_4 + p_3^2q_4 - 2p_4^2q_4 - 2p_4p_1q_1 - 2p_4p_2q_2 - 2p_4p_3q_3 \end{cases}$$

$$\begin{cases} c_1^{[3]} = -q_{1,xx} - p_1q_1^2 + p_1q_2^2 + p_1q_3^2 + 2p_1q_4^2 - 2q_1p_2q_2 - 2q_1p_3q_3 - 4q_1p_4q_4 \\ c_2^{[3]} = -q_{2,xx} + p_2q_1^2 - p_2q_2^2 + p_2q_3^2 + 2p_2q_4^2 - 2q_2p_1q_1 - 2q_2p_3q_3 - 4q_2p_4q_4 \\ c_3^{[3]} = -q_{3,xx} + p_3q_1^2 + p_3q_2^2 - p_3q_3^2 + 2p_3q_4^2 - 2q_3p_1q_1 - 2q_3p_2q_2 - 4q_3p_4q_4 \\ c_4^{[3]} = -q_{4,xx} + p_4q_1^2 + p_4q_2^2 + p_4q_3^2 - 2p_4q_4^2 - 2q_4p_1q_1 - 2q_4p_2q_2 - 2q_4p_3q_3 \end{cases}$$

$$\begin{cases} d_{m,n}^{[3]} = i\left[-p_m q_{n,x} + p_{m,x}q_n + p_n q_{m,x} - p_{n,x}q_m\right] \\ a^{[3]} = i\left[p_{1,x}q_1 - p_1q_{1,x} + p_{2,x}q_2 - p_2q_{2,x} + p_{3,x}q_3 - p_3q_{3,x} + 2p_{4,x}q_4 - 2p_4q_{4,x}\right] \end{cases}$$

and

$$\begin{cases} b_1^{[4]} = i\big[\, p_{1,xxx} + 3p_1p_{1,x}q_1 - 3p_2p_{2,x}q_1 - 3p_3p_{3,x}q_1 - 6p_4p_{4,x}q_1 + 3p_{1,x}p_2q_2 \\ \quad + 3p_{1,x}p_3q_3 + 6p_{1,x}p_4q_4 + 3p_1p_{2,x}q_2 + 3p_1p_{3x}q_3 + 6p_1p_{4,x}q_4 \,\big] \\ b_2^{[4]} = i\big[\, p_{2,xxx} - 3p_1p_{1,x}q_2 + 3p_2p_{2,x}q_2 - 3p_3p_{3,x}q_2 - 6p_4p_{4,x}q_2 + 3p_{2,x}p_1q_1 \\ \quad + 3p_{2,x}p_3q_3 + 6p_{2,x}p_4q_4 + 3p_2p_{1,x}q_1 + 3p_2p_{3,x}q_3 + 6p_2p_{4,x}q_4 \,\big] \\ b_3^{[4]} = i\big[\, p_{3,xxx} - 3p_1p_{1,x}q_3 - 3p_2p_{2,x}q_3 + 3P_3p_{3,x}q_3 - 6p_4p_{4x}q_3 + 3p_{3x}p_1q_1 \\ \quad + 3p_{3,x}p_2q_2 + 6p_{3,x}p_4q_4 + 3p_3p_{1,x}q_1 + 3p_3p_{2,x}q_2 + 6p_3p_{4x}q_4 \,\big] \\ b_4^{[4]} = i\big[\, p_{4,xxx} - 3p_1p_{1,x}q_4 - 3p_2p_{2x}q_4 - 3p_3p_{3,x}q_4 + 6p_4p_{4,x}q_4 + 3p_{4,x}p_1q_1 \\ \quad + 3p_{4,x}p_2q_2 + 3p_{4,x}p_3q_3 + 3p_4p_{1,x}q_1 + 3p_4p_{2,x}q_2 + 3p_4p_{3,x}q_3 \,\big] \end{cases}$$

$$\begin{cases} c_1^{[4]} = i\big[-q_{1,xxx} - 3q_1q_{1x}p_1 + 3q_2q_{2,x}p_1 + 3q_3q_{3,x}p_1 + 6q_4q_4p_1 - 3q_{1,x}p_2q_2 \\ \quad - 3q_{1,x}p_3q_3 - 6q_{1,x}p_4q_4 - 3q_1q_{2x}p_2 - 3q_1q_{3,x}p_3 - 6q_1q_{4,x}p_4 \,\big] \\ c_2^{[4]} = i\big[-q_{2,xxx} + 3p_2q_1q_{1,x} - 3p_2q_2q_{2,x} + 3p_2q_3q_{3,x} + 6p_2q_4q_{4,x} - 3q_{2,x}p_1q_1 \\ \quad - 3q_{2,x}p_3q_3 - 6q_{2,x}p_4q_4 - 3q_2q_{1,x}p_1 - 3q_2q_{3,x}p_3 - 6q_2q_{4,x}p_4 \,\big] \\ c_3^{[4]} = i\big[-q_{3,xxx} + 3p_3q_1q_{1,x} + 3p_3q_2q_{2,x} - 3p_3q_3q_{3,x} + 6p_3q_4q_{4x} - 3q_{3x}p_1q_1 \\ \quad - 3q_{3x}p_2q_2 - 6q_{3,x}p_4q_4 - 3q_3q_{1,x}p_6 - 3q_3q_{2,x}p_2 - 6q_3q_{4,x}p_4 \,\big] \\ c_4^{[4]} = i\big[-q_{4,xxx} + 3p_4q_1q_{1,x} + 3p_4q_2q_{2,x} + 3p_4q_3q_{3,x} - 6p_4q_4q_{4,x} - 3q_{4,x}p_1q_1 \\ \quad - 3q_{4,x}p_2q_2 + 3q_{4,x}p_3q_3 - 3q_4q_{1,x}p_1 - 3q_4q_{2,x}p_2 - 3q_4q_{3,x}p_3 \,\big] \end{cases}$$

$$\begin{aligned} a^{[4]} = {} & -p_{1,x}q_{1,x} - p_{2,x}q_{2,x} - p_{3,x}q_{3,x} - 2p_{4,x}q_{4,x} + p_1q_{1,x}x + q_1p_{1,xx} + p_2q_{2,xx} + q_2p_{2,xx} \\ & + p_3q_{3,xx} + q_3p_{3,xx} + 2p_4q_{4,xx} + 2q_4p_{4,xx} + \frac{3}{2}(p_1q_1)^2 - \frac{3}{2}(p_2q_1)^2 - \frac{3}{2}(p_3q_1)^2 \\ & - 3(p_4q_1)^2 - \frac{3}{2}(p_1q_2)^2 + \frac{3}{2}(p_2q_2)^2 - \frac{3}{2}(p_3q_2)^2 - 3(p_4q_2)^2 - \frac{3}{2}(p_1q_3)^2 \\ & - \frac{3}{2}(p_2q_3)^2 + \frac{3}{2}(p_3q_3)^2 - 3(p_4q_3)^2 + 6(p_1p_2q_1q_2) + 6(p_1p_3q_1q_3) \\ & + 6(p_2p_3q_2q_3) + 12(p_1p_4q_1q_4) + 12\dot{p}_2p_4q_2q_4 + 12(p_3p_4q_3q_4) \end{aligned}$$

$$\begin{cases} d_{1,2}^{[4]} = p_{1x}q_{2x} + p_{2x}q_{1x} - p_1q_{2xx} - p_{1xx}q_2 - p_{2xx}q_1 - p_2q_{1xx} + 3p_1^2q_1q_2 - 3p_2^2q_1q_2 \\ \quad + 3p_1p_2q_2^2 - 3p_1p_2q_1^2 + 3p_3q_3(p_1q_2 - p_2q_1) + 6p_4q_4(p_1q_2 - p_2q_1) \\ d_{1,3}^{[4]} = p_{1,x}q_{3,x} + p_{3,x}q_{1,x} - p_1q_{3,xx} - p_{1,xx}q_3 - p_{3,xx}q_1 - p_3q_{1,xx} + 3p_1^2q_1q_3 - 3p_3^2q_1q_3 \\ \quad + 3p_1p_3q_3^2 - 3p_1p_3q_1^2 + 3p_2q_2(p_1q_3 - p_3q_1) + 6p_4q_4(p_1q_3 - p_3q_1) \\ d_{1,4}^{[4]} = p_{1,x}q_{4,x} + p_{4,x}q_{1,x} - p_1q_{4,xx} - p_{1,xx}q_4 - p_{4,xx}q_1 - p_4q_{1,xx} + 3p_1^2q_1q_4 - 6p_4^2q_1q_4 \\ \quad + 6p_1p_4q_4^2 - 3p_1p_4q_1^2 + 3p_2q_2(p_1q_4 - p_4q_1) + 3p_3q_3(p_1q_4 - p_4q_1) \\ d_{2,3}^{[4]} = p_{2x}q_{3,x} + p_{3,x}q_{2,x} - p_2q_{3,xx} - p_{2,xx}q_3 - p_{3,xx}q_2 - p_3q_{2,xx} + 3p_2^2q_2q_3 - 3p_3^2q_2q_3 \\ \quad + 3p_2p_3q_3^2 - 3p_2p_3q_2^2 + 3p_1q_1(p_2q_3 - p_3q_2) + 6p_4q_4(p_2q_3 - p_3q_2) \\ d_{2,4}^{[4]} = p_{2,x}q_{4,x} + p_{4,x}q_{2,x} - p_2q_{4,xx} - p_{2,xx}q_4 - p_{4,xx}q_2 - p_4q_{2,xx} + 3p_2^2q_2q_4 - 6p_4^2q_2q_4 \\ \quad + 6p_2p_4q_4^2 - 3p_2p_4q_2^2 + 3p_1q_1(p_2q_4 - p_4q_2) + 3p_3q_3(p_2q_4 - p_4q_2) \\ d_{3,4}^{[4]} = p_{3,x}q_{4,x} + p_{4,x}q_{3,x} - p_3q_{4,xx} - p_{3,xx}q_4 - p_{4,xx}q_3 - p_4q_{3,xx} + 3p_3^2q_3q_4 - 6p_4^2q_3q_4 \\ \quad + 6p_3p_4q_4^2 - 3p_3p_4q_3^2 + 3p_1q_1(p_3q_4 - p_4q_3) + 3p_2q_2(p_3q_4 - p_4q_3) \end{cases}$$

Next, we present the temporal matrix spectral problem:

$$-i\psi_t = V^{[g]} = V^{[g]}(u,\lambda)\psi, \quad V^{[g]} = \left(\lambda^{[g]}W\right)_+ = \sum_{s=0}^{g}\lambda^s W^{[g-s]}, \quad g \geq 0, \quad (2.13)$$

which is the complementary part of the Lax pair associated with the matrix spectral problem.

The compatibility conditions between the spatial (2.2) and temporal (2.13) matrix spectral problems, are represented by the zero curvature equation:

$$U_{t_g} - V_x^{[g]} + i\left[U, V^{[g]}\right] = 0, \quad g \geq 0 \tag{2.14}$$

and consequently generate the eight-component integrable hierarchy:

$$u_{t_g} = K^{[g]} = \left(ib_1^{[g+1]}, ib_2^{[g+1]}, ib_3^{[g+1]}, ib_4^{[g+1]}, -ic_1^{[g+1]}, -ic_2^{[g+1]}, -ic_3^{g+1}, -ic_4^{g+1}\right)^{\mathrm{T}}, \quad g \geq 0 \tag{2.15}$$

or precisely,

$$p_{j_{t_g}} = ib_j^{[g+1]}, \quad q_{j_{t_g}} = -ic_j^{[g+1]}, \quad g \geq 0 \tag{2.16}$$

resulting in the nonlinear Schrodinger and Koreteweg-De Vries equations, which represent the initial two nonlinear examples in the above eight component integrable hierarchy:

$$\begin{cases} ip_{1_{t_2}} = -p_{1,xx} + p_1^2 q_1 - p_2^2 q_1 - p_3^2 q_1 - 2p_4^2 q_1 + 2p_1 p_2 q_2 + 2p_1 p_3 q_3 + 4p_1 p_4 q_4 \\ ip_{2_{t_2}} = -p_{2xx} - p_1^2 q_2 + p_2^2 q_2 - p_3^2 q_2 - 2p_4^2 q_2 + 2p_2 p_1 q_1 + 2p_2 p_3 q_3 + 4p_2 p_4 q_4 \\ ip_{3_{t_2}} = -p_{3,xx} - p_1^2 q_3 - p_2^2 q_3 + p_3^2 q_3 - 2p_4^2 q_3 + 2p_3 p_1 q_1 + 2p_3 p_2 q_2 + 4p_3 p_4 q_4 \\ ip_{4_{t_2}} = -p_{4,xx} - p_1^2 q_4 - p_2^2 q_4 - p_3^2 q_4 + 2p_4^2 q_4 + 2p_4 p_1 q_1 + 2p_4 p_2 q_2 + 2p_4 p_3 q_3 \\ iq_{1_{t_2}} = -q_{1,xx} - p_1 q_1^2 + p_1 q_2^2 + p_1 q_3^2 + 2p_1 q_4^2 - 2q_1 p_2 q_2 - 2q_1 p_3 q_3 - 4q_1 p_4 q_4 \\ iq_{2_{t_2}} = -q_{2,xx} + p_2 q_1^2 - p_2 q_2^2 + p_2 q_3^2 + 2p_2 q_4^2 - 2q_2 p_1 q_1 - 2q_2 p_3 q_3 - 4q_2 p_4 q_4 \\ iq_{3_{t_2}} = -q_{3,xx} + p_3 q_1^2 + p_3 q_2^2 - p_3 q_3^2 + 2p_3 q_4^2 - 2q_3 p_1 q_1 - 2q_3 p_2 q_2 - 4q_3 p_4 q_4 \\ iq_{4_{t_2}} = -q_{4,xx} + p_4 q_1^2 + p_4 q_2^2 + p_4 q_3^2 - 2p_4 q_4^2 - 2q_4 p_1 q_1 - 2q_4 p_2 q_2 + 2q_4 p_3 q_3 \end{cases} \tag{2.17}$$

and

$$\begin{cases} p_{1_{t_3}} = -p_{1,xxx} - 3p_1 p_{1,x} q_1 + 3p_2 p_{2,x} q_1 + 3p_3 p_{3x} q_1 + 6p_4 q_{4x} q_1 - 3p_{1,x} p_2 q_2 \\ \quad - 3p_{1,x} p_3 q_3 - 6p_{1,x} p_4 q_4 - 3p_1 p_{2,x} q_2 - 3p_1 p_{3,x} q_3 - 6p_1 p_{4,x} q_4 \\ p_{2_{t_3}} = -p_{2,xxx} + 3p_1 p_{1,x} q_2 - 3p_2 p_{2,x} q_2 + 3p_3 p_{3,x} q_2 + 6p_4 p_{4,x} q_2 - 3p_{2,x} p_1 q_1 \\ \quad - 3p_{2,x} p_3 q_3 - 6p_{2,x} p_4 q_4 - 3p_2 p_{1,x} q_1 - 3p_2 p_{3,x} q_3 - 6p_2 p_{4,x} q_4 \\ p_{3_{t_3}} = -p_{3,xxx} + 3p_1 p_{1,x} q_3 + 3p_2 p_{2,x} q_3 - 3p_3 p_{3,x} q_3 + 6p_4 p_{4,x} q_3 - 3p_{3,x} p_1 q_1 \\ \quad - 3p_{3,x} p_2 q_2 - 6p_{3,x} p_4 q_4 - 3p_3 p_{1,x} q_1 - 3p_3 p_{2,x} q_2 - 6p_3 p_{4,x} q_4 \\ p_{4_{t_3}} = -p_{4,xxx} + 3p_1 p_{1,x} q_4 + 3p_2 p_{2,x} q_4 + 3p_3 p_{3,x} q_4 - 6p_4 p_{4,x} q_4 - 3p_{4,x} p_1 q_1 \\ \quad - 3p_{4,x} p_2 q_2 - 3p_{4,x} p_3 q_3 - 3p_4 p_{1,x} q_1 - 3p_4 p_{2,x} q_2 - 3p_4 p_{3,x} q_3 \\ q_{1_{t_3}} = -q_{1,xxx} - 3q_1 q_{1,x} p_1 + 3q_2 q_{2,x} p_1 + 3q_3 q_{3,x} p_1 + 6q_4 q_{4,x} p_1 - 3q_{1,x} p_2 q_2 \\ \quad - 3q_{1,x} p_3 q_3 - 6q_{1,x} p_4 q_4 - 3q_1 q_{2,x} p_2 - 3q_1 q_{3,x} p_3 - 6q_1 q_{4,x} p_4 \\ q_{2_{t_3}} = -q_{2,xxx} + 3p_2 q_1 q_{1,x} - 3p_2 q_2 q_{2,x} + 3p_2 q_3 q_{3,x} + 6p_2 q_4 q_{4,x} - 3q_{2,x} p_1 q_1 \\ \quad - 3q_{2,x} p_3 q_3 - 6q_{2,x} p_4 q_4 - 3q_2 q_{1,x} p_1 - 3q_2 q_{3,x} p_3 - 6q_2 q_{4,x} p_4 \\ q_{3_{t_3}} = -q_{3,xxx} + 3p_3 q_1 q_{1,x} + 3p_3 q_2 q_{2,x} - 3p_3 q_3 q_{3,x} + 6p_3 q_4 q_{4,x} - 3q_{3,x} p_1 q_1 \\ \quad - 3q_{3,x} p_2 q_2 - 6q_{3,x} p_4 q_4 - 3q_3 q_{1,x} p_1 - 3q_3 q_{2,x} p_2 - 6q_3 q_{4,x} p_4 \\ q_{4_{t_3}} = -q_{4,xxx} + 3p_4 q_1 q_x + 3p_4 q_2 q_{2,x} + 3p_4 q_3 q_{3,x} - 6p_4 q_4 q_{4,x} - 3q_{4,x} p_1 q_1 \\ \quad - 3q_{4,x} p_2 q_2 - 3q_{4,x} p_3 q_3 - 3q_4 q_{1,x} p_1 - 3q_4 q_{2,x} p_2 - 3q_4 q_{3,x} p_3 \end{cases} \tag{2.18}$$

3. Hamiltonian Structures

To formulate Hamiltonian structures for the integrable equations in (2.14), we apply the trace identity

$$\frac{\delta}{\delta u}\int \mathrm{tr}\left(W\frac{\partial U}{\partial \lambda}\right)\mathrm{d}x = \lambda^{-\alpha}\frac{\partial}{\partial \lambda}\lambda^{\alpha}\mathrm{tr}\left(W\frac{\partial U}{\partial u}\right),$$

where α is a constant. Further, we have

$$\mathrm{tr}\left(W\frac{\partial U}{\partial \lambda}\right)=2a,\quad \mathrm{tr}\left(W\frac{\partial U}{\partial u}\right)=\left(2c_1,2c_2,2c_3,4c_4,2b_1,2b_2,2b_3,4b_4\right)^{\mathrm{T}}$$

and thus, we obtain

$$\frac{\delta}{\delta u}\int \lambda^{-s-1}a^{[s+1]}\mathrm{d}x = \lambda^{-\alpha}\frac{\partial}{\partial \lambda}\lambda^{\alpha-s}\left(c_1^{[s]},c_2^{[s]},c_3^{[s]},2c_4^{[s]},b_1^{[s]},b_2^{[s]},b_3^{[s]},2b_4^{[s]}\right)^{\mathrm{T}},\quad s\geq 0. \tag{3.1}$$

For $s=2$, with $\alpha=0$, we obtain:

$$\frac{\delta}{\delta u}\int H^{[s]}\mathrm{d}x=\left(c_1^{[s]},c_2^{[s]},c_3^{[s]},2c_4^{[s]},b_1^{[s]},b_2^{[s]},b_3^{[s]},2b_4^{[s]}\right)^{\mathrm{T}}, \tag{3.2}$$

with the Hamiltonian functionals determined by:

$$H^{[s]}=-\int\frac{a^{[s+1]}}{s}\mathrm{d}x,\quad s\geq 1, \tag{3.3}$$

of which the first three functionals are:

$$\begin{aligned}
H^{[1]} &= \int p_1q_1+p_2q_2+p_3q_3+2p_4q_4\mathrm{d}x\\
H^{[2]} &= \int -\frac{i}{2}\left[p_{1,x}q_1-p_1q_{1,x}+p_{2,x}q_2-p_2q_{2,x}+p_{3,x}q_3-p_3q_{3,x}+2p_{4,x}q_4-2p_4q_{4,x}\right]\mathrm{d}x\\
H^{[3]} &= \int\left[\frac{1}{3}\left(p_{1,x}q_{1,x}+p_{2,x}q_{2,x}+p_{3,x}q_{3,x}+2p_{4,x}q_{4,x}-p_1q_{1,xx}-q_1p_{1,xx}-p_2q_{2,xx}-q_2p_{2,xx}\right.\right.\\
&\quad -p_3q_{3,xx}-q_3p_{3,xx}-2p_4q_{4,xx}-2q_4p_{4,xx}\left.\right)-\frac{1}{2}\left(-p_1^2q_1^2+p_2^2q_1^2+p_3^2q_1^2+p_4^2q_1^2\right.\\
&\quad +p_1^2q_2^2-p_2^2q_2^2+p_3^2q_2^2+p_4^2q_2^2+p_1^2q_3^2+p_2^2q_3^2-p_3^2q_3^2+p_4^2q_3^2\left.\right)-2p_1p_2q_1q_2\\
&\quad \left.-2p_1p_3q_1q_3-q_2p_2p_3q_2q_3-4p_1p_4q_1q_4-4p_2p_4q_2q_4-4p_3p_4q_3q_4\right]\mathrm{d}x
\end{aligned} \tag{3.4}$$

Following these equivalences, we can then establish Hamiltonian structures for the integrable hierarchy (2.14):

$$u_{t_g}=K^{[g]}=\left(ib_1^{[g+1]},ib_2^{[g+1]},ib_3^{[g+1]},ib_4^{[g+1]},-ic_1^{[g+1]},-ic_2^{[g+1]},-ic_3^{[g+1]}-ic_4^{[g+1]}\right)^{\mathrm{T}}=J\frac{\delta H^{[g+1]}}{\delta u},$$

where J, the Hamiltonian operator:

$$J=\begin{bmatrix}
0 & 0 & 0 & 0 & i & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & i & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & i & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2i\\
-i & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -i & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -i & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -2i & 0 & 0 & 0 & 0
\end{bmatrix}, \tag{3.5}$$

and $H^{[r]}$ defined by (3.3), form the Hamiltonian framework, which establishes a connection between symmetries and conserved quantities.

Innumerably many mutually commuting symmetries:

$$\left[\left[K_{r_1}, K_{r_2}\right]\right] = K'_{r_1}(u)\left[K_{r_2}\right] - K'_{r_2}(u)\left[K_{r_1}\right] = 0, \quad r_1, r_2 \geq 0., \tag{3.6}$$

are guaranteed by a Lax operator:

$$\left[\left[V^{[r_1]}, V^{(r_2)}\right]\right] = V^{[r_1]\prime}(u)\left[K^{[r_2]}\right] - V^{[r_2]\prime}(u)\left[K^{[r_1]}\right] + \left[V^{[r_1]}, V^{[r_2]}\right] = 0, \quad r_1, r_2 \geq 0, \tag{3.7}$$

This relationship arises as an outcome of the isospectral zero curvature equation [4].

Based on the skew-symmetric nature of the Hamiltonian operator J and the recursion relation of the hierarchy, the conserved functionals commute under the Poisson bracket [3]:

$$\left\{H^{[r_1]}, H^{[r_2]}\right\}_J = \int \frac{\delta H^{[r_1]}}{\delta u}^{\mathrm{T}} J \frac{\delta H^{[r_2]}}{\delta u} \mathrm{d}x = 0, \quad r_1, r_2 \geq 0. \tag{3.8}$$

Finally, a bi-Hamiltomian structure [5] can also be constructed by combining J with a recursion relation for K_s derived from (2.16) [6].

4. Conclusion

An eight component integrable hierarchy has been introduced from a reduced seventh order matrix spectral problem that is related to a special Lie algebra [7] of the general linear algebra [8]. Hamiltonian structures have been constructed for the hierarchy and the first three Hamiltonian functionals have been introduced. The difficulty is a repeat of the fourth pair of potentials and such an exploration will help one gain a deeper discernment of their mathematical structures. Exploring the structure of solitons in the resulting integrable systems, with the inverse scattering transform [9] [10], Hirota's direct method or backlund transformations [6] would be highly significant.

The exploration of integrable equations constitutes a dynamic and stimulating realm of research, with the capacity to unveil insights into diverse physical phenomena, such as nonlinear optics, plasma physics, Bose-Einstein Condensates and shallow water waves. There is still much to learn about these intriguing mathematical systems. The methods employed to investigate their mathematical structures will remain a vibrant area of research for the foreseeable future.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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