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Two integrable couplings of a generalized D-Kaup–Newell hierarchy and their Hamiltonian and bi-Hamiltonian structures[☆]

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ABSTRACT

We enlarge the spectral problem of a generalized D-Kaup–Newell (D-KN) spectral problem. Solving the enlarged zero-curvature equations, we produce integrable couplings. A reduction of the spectral matrix leads to a second integrable coupling system. Next, bilinear forms that are symmetric, ad-invariant, and non-degenerate on the given non-semisimple matrix Lie algebra are computed to employ the variational identity. The variational identity is then applied to the original enlarged spectral problem of a generalized D-KN hierarchy and the reduced problem. Hamiltonian structures are presented, as well as a bi-Hamiltonian formulation of the reduced problem. Both hierarchies have infinitely many commuting high-order symmetries and conserved densities, i.e., are Liouville integrable.

1. Introduction

The finding of new integrable couplings has become an important area of research in mathematical physics [1–24] and their study will aid in the classification of multi-component integrable systems. Originally, the study of centerless Virasoro symmetry algebras of integrable systems revealed integrable couplings [1,2]. Given an integrable system $u_t = K(u)$, an integrable coupling of the system is a triangular system of the form

$$\bar{u}_t = [u_t, v_t]^T = \bar{K}(\bar{u}) = [K(u), S(u, v)]^T, \tag{1}$$

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where potentials u and v are scalar functions or vector functions with dependent variables $\bar{x} = (t, x_1, x_2, \dots)$ and $\bar{u} = [u, v]^T$. It is important that the new differential equations in the bigger system (1) involve u and all of its derivatives, i.e., $\frac{\partial S}{\partial u} \neq 0$, to avoid triviality. Integrable couplings were first constructed through perturbations [1–3], then the spectral matrices were enlarged [4–6], and, in 2006, the connection between integrable couplings and semi-direct sums of Lie algebras was realized [7,8]. Very recently, a novel kind of AKNS integrable couplings was analyzed [14]. This paper has an enlarged spectral matrix with all sub-matrices depending on λ. This technique has only be seen recently and can lead to Darboux transformations for explicit solutions [25,26].

A generalized D-KN hierarchy was derived from the following isospectral problem

$$\phi_x = U(\lambda, u)\phi = \begin{bmatrix} \lambda^2 - r_1 & \lambda p_1 + s_1 \\ \lambda q_1 + v_1 & -\lambda^2 + r_1 \end{bmatrix} \phi, \tag{2}$$

where $u = [p_1, q_1, r_1, s_1, v_1]^T$ are potentials, $\phi = [\phi_1, \phi_2]^T$ and $U \in sl(2, \mathbb{R})$. The hierarchy is integrable in the Liouville sense and a reduction to the spectral problem (2) is bi-Hamiltonian [27]. The spectral matrix (2) is a generalization of the D-KN spectral matrix which is found by letting $s_1 = v_1 = 0$. The AKNS hierarchy [28] may be found from (2) by letting $p_1 = q_1 = r_1 = 0$ and choosing a suitable Laurent expansion. This new generalized hierarchy contains equations from both the AKNS and the D-KN hierarchies through reductions as well as many new interesting equations.

Two major sections complete this paper: integrable couplings and Hamiltonian structures. In the integrable couplings section, an enlarged spectral matrix (2) is presented where all sub-matrices are dependent on λ. Solving the zero-curvature equation, we find a system of recursive relations and prove they are local. We use this to present the hierarchy of integrable couplings. Next, a reduction of the enlarged spectral matrix leads to a second integrable coupling system. The section of Hamiltonian structures follows where non-degenerate, ad-invariant, symmetric bilinear forms are found. The bilinear forms along with the variational identity produce Hamiltonian structures of generalized D-KN integrable couplings and its reduced integrable couplings. A bi-Hamiltonian structure is found for the reduced integrable couplings. We discover infinitely many commuting high-order symmetries and conserved functionals for both hierarchies and, thus, their Liouville integrability.

2. Integrable couplings

In order to simplify notation, we define a triangular block matrix as follows:

$$M(A_1, A_2) = \begin{bmatrix} A_1 & A_2 \\ 0 & A_1 \end{bmatrix}. \tag{3}$$

It can easily be shown that matrices of this form are closed under matrix multiplication, i.e., constitute a Lie algebra. The associated matrix loop algebra $\tilde{\mathfrak{g}}(\lambda)$ is formed by all block matrices of the type:

$$\tilde{\mathfrak{g}}(\lambda) = \{M(A_1, A_2) |\ M \text{ defined by (3), entries of } A_i \text{ are Laurent series in } \lambda\}. \tag{4}$$

We will use this notation throughout this paper.

2.1. Generalized D-KN integrable couplings

A spectral matrix is chosen from $\tilde{\mathfrak{g}}(\lambda)$ as

$$\bar{U} = \bar{U}(\bar{u}, \lambda) = M(U(\lambda, u), U_1(\lambda, v)), \tag{5}$$

where $\bar{u} = [u, v]^T, u = [p_1, q_1, r_1, s_1, v_1]^T, v = [p_2, q_2, r_2, s_2, v_2]^T$ are potentials. U is from the generalized D-KN spectral problem (2) and

$$U_1 = U(\lambda, v) = \begin{bmatrix} \lambda^2 - r_2 & \lambda p_2 + s_2 \\ \lambda q_2 + v_2 & -\lambda^2 + r_2 \end{bmatrix}. \tag{6}$$

The isospectral problem is

$$\bar{\phi}_x = \bar{U}\bar{\phi}, \tag{7}$$

where $\bar{\phi} = [\psi, \phi]^T, \psi = [\psi_1, \psi_2]^T$ and $\phi = [\phi_1, \phi_2]^T$. Note that U is the same matrix as (2).

Assume that the solution to the stationary zero-curvature equation, $\bar{W}_x = [\bar{U}, \bar{W}]$, is of the form

$$\bar{W} = \begin{bmatrix} W & W_1 \\ 0 & W \end{bmatrix} \in \tilde{\mathfrak{g}}(\lambda), W = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}, W_1 = \begin{bmatrix} e & f \\ g & -e \end{bmatrix}, \tag{8}$$

then we get the following matrix formulas

$$\begin{cases} W_x = UW - WU, \\ W_{1,x} = U_1 W - W U_1 + U W_1 - W_1 U. \end{cases} \tag{9}$$

Solving these two formulas, we get the differential equations

$$\begin{cases} a_x = -q_1 b\lambda + p_1 c\lambda - v_1 b + s_1 c, \\ b_x = -2p_1 a\lambda + 2b\lambda^2 - 2s_1 a - 2r_1 b, \\ c_x = 2q_1 a\lambda - 2c\lambda^2 + 2v_1 a + 2r_1 c, \\ e_x = p_1 g\lambda + p_2 c\lambda - q_2 b\lambda - q_1 f\lambda + s_1 g + s_2 c - v_1 f - v_2 b, \\ f_x = 2b\lambda^2 + 2f\lambda^2 - 2p_1 e\lambda - 2p_2 a\lambda - 2r_1 f - 2r_2 b - 2s_1 e - 2s_2 a, \\ g_x = -2c\lambda^2 - 2g\lambda^2 + 2q_1 e\lambda + 2q_2 a\lambda + 2r_1 g + 2r_2 c + 2v_1 e + 2v_2 a. \end{cases} \tag{10}$$

By assuming a, b, c, e, f, g, have the following Laurent series expansions

$$\begin{aligned} a &= \sum_{i=0}^{\infty} a_i \lambda^{-i}, & b &= \sum_{i=0}^{\infty} b_i \lambda^{-i}, & c &= \sum_{i=0}^{\infty} c_i \lambda^{-i}, \\ e &= \sum_{i=0}^{\infty} e_i \lambda^{-i}, & f &= \sum_{i=0}^{\infty} f_i \lambda^{-i}, & g &= \sum_{i=0}^{\infty} g_i \lambda^{-i}, \end{aligned} \tag{11}$$

and substituting (11) into (10), we have the recursion relations

$$\begin{cases} b_{i+1} &= \frac{b_{i-1,x}}{2} + p_1 a_i + s_1 a_{i-1} + r_1 b_{i-1}, \\ c_{i+1} &= -\frac{c_{i-1,x}}{2} + q_1 a_i + v_1 a_{i-1} + r_1 c_{i-1}, \\ a_{i+1,x} &= -q_1 \frac{b_{i,x}}{2} - p_1 \frac{c_{i,x}}{2} + (p_1 v_1 - q_1 s_1) a_i - q_1 r_1 b_i + p_1 r_1 c_i + s_1 c_{i+1} \\ & - v_1 b_{i+1}, \\ f_{i+1} &= \frac{f_{i-1,x}}{2} - b_{i+1} + p_2 a_i + p_1 e_i + s_2 a_{i-1} + s_1 e_{i-1} + r_2 b_{i-1} + r_1 f_{i-1}, \\ g_{i+1} &= -\frac{g_{i-1,x}}{2} - c_{i+1} + q_2 a_i + q_1 e_i + v_2 a_{i-1} + v_1 e_{i-1} + r_2 c_{i-1} + r_1 g_{i-1}, \\ e_{i+1,x} &= -\frac{g_{i,x}}{2} p_1 - \frac{f_{i,x}}{2} q_1 + (p_2 - p_1)[-\frac{c_{m,x}}{2} + v_1 a_m + r_1 c_m] \\ & + (q_1 - q_2)[\frac{b_{m,x}}{2} + s_1 a_m + r_1 b_m] + s_1 g_{i+1} + s_2 c_{i+1} - v_1 f_{i+1} - v_2 b_{i+1} \\ & + (p_1 v_2 - q_1 s_2) a_i + (p_1 v_1 - q_1 s_1) e_i + p_1 r_2 c_i - q_1 r_2 b_i \\ & + p_1 r_1 g_i - q_1 r_1 f_i, \end{cases} \tag{12}$$

for all $i \geq 1$ with initial values

$$\begin{aligned} &a_0 = \alpha, \quad b_0 = c_0 = 0, \quad a_1 = 0, \quad b_1 = \alpha p_1, \quad c_1 = \alpha q_1, \\ &e_0 = \beta, \quad f_0 = g_0 = 0, \quad e_1 = 0, \quad f_1 = (\beta - \alpha) p_1 + p_2 \alpha, \quad g_1 = (\beta - \alpha) q_1 + q_2 \alpha, \end{aligned} \tag{13}$$

and the conditions for integration

$$\begin{cases} a_i|_{u=0} = b_i|_{u=0} = c_i|_{u=0} = 0, & i \geq 1, \\ e_i|_{u=0} = f_i|_{u=0} = g_i|_{u=0} = 0, & i \geq 1, \end{cases} \tag{14}$$

which determine the sequence of $\{a_i, b_i, c_i, e_i, f_i, g_i | i \geq 0\}$ uniquely. For $i = 2, 3$, we have the following:

$$\begin{aligned}
&b_2 = \alpha s_1, \qquad c_2 = \alpha v_1, \qquad a_2 = -\alpha \frac{1}{2} p_1 q_1, \\
&f_2 = (\beta - \alpha)s_1 + \alpha s_2, \qquad e_2 = (\alpha q_1 + \frac{1}{2}\beta q_1 - \frac{1}{2} q_2)p_1 - \frac{1}{2}\alpha p_2 q_1, \\
&g_2 = (\beta - \alpha)v_1 + \alpha v_2; \\
&b_3 = \alpha \frac{1}{2}(-p_1^2 q_1 + 2p_1 r_1 + p_{1,x}), \qquad c_3 = -\alpha \frac{1}{2}(q_1^2 p_1 - 2q_1 r_1 + q_{1,x}), \\
&a_3 = -\alpha \frac{1}{2}(p_1 v_1 + q_1 s_1), \\
&f_3 = \frac{1}{2}[(\beta - 2\alpha)p_{1,x} + \alpha p_{2,x} + (\beta + 3\alpha)p_1^2 q_1 + (2\beta - 4\alpha)r_1 p_1 \\
&\quad - \alpha(p_2 q_1 p_1 - 2r_2 p_1 + q_2 p_1^2) + 2\alpha p_2 r_1], \\
&g_3 = -\frac{1}{2}[(\beta - 2\alpha)q_{1,x} + \alpha q_{2,x} - (-\beta + 3\alpha)q_1^2 p_1 \alpha - (2\beta - 4\alpha)r_1 q_1 \\
&\quad + \alpha(q_2 p_1 q_1 - 2r_2 q_1 + p_2 q_1^2) - 2\alpha q_2 r_1], \\
&e_3 = (v_1 \alpha - \frac{1}{2} v_1 \beta - \frac{1}{2} v_2 \alpha)p_1 + (s_1 \alpha - \frac{1}{2} s_1 \beta - \frac{1}{2} s_2 \alpha)q_1 - \frac{1}{2} p_2 v_1 \alpha - \frac{1}{2} q_2 s_1 \alpha.
\end{aligned}$$

All $\{a_i, b_i, c_i, e_i, f_i, g_i | i \geq 0\}$ can be proven as differential polynomials of $\bar{u}$ with respect to x.

Proposition 2.1. *Let $\{a_i, b_i, c_i, e_i, f_i, g_i | i = 0, 1\}$ be given by Eq. (13). Then all functions $\{a_i, b_i, c_i, e_i, f_i, g_i | i \geq 0\}$ determined by Eq. (12) with the conditions (14) are differential polynomials in $\bar{u}$ with respect to x, and thus, are local.*

Proof. We compute from the enlarged stationary zero-curvature equation, $\bar{W}_x = [\bar{U}, \bar{W}]$,

$$\frac{d}{dx}\mathrm{tr}(\bar{W}^2) = 2\mathrm{tr}(\bar{W}\bar{W}_x) = 2\mathrm{tr}(\bar{W}[\bar{U}, \bar{W}]) = 2(\mathrm{tr}(\bar{W}^2\bar{U}) - \mathrm{tr}(\bar{W}^2\bar{U})) = 0, \tag{15}$$

and seeing that the $\mathrm{tr}(\bar{W}^2) = 4(a^2 + bc)$, we have

$$a^2 + bc = (a^2 + bc)|_{u=0} = \alpha^2, \tag{16}$$

following from the initial data (13). Now, we use (11), the Laurent expansions of a, b, c, to give

$$a_i = \frac{\alpha}{2} - \frac{1}{2\alpha} \sum_{k+l=i, k, l \geq 1} a_k a_l - \frac{1}{2\alpha} \sum_{k+l=i, k, l \geq 0} b_k c_l, i \geq 1. \tag{17}$$

Based on the recursion relation above (17) and the previous (12), we use mathematical induction to see that all functions $\{a_i, b_i, c_i, i \geq 0\}$ are differential polynomials in u with respect to x, and therefore, are local.

Now, we have

$$\begin{aligned}
\frac{d}{dx}(2ae + fc + gb) =& 2a_x e + 2ae_x + f_x c + f c_x + g_x b + g b_x \\
=& 2e(-q_1 b\lambda + p_1 c\lambda - v_1 b + s_1 c) + 2a(p_1 g\lambda + p_2 c\lambda \\
&- q_2 b\lambda - q_1 f\lambda + s_1 g + s_2 c - v_1 f - v_2 b) + c(2b\lambda^2 \\
&+ 2f\lambda^2 - 2p_1 e\lambda - p_2 a\lambda - 2r_1 f - 2r_2 b - 2s_1 e - 2s_2 a)
\end{aligned}$$

$$+ f(2q_1a\lambda - 2c\lambda^2 + 2v_1a + 2r_1c) + b(-2c\lambda^2 - 2g\lambda^2 + 2q_1e\lambda + q_2a\lambda + 2r_1g + 2r_2c + 2v_1e + 2v_2a) + g(-2p_1a\lambda + 2b\lambda^2 - 2s_1a - 2r_1b) = 0.$$

Similarly, we get

$$2ae + fc + gb = (2ae + fc + gb)|_{\bar{u}=0} = \alpha\beta.$$

Therefore, using the Laurent expansions of a, b, c, e, f, and g in (11), we have

$$e_i = \beta - \frac{\beta}{\alpha}a_i - \frac{1}{2\alpha}\sum_{k+l=i,k,l\geq 0} f_kc_l - \frac{1}{2\alpha}\sum_{k+l=i,k,l\geq 0} g_kb_l - \frac{1}{\alpha}\sum_{k+l=i,k,l\geq 1} a_ke_l, \tag{18}$$

for all $i \geq 1$. Using the localness of $\{a_i, b_i, c_i|i \geq 0\}$ and the recursive relations (12) and (18), we may see through mathematical induction that all functions $\{e_i, f_i, g_i|i \geq 0\}$ are differential polynomials in $\bar{u}$ with respect to x. This completes the proof. ■

Now, we need to solve the zero-curvature equations,

$$\bar{U}_{t_m} - \bar{V}_x^{[m]} + [\bar{U}, \bar{V}^{[m]}] = 0, \quad m \geq 0, \tag{19}$$

which are the compatibility conditions between (7) and the temporal problems,

$$\bar{\phi}_{t_m} = \bar{V}^{[m]}\bar{\phi} = \bar{V}^{[m]}(\bar{u}, \lambda)\bar{\phi}, \quad m \geq 0. \tag{20}$$

In order to do this, we introduce a series of Lax operators

$$\bar{V}^{[m]}(\bar{u}, \lambda) = (\lambda^m\bar{W})_+. \tag{21}$$

After solving (19), we generate a hierarchy of soliton equations, for all $m \geq 0$,

$$\bar{u}_{t_m} = \bar{K}_m = \begin{bmatrix} 2b_{m+1} \\ -2c_{m+1} \\ q_1b_{m+1} - p_1c_{m+1} \\ -2p_1a_{m+1} + 2b_{m+2} \\ 2q_1a_{m+1} - 2c_{m+2} \\ 2f_{m+1} + 2b_{m+1} \\ -2g_{m+1} - 2c_{m+1} \\ q_1f_{m+1} + q_2b_{m+1} - p_1g_{m+1} - p_2c_{m+1} \\ -2p_1e_{m+1} - 2p_2a_{m+1} + 2b_{m+2} + 2f_{m+2} \\ 2q_1e_{m+1} + 2q_2a_{m+1} - 2c_{m+2} - 2g_{m+2} \end{bmatrix}. \tag{22}$$

We have

$$\bar{K}_m = \bar{\Phi}\bar{K}_{m-1} = \bar{\Phi}^m\bar{K}_0, \quad m \geq 0, \tag{23}$$

where

$$\bar{\Phi} = \begin{bmatrix} \Phi & 0 \\ \Phi_1 - \Phi & \Phi \end{bmatrix}. \tag{24}$$

where Φ is the recursion operator of the original system $u_t = K(u)$ equal to

$$\begin{bmatrix} \begin{matrix}-p_1\partial^{-1}v_1\\ -s_1\partial^{-1}q_1\end{matrix} & \begin{matrix}-p_1\partial^{-1}s_1\\ -s_1\partial^{-1}p_1\end{matrix} & 2s_1\partial^{-1} & 1-p_1\partial^{-1}q_1 & -p_1\partial^{-1}p_1 \\ \begin{matrix}q_1\partial^{-1}v_1\\ +v_1\partial^{-1}q_1\end{matrix} & \begin{matrix}q_1\partial^{-1}s_1\\ +v_1\partial^{-1}p_1\end{matrix} & -2v_1\partial^{-1} & q_1\partial^{-1}q_1 & 1+q_1\partial^{-1}p_1 \\ \begin{matrix}p_1v_1\partial^{-1}\frac{q_1}{2}\\ -q_1s_1\partial^{-1}\frac{q_1}{2}\end{matrix} & \begin{matrix}p_1v_1\partial^{-1}\frac{p_1}{2}\\ -q_1s_1\partial^{-1}\frac{p_1}{2}\end{matrix} & \begin{matrix}-p_1v_1\partial^{-1}\\ +q_1s_1\partial^{-1}\end{matrix} & \frac{q_1}{2} & \frac{p_1}{2} \\ \begin{matrix}\frac{1}{2}\partial+r_1-s_1\partial^{-1}v_1\\ -p_1r_1\partial^{-1}q_1\\ -\partial p_1\partial^{-1}\frac{q_1}{2}\end{matrix} & \begin{matrix}-s_1\partial^{-1}s_1\\ -\partial p_1\partial^{-1}\frac{p_1}{2}\\ -p_1r_1\partial^{-1}p_1\end{matrix} & \begin{matrix}2p_1r_1\partial^{-1}\\ +\partial p_1\partial^{-1}\end{matrix} & -s_1\partial^{-1}q_1 & -s_1\partial^{-1}p_1 \\ \begin{matrix}v_1\partial^{-1}v_1\\ +\partial q_1\partial^{-1}\frac{q_1}{2}\\ +q_1r_1\partial^{-1}q_1\end{matrix} & \begin{matrix}-\frac{1}{2}\partial+r_1+v_1\partial^{-1}s_1\\ +q_1r_1\partial^{-1}p_1\\ +\partial q_1\partial^{-1}\frac{p_1}{2}\end{matrix} & \begin{matrix}-2q_1r_1\partial^{-1}\\ -\partial q_1\partial^{-1}\end{matrix} & v_1\partial^{-1}q_1 & v_1\partial^{-1}p_1 \end{bmatrix}, \tag{25}$$

and Φ_1 is the supplemental matrix differential operator with entries

$$\begin{aligned}
&[\Phi_1]_{11} = -p_1\partial^{-1}v_2 - p_2\partial^{-1}v_1 - s_2\partial^{-1}q_1 - s_1\partial^{-1}q_2,\\
&[\Phi_1]_{12} = -p_1\partial^{-1}s_2 - p_2\partial^{-1}s_1 - s_2\partial^{-1}p_1 - s_1\partial^{-1}p_2,\\
&[\Phi_1]_{13} = 2s_2\partial^{-1} + 2s_1\partial^{-1}, [\Phi_1]_{14} = 1 - p_1\partial^{-1}q_2 - p_2\partial^{-1}q_1,\\
&[\Phi_1]_{15} = -p_1\partial^{-1}p_2 - p_2\partial^{-1}p_1,\\
&[\Phi_1]_{21} = q_1\partial^{-1}v_2 + q_2\partial^{-1}v_1 + v_1\partial^{-1}q_2 + v_2\partial^{-1}q_1,\\
&[\Phi_1]_{22} = q_1\partial^{-1}s_2 + q_2\partial^{-1}s_1 + v_1\partial^{-1}p_2 + v_2\partial^{-1}p_1,\\
&[\Phi_1]_{23} = -2v_1\partial^{-1} - 2v_2\partial^{-1},\\
&[\Phi_1]_{24} = q_1\partial^{-1}q_2 + q_2\partial^{-1}q_1, [\Phi_1]_{25} = 1 + q_1\partial^{-1}p_2 + q_2\partial^{-1}p_1,\\
&[\Phi_1]_{31} = \frac{(v_1p_2 - q_2s_1)}{2}\partial^{-1}q_1 + \frac{(p_1v_2 - q_1s_2)}{2}\partial^{-1}q_1 - \frac{(p_1v_1 - q_1s_1)}{2}\partial^{-1}q_1\\
&+ \frac{(p_1v_1 - q_1s_1)}{2}\partial^{-1}q_2,\\
&[\Phi_1]_{32} = \frac{(v_1p_2 - q_2s_1)}{2}\partial^{-1}p_1 + \frac{(p_1v_2 - q_1s_2)}{2}\partial^{-1}p_1 - \frac{(p_1v_1 - q_1s_1)}{2}\partial^{-1}p_1\\
&+ \frac{(p_1v_1 - q_1s_1)}{2}\partial^{-1}p_2,\\
&[\Phi_1]_{33} = -(p_1v_2 - q_1s_2)\partial^{-1} - (p_2v_1 - q_2s_1)\partial^{-1},\\
&[\Phi_1]_{34} = \frac{q_2}{2}, [\Phi_1]_{35} = \frac{p_2}{2},\\
&[\Phi_1]_{41} = r_2 - s_1\partial^{-1}v_2 - s_2\partial^{-1}v_1 - \partial\frac{(p_2 - p_1)}{2}\partial^{-1}q_1 - (r_1p_2 + p_1r_2)\partial^{-1}q_1\\
&+ p_1r_1\partial^{-1}q_1 - r_1p_1\partial^{-1}q_2 - \partial\frac{p_1}{2}\partial^{-1}q_2,\\
&[\Phi_1]_{42} = -s_1\partial^{-1}s_2 - s_2\partial^{-1}s_1 - \partial\frac{(p_2 - p_1)}{2}\partial^{-1}p_1 - (r_1p_2 + p_1r_2)\partial^{-1}p_1\\
&+ p_1r_1\partial^{-1}p_1 - r_1q_1\partial^{-1}p_2 - \partial\frac{p_1}{2}\partial^{-1}p_2,
\end{aligned}$$

$$
\begin{aligned}
&[\Phi_1]_{43} = \partial(p_2 - p_1)\partial^{-1} + 2(r_1p_2 + p_1r_2)\partial^{-1} + \partial p_1\partial^{-1},\\
&[\Phi_1]_{44} = -s_1\partial^{-1}q_2 - s_2\partial^{-1}q_1, [\Phi_1]_{45} = -s_1\partial^{-1}p_2 - s_2\partial^{-1}p_1,\\
&[\Phi_1]_{51} = v_1\partial^{-1}v_2 + v_2\partial^{-1}v_1 + \partial\frac{(q_1-q_2)}{2}\partial^{-1}q_1 + (r_1q_2 + q_1r_2)\partial^{-1}q_1\\
&- q_1r_1\partial^{-1}q_1 + r_1q_1\partial^{-1}q_2 - \partial\frac{q_1}{2}\partial^{-1}q_2,\\
&[\Phi_1]_{52} = r_2 + v_1\partial^{-1}s_2 + v_2\partial^{-1}s_1 + \partial\frac{(q_1-q_2)}{2}\partial^{-1}p_1 + (r_1q_2 + q_1r_2)\partial^{-1}p_1\\
&- q_1r_1\partial^{-1}p_1 + r_1q_1\partial^{-1}p_2 - \partial\frac{q_1}{2}\partial^{-1}p_2,\\
&[\Phi_1]_{53} = -\partial(q_1 - q_2)\partial^{-1} - 2(r_1q_2 + q_1r_2)\partial^{-1} + \partial q_1\partial^{-1},\\
&[\Phi_1]_{54} = v_1\partial^{-1}q_2 + v_2\partial^{-1}q_1, [\Phi_1]_{55} = v_1\partial^{-1}p_2 + v_2\partial^{-1}p_1,
\end{aligned}
\tag{26}
$$

with $\partial = \frac{\partial}{\partial x}$ and ∂^{-1} as the inverse operator of ∂.

2.2. A specific reduction with two less potentials

A spectral matrix, $\bar{U}$, chosen from $\tilde{\mathfrak{g}}(\lambda)$, is of the form:

$$\bar{U} = M(U(\lambda, u), U_1(\lambda, v)), \tag{27}$$

where r_1, r_2 from (5) are replaced with $\widetilde{r_1} = \frac{1}{2}p_1q_1, \widetilde{r_2} = \frac{1}{2}(p_1q_2 + p_2q_1 - p_1q_1)$, and $\bar{u} = (u, v)^T$, $u = [p_1, q_1, s_1, v_1]^T, v = [p_2, q_2, s_2, v_2]^T$ are potentials. The corresponding spacial spectral problem is

$$\bar{\phi}_x = \bar{U}(\bar{u}, \lambda)\bar{\phi}, \tag{28}$$

where $\bar{\phi} = [\psi, \phi]^T, \psi = [\psi_1, \psi_2]^T$ and $\phi = [\phi_1, \phi_2]^T$.

Again, we assume that the solution to the stationary zero-curvature equation, $\bar{W}_x = [\bar{U}, \bar{W}]$, is of the form as (8). Solving the stationary zero-curvature equation (19), we have the following differential equations as (10) with $r_1 = \widetilde{r_1}$ and $r_2 = \widetilde{r_2}$. By assuming a, b, c, e, f, g, have the Laurent expansions (11), we have the recursion relations

$$
\begin{cases}
b_{i+1} &= \frac{b_{i-1,x}}{2} + p_1a_i + s_1a_{i-1} + \frac{1}{2}p_1q_1b_{i-1},\\
c_{i+1} &= -\frac{c_{i-1,x}}{2} + q_1a_i + v_1a_{i-1} + \frac{1}{2}p_1q_1c_{i-1},\\
a_{i+1,x} &= -q_1\frac{b_{i,x}}{2} - p_1\frac{c_{i,x}}{2} + (p_1v_1 - q_1s_1)a_i - \frac{1}{2}p_1q_1^2b_i\\
&\quad + \frac{1}{2}p_1^2q_1c_i + s_1c_{i+1} - v_1b_{i+1},\\
f_{i+1} &= \frac{f_{i-1,x}}{2} - b_{i+1} + p_2a_i + p_1e_i + s_2a_{i-1} + s_1e_{i-1}\\
&\quad + \frac{1}{2}(p_1q_2 + p_2q_1 - p_1q_1)b_{i-1} + \frac{1}{2}p_1q_1f_{i-1},\\
g_{i+1} &= -\frac{g_{i-1,x}}{2} - c_{i+1} + q_2a_i + q_1e_i + v_2a_{i-1} + v_1e_{i-1}\\
&\quad + \frac{1}{2}(p_1q_2 + p_2q_1 - p_1q_1)c_{i-1} + \frac{1}{2}p_1q_1g_{i-1},\\
e_{i+1,x} &= -\frac{p_1g_{i,x}}{2} - \frac{q_1f_{i,x}}{2} + (p_2 - p_1)[-\frac{c_{m,x}}{2} + v_1a_m + \frac{1}{2}p_1q_1c_m]\\
&\quad + (q_1 - q_2)[\frac{b_{m,x}}{2} + s_1a_m + \frac{1}{2}p_1q_1b_m]\\
&\quad + s_1g_{i+1} + s_2c_{i+1} - v_1f_{i+1} - v_2b_{i+1} + (p_1v_2 - q_1s_2)a_i\\
&\quad + (p_1v_1 - q_1s_1)e_i + \frac{1}{2}p_1(p_1q_2 + p_2q_1 - p_1q_1)c_i\\
&\quad - \frac{1}{2}q_1(p_1q_2 + p_2q_1 - p_1q_1)b_i + \frac{1}{2}p_1^2q_1g_i - \frac{1}{2}p_1q_1^2f_i,
\end{cases}
\tag{29}
$$

for all $i \geq 1$ with the same initial values (13) and conditions for integration (14) which determine the sequence of $\{a_i, b_i, c_i, e_i, f_i, g_i | i \geq 0\}$ uniquely. All $\{a_i, b_i, c_i, e_i, f_i, g_i\}$ can be proven as differential polynomials of $\bar{u}$ with respect to x.

Proposition 2.2. *Let $\{a_i, b_i, c_i, e_i, f_i, g_i | i = 0, 1\}$ be given by Eq. (13). Then all functions $\{a_i, b_i, c_i, e_i, f_i, g_i | i \geq 0\}$ determined by Eqs. (29) with the conditions (14) are differential polynomials in $\bar{u}$ with respect to x, and thus, are local.*

Proof. For brevity, we leave the proof out. It is similar to Proposition 2.1. ■

We solve the zero-curvature equations (19) with the Lax matrices (21) to generate a hierarchy of soliton equations for all $m \geq 0$,

$$\bar{u}_{t_m} = \bar{K}_m = \begin{bmatrix} 2b_{m+1} \\ -2c_{m+1} \\ -2p_1 a_{m+1} + 2b_{m+2} \\ 2q_1 a_{m+1} - 2c_{m+2} \\ 2f_{m+1} + 2b_{m+1} \\ -2g_{m+1} - 2c_{m+1} \\ -2p_1 e_{m+1} - 2p_2 a_{m+1} + 2b_{m+2} + 2f_{m+2} \\ 2q_1 e_{m+1} + 2q_2 a_{m+1} - 2c_{m+2} - 2g_{m+2} \end{bmatrix}. \tag{30}$$

We have

$$\bar{K}_m = \bar{\Phi} \bar{K}_{m-1} = \bar{\Phi}^m \bar{K}_0, \quad m \geq 0, \tag{31}$$

where $\bar{\Phi}$ is a recursion operator determined from (29) and given by

$$\bar{\Phi} = \begin{bmatrix} \Phi & 0 \\ \Phi_1 - \Phi & \Phi \end{bmatrix}. \tag{32}$$

The matrix blocks of $\bar{\Phi}$ are defined by

$$\Phi = \begin{bmatrix} -p_1 \partial^{-1} v_1 & -p_1 \partial^{-1} s_1 & 1 - p_1 \partial^{-1} q_1 & -p_1 \partial^{-1} p_1 \\ q_1 \partial^{-1} v_1 & q_1 \partial^{-1} s_1 & q_1 \partial^{-1} q_1 & 1 + q_1 \partial^{-1} p_1 \\ \frac{1}{2}\partial + \widetilde{r_1} - s_1 \partial^{-1} v_1 & -s_1 \partial^{-1} s_1 & -s_1 \partial^{-1} q_1 & -s_1 \partial^{-1} p_1 \\ v_1 \partial^{-1} v_1 & -\frac{1}{2}\partial + \widetilde{r_1} + v_1 \partial^{-1} s_1 & v_1 \partial^{-1} q_1 & v_1 \partial^{-1} p_1 \end{bmatrix}, \tag{33}$$

and

$$\Phi_1 = \begin{bmatrix} \begin{matrix} -p_1 \partial^{-1} v_2 \\ -p_2 \partial^{-1} v_1 \end{matrix} & \begin{matrix} -p_1 \partial^{-1} s_2 \\ -p_2 \partial^{-1} s_1 \end{matrix} & \begin{matrix} 1 - p_1 \partial^{-1} q_2 \\ -p_2 \partial^{-1} q_1 \end{matrix} & \begin{matrix} -p_1 \partial^{-1} p_2 \\ -p_2 \partial^{-1} p_1 \end{matrix} \\ \begin{matrix} q_1 \partial^{-1} v_2 \\ +q_2 \partial^{-1} v_1 \end{matrix} & \begin{matrix} q_1 \partial^{-1} s_2 \\ +q_2 \partial^{-1} s_1 \end{matrix} & \begin{matrix} q_1 \partial^{-1} q_2 \\ +q_2 \partial^{-1} q_1 \end{matrix} & \begin{matrix} 1 + q_1 \partial^{-1} p_2 \\ +q_2 \partial^{-1} p_1 \end{matrix} \\ \begin{matrix} \widetilde{r_2} - s_1 \partial^{-1} v_2 \\ -s_2 \partial^{-1} v_1 \end{matrix} & \begin{matrix} -s_1 \partial^{-1} s_2 \\ -s_2 \partial^{-1} s_1 \end{matrix} & \begin{matrix} -s_1 \partial^{-1} q_2 \\ -s_2 \partial^{-1} q_1 \end{matrix} & \begin{matrix} -s_1 \partial^{-1} p_2 \\ -s_2 \partial^{-1} p_1 \end{matrix} \\ \begin{matrix} v_1 \partial^{-1} v_2 \\ +v_2 \partial^{-1} v_1 \end{matrix} & \begin{matrix} \widetilde{r_2} + v_2 \partial^{-1} s_1 \\ +v_1 \partial^{-1} s_2 \end{matrix} & \begin{matrix} v_1 \partial^{-1} q_2 \\ +v_2 \partial^{-1} q_1 \end{matrix} & \begin{matrix} v_1 \partial^{-1} p_2 \\ +v_2 \partial^{-1} p_1 \end{matrix} \end{bmatrix}, \tag{34}$$

where $\widetilde{r_1} = \frac{1}{2} p_1 q_1$ and $\widetilde{r_2} = \frac{1}{2}(p_1 q_2 + p_2 q_1 - p_1 q_1)$.

A specific example can be found from the reduced hierarchy of integrable couplings (30) when $m = 6$ by setting the eight potentials and α and β to be the following: $\{p_1 = q_1 = 0, s_1 = u, v_1 = -u, p_2 = q_2 = v, s_2 = w, v_2 = r, \alpha = -4, \beta = -8\}$. We find a coupled mKdV [10,11] system of equations:

$$\begin{cases} u_t = -u_{xxx} - 6u^2 u_x, \\ v_t = -v_{xxx} - 4uvu_x - 2u^2 v_x + (4r - 4w)u^2, \\ w_t = -w_{xxx} + u_{xxx} + (6u^2 + (4v - 4w)u)u_x - 4u^2 v_x - 2u^2 w_x, \\ r_t = -r_{xxx} - u_{xxx} + (-6u^2 + (-4r + 4v)u)u_x - 2u^2 r_x - 2u^2 v_x. \end{cases} \tag{35}$$

3. Hamiltonian structures

3.1. Constructing bilinear forms over a non-semisimple Lie algebra

For generating Hamiltonian structures for the integrable couplings in (22) and (30), we use the variational identity over the non-semisimple Lie algebra $\tilde{\mathfrak{g}}(\lambda)$ [9,12,13]. The variational identity is a generalization of the trace identity and may be applied to non-semisimple Lie algebras which is a limitation of the trace identity [9,29]. The idea is to use non-degenerate, symmetric, and ad-invariant bilinear forms on the Lie algebra. We begin with a construction of the bilinear forms on $\tilde{\mathfrak{g}}(\lambda)$ by rewriting $\tilde{\mathfrak{g}}(\lambda)$ into a vector form.

The isomorphism

$$\sigma : \tilde{\mathfrak{g}}(\lambda) \to \mathbb{R}^6, A \mapsto (a_1, \dots, a_6)^T, \tag{36}$$

where

$$A = M(A_1, A_2) \in \tilde{\mathfrak{g}}(\lambda), \quad A_i = \begin{bmatrix} a_{3i-2} & a_{3i-1} \\ a_{3i} & -a_{3i-2} \end{bmatrix}, i = 1, 2, \tag{37}$$

and a constant symmetric matrix,

$$F = F_1 \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + F_2 \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, F_i = \begin{bmatrix} 2\eta_i & 0 & 0 \\ 0 & 0 & \eta_i \\ 0 & \eta_i & 0 \end{bmatrix}, i = 1, 2, \tag{38}$$

where $\otimes$ is the Kronecker product and arbitrary constants η_1 and η_2, we furnish the bilinear forms on $\tilde{\mathfrak{g}}(\lambda)$ defined as

$$\begin{aligned} \langle A, B \rangle_{\tilde{\mathfrak{g}}(\lambda)} =& \langle \sigma(A), \sigma(B) \rangle_{\mathbb{R}^6} \\ =&(a_1, \dots, a_6) F (b_1, \dots, b_6)^T \\ =&(2a_1b_1 + a_2b_3 + a_3b_2)\eta_1 + (2a_1b_4 + a_2b_6 + a_3b_5 \\ &+ 2a_4b_1 + a_5b_3 + a_6b_2)\eta_2. \end{aligned} \tag{39}$$

The bilinear forms (39) are symmetric and ad-invariant due to the isomorphism σ. The bilinear forms, defined by (39), are non-degenerate if and only if the determinant of F is not zero, i.e.,

$$det(F) = -4\eta_2^6 \neq 0. \tag{40}$$

Therefore, we choose $\eta_2 \neq 0$ to obtain the required non-degenerate, symmetric, and ad-invariant bilinear forms over the enlarged matrix loop algebra $\tilde{\mathfrak{g}}(\lambda)$. For simplicity, we choose $\eta_1 = 0$ and $\eta_2 = 1$.

3.2. Hamiltonian structures of generalized D-KN integrable couplings

Now, we begin with the enlarged spectral matrix of a generalized D-KN hierarchy (7) and compute

$$\langle \bar{W}, \bar{U}_\lambda \rangle_{\tilde{\mathfrak{g}}(\lambda)} = (4a + 4e)\lambda + fq_1 + bq_2 + cp_2 + gp_1, \tag{41}$$

and

$$\langle \bar{W}, \bar{U}_{\bar{u}} \rangle_{\tilde{\mathfrak{g}}(\lambda)} = [g\lambda, f\lambda, -2e, g, f, c\lambda, b\lambda, -2a, c, b]^T. \tag{42}$$

Substituting the Laurent series and comparing powers of λ, we have

$$\begin{aligned} &\frac{\delta}{\delta \bar{u}} \int \frac{(4a_{m+2} + 4e_{m+2}) + f_{m+1}q_1 + b_{m+1}q_2 + c_{m+1}p_2 + g_{m+1}p_1}{m} dx = \\ &[g_{m+1}, f_{m+1}, -2e_m, g_m, f_m, c_{m+1}, b_{m+1}, -2a_m, c_m, b_m]^T, \ m \geq 1. \end{aligned} \tag{43}$$

A long calculation involving the recursion relations (12) shows that

$$\frac{\delta\bar{\mathcal{H}}_{m+1}}{\delta\bar{u}} = \bar{\Psi}\frac{\delta\bar{\mathcal{H}}_m}{\delta\bar{u}}, \tag{44}$$

where

$$\bar{\Psi} = \bar{\Phi}^{\dagger} = \begin{bmatrix} \Phi^{\dagger} & (\Phi_1 - \Phi)^{\dagger} \\ 0 & \Phi^{\dagger} \end{bmatrix}, \tag{45}$$

with Φ and Φ_1 from (24), (25) and (26), respectively. We consequently obtain Hamiltonian structures for the hierarchy of integrable couplings (22), i.e.,

$$\bar{u}_{t_m} = \bar{J}\frac{\delta\bar{\mathcal{H}}_m}{\delta\bar{u}}, \quad m \geq 0, \tag{46}$$

with the Hamiltonian functionals,

$$\bar{\mathcal{H}}_m = \int \frac{(4a_{m+2} + 4e_{m+2}) + f_{m+1}q_1 + b_{m+1}q_2 + c_{m+1}p_2 + g_{m+1}p_1}{m} dx, \tag{47}$$

for $m \geq 1$, and

$$\bar{\mathcal{H}}_0 = \int [(\beta - \alpha)p_1q_1 + \alpha(p_1q_2 + p_2q_1) - 2\beta r_1 - 2\alpha r_2]dx, \tag{48}$$

calculated directly from $[g_1, f_1, -2e_0, g_0, f_0, c_1, b_1, -2a_0, c_0, b_0]^T$. The Hamiltonian operator in (46) is the block matrix of the form:

$$\bar{J} = \begin{bmatrix} 0 & J_1 \\ J_1 & J_2 \end{bmatrix}, \tag{49}$$

where

$$J_1 = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\partial & s_1 & -v_1 \\ 0 & 0 & -s_1 & 0 & \partial + 2r_1 \\ 0 & 0 & v_1 & \partial - 2r_1 & 0 \end{bmatrix}, J_2 = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s_2 & -v_2 \\ 0 & 0 & -s_2 & 0 & 2r_2 \\ 0 & 0 & v_2 & -2r_2 & 0 \end{bmatrix}. \tag{50}$$

As a direct result of the Hamiltonian structures (46), the recursion structure (23) and (44), and the property $\bar{J}\bar{\Psi} = \bar{\Psi}^{\dagger}\bar{J}$, the hierarchy (22) has the following commutativity of flows

$$\{\bar{\mathcal{H}}_k, \bar{\mathcal{H}}_l\}_{\bar{J}} = \int \left(\frac{\delta\bar{\mathcal{H}}_k}{\delta\bar{u}}\right)^T \bar{J}\frac{\delta\bar{\mathcal{H}}_l}{\delta\bar{u}} dx = 0. \tag{51}$$

We also have the commutativity of high-order symmetries for $\{\bar{K}_n\}$, i.e.,

$$[\bar{K}_k, \bar{K}_l] = \bar{K}_k'(\bar{u})[\bar{K}_l] - \bar{K}_l'(\bar{u})[\bar{K}_k] = 0, \quad k, l \geq 0. \tag{52}$$

Therefore, the hierarchy (22) is Liouville integrable, as expected.

3.3. Bi-Hamiltonian structures of the reduced integrable couplings

Next, we focus on the reduced spectral matrix (28) and compute

$$\langle\bar{W}, \bar{U}_\lambda\rangle_{\tilde{\mathfrak{g}}(\lambda)} = (4a + 4e)\lambda + fq_1 + bq_2 + cp_2 + gp_1, \tag{53}$$

and

$$\begin{aligned}\langle\bar{W}, \bar{U}_{\bar{u}}\rangle_{\tilde{\mathfrak{g}}(\lambda)} =& [(a - e)q_1 - aq_2 + g\lambda, (a - e)p_1 - ap_2 + f\lambda, g, f, \\ & - aq_1 + c\lambda, -ap_1 + b\lambda, c, b]^T.\end{aligned} \tag{54}$$

Again, we compare powers of λ after substituting the Laurent series for a, b, c, e, f, g to get

$$\begin{aligned}\frac{\delta}{\delta\bar{u}}\int\frac{(4a_{m+2}+4e_{m+2})+f_{m+1}q_1+b_{m+1}q_2+c_{m+1}p_2+g_{m+1}p_1}{m}dx = \\ [(a_m-e_m)q_1-a_mq_2+g_{m+1},(a_m-e_m)p_1-a_mp_2+f_{m+1},g_m,f_m \\ -a_mq_1+c_{m+1},-a_mp_1+b_{m+1},c_m,b_m]^T, \;\; m\geq 1.\end{aligned} \tag{55}$$

Now using the recursion relations (29), we have

$$\frac{\delta\bar{\mathcal{H}}_{m+1}}{\delta\bar{u}} = \bar{\Psi}\frac{\delta\bar{\mathcal{H}}_m}{\delta\bar{u}}, \tag{56}$$

where

$$\bar{\Psi} = \bar{\Phi}^\dagger = \begin{bmatrix}\Phi^\dagger & (\Phi_1-\Phi)^\dagger \\ 0 & \Phi^\dagger\end{bmatrix}, \tag{57}$$

with Φ and Φ_1 from (33) and (34), respectively.

We finally obtain the bi-Hamiltonian structure for the hierarchy of integrable couplings (30),

$$\bar{u}_{t_m} = \bar{J}\frac{\delta\bar{\mathcal{H}}_{m+1}}{\delta\bar{u}} = \bar{M}\frac{\delta\bar{\mathcal{H}}_m}{\delta\bar{u}}, \;\; m\geq 0, \tag{58}$$

with the Hamiltonian functionals

$$\bar{\mathcal{H}}_m = \int\frac{(4a_{m+2}+4e_{m+2})+f_{m+1}q_1+b_{m+1}q_2+c_{m+1}p_2+g_{m+1}p_1}{m}dx, \tag{59}$$

for $m \geq 1$, and the block Hamiltonian operators $\bar{J}$ given by

$$\bar{J} = \begin{bmatrix}0 & J \\ J & J\end{bmatrix}, J = \begin{bmatrix}0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0\end{bmatrix}, \tag{60}$$

and $\bar{M} = \bar{\Phi}\bar{J}$ where $\bar{J}$ is above (60) and $\bar{\Phi}$ is the recursion operator (32) for the reduced integrable couplings (30). Recall, a bi-Hamiltonian property means that $\bar{J}$ and $\bar{M}$ constitute a Hamiltonian pair, or, $\bar{N} = \alpha\bar{J}+\beta\bar{M}$, for any $\alpha, \beta \in \mathbb{R}$, is a Hamiltonian operator. As a direct result of the bi-Hamiltonian structure (58), we can say that the soliton hierarchy (30) is integrable in the Liouville sense with

$$\begin{cases}\{\bar{\mathcal{H}}_k,\bar{\mathcal{H}}_l\}_{\bar{J}} = & \int\left(\frac{\delta\bar{\mathcal{H}}_k}{\delta\bar{u}}\right)^T\bar{J}\frac{\delta\bar{\mathcal{H}}_l}{\delta\bar{u}}dx = 0, \\ \{\bar{\mathcal{H}}_k,\bar{\mathcal{H}}_l\}_{\bar{M}} = & \int\left(\frac{\delta\bar{\mathcal{H}}_k}{\delta\bar{u}}\right)^T\bar{M}\frac{\delta\bar{\mathcal{H}}_l}{\delta\bar{u}}dx = 0,\end{cases} \tag{61}$$

and

$$[\bar{K}_k,\bar{K}_l] = \bar{K}_k'(\bar{u})[\bar{K}_l]-\bar{K}_l'(\bar{u})[\bar{K}_k] = 0, \quad k,l\geq 0. \tag{62}$$

4. Concluding remarks

We have introduced a newly studied spectral matrix that is a generalization of the D-Kaup–Newell spectral problems. Integrable couplings resulted from enlarging this spectral matrix and solving zero-curvature equations. The Hamiltonian structures of the integrable couplings were constructed and presented. The reduced hierarchy of integrable couplings was found to be bi-Hamiltonian and both hierarchies are Liouville integrable. These enlarged hierarchies are both new and different from one another. Many new equations found in these hierarchies have been found to be integrable like the coupled mKdV presented in (35).

This paper uses a relatively new idea of having $\frac{\partial U_1}{\partial \lambda} \neq 0$ in the enlarged spectral matrix $\bar{U}$ where most integrable couplings are generated through perturbations and $\frac{\partial U_1}{\partial \lambda} = 0$. Although the calculations are more difficult and tedious, many new applications may arise from integrable couplings starting from enlarged spectral matrices of this form. One application is the Darboux transformation method for the construction of solutions to integrable couplings [25,26]. This construction also creates new integrable systems associated with non-semisimple Lie algebras and brings us new insightful thoughts to classify integrable systems from an algebraic point of view.

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