Bull. Malays. Math. Sci. Soc. (2019) 42:1921–1935
https://doi.org/10.1007/s40840-017-0580-y

BULLETIN of the
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

Bi-Integrable Couplings Associated with so(3, $\mathbb{R}$)

Morgan McAnally[1] · Wen-Xiu Ma[1,2,3,4,5]

Received: 10 April 2017 / Revised: 10 November 2017 / Published online: 29 November 2017

Abstract By a class of zero curvature equations over a non-semisimple matrix loop algebra, we generate a new hierarchy of bi-integrable couplings for a soliton hierarchy associated with so(3, $\mathbb{R}$). The bi-Hamiltonian structures are found by the associated variational identity, which imply that all the presented coupling systems possess infinitely many commuting symmetries and conserved functionals and, thus, are Liouville integrable.

Keywords Integrable coupling · Matrix loop algebra · Hamiltonian structure

Communicated by Ahmad Izani Md. Ismail.

This research was supported by the University of South Florida.

✉ Morgan McAnally
mamcanal@mail.usf.edu

Wen-Xiu Ma
mawx@cas.usf.edu

1 University of South Florida, Tampa, FL 33620-5700, USA

2 Department of Mathematics,Zhejiang Normal University, Jinhua 321004, Zhejiang, People's Republic of China

3 College of Mathematics and Systems Science,Shandong University of Science and Technology, Qingdao 266590, Shandong, People's Republic of China

4 Department of Mathematical Sciences, International Institute for Symmetry Analysis and Mathematical Modelling, North-West University, Mafikeng Campus, Private Bag X2024, Mmabatho 2735, South Africa

5 College of Mathematics and Physics, Shanghai University of Electric Power, Shanghai 200090, People's Republic of China

Mathematics Subject Classification 37K05 · 37K10 · 35Q53

1 Introduction

The study of solitons in regard to integrable systems has facilitated a deeper understanding of mathematics and physics. Many well-known nonlinear partial differential equations have been found to have soliton solutions, for example, the Korteweg–de Vries equation and the sine-Gordon equation. It is known that zero curvature equations associated with simple Lie algebras generate classical integrable systems [1], and semisimple Lie algebras generate non-coupled systems of classical integrable systems. It is our business to further develop the study of non-semisimple Lie algebras in relation to integrable couplings. Soliton hierarchies, and specifically, integrable couplings and bi-integrable couplings, provide valuable new insights into the classification of multi-component integrable systems [2–6].

It is known that zero curvature equations on semidirect sums of matrix loop algebras generate integrable couplings [7,8], and the associated variational identity [9,10] is used to furnish Hamiltonian structures and bi-Hamiltonian structures of the resulting integrable couplings and bi-integrable couplings [11–17]. An important step in generating Hamiltonian structures is to search for non-degenerate, symmetric, and ad-invariant bilinear forms on the underlying loop algebras [13,18] as the trace identity proposed by Gui-Zhang Tu [18,19] is ineffective for non-semisimple Lie algebras which possess a degenerate Killing form. Semidirect sums of loop algebras bring various interesting integrable couplings and bi-integrable couplings [20–24], including higher-dimensional local bi-Hamiltonian integrable couplings [25–29], greatly enriching multi-component integrable systems. Recently, it has been of interest to study new integrable couplings and bi-integrable couplings generated from spectral problems associated with so(3, $\mathbb{R}$) [14].

Integrable couplings enlarge an original integrable system and often times retain its properties [2,4]. Bi-integrable couplings then take the integrable coupling system and enlarge that system. Again, the original properties frequently are maintained. An important feature is if a soliton hierarchy has infinitely many commuting symmetries and conserved densities, the integrable coupling and then bi-integrable coupling generally will too [14–17,30,31]. A bi-integrable coupling system is a natural way of extending a well-behaved integrable system. We show that the bi-integrable couplings of an original spectral problem associated with so(3, $\mathbb{R}$) will preserve bi-Hamiltonian structures, i.e., Liouville integrability, of the integrable couplings associated with the same spectral problem [32].

A zero curvature representation of a system of the form

$$u_t = K(u) = K(x, t, u, u_x, u_{xx}, \ldots), \tag{1}$$

where u is a column vector of dependent variables and means there exists a Lax pair [33] $U = U(u, \lambda)$ and $V = V(u, \lambda)$ in a matrix loop algebra such that the zero curvature equation,

$$U_t - V_x + [U, V] = 0, \tag{2}$$

will generate system (1) [19]. The integrable coupling of system (1) is an integrable system of the form ([25,26] for definition):

$$\bar{u}_t = \bar{K}_1(\bar{u}) = \begin{bmatrix} K(u) \\ S(u, u_1) \end{bmatrix}, \qquad \bar{u} = \begin{bmatrix} u \\ u_1 \end{bmatrix}, \tag{3}$$

where u_1 is a new column vector of dependent variables. An integrable system of the form

$$\bar{u}_t = \bar{K}_1(\bar{u}) = \begin{bmatrix} K(u) \\ S_1(u, u_1) \\ S_2(u, u_1, u_2) \end{bmatrix}, \qquad \bar{u} = \begin{bmatrix} u \\ u_1 \\ u_2 \end{bmatrix}, \tag{4}$$

is called a bi-integrable coupling of (1). Note that in (4), S_2 depends on u_2, but S_1 does not. Now, we use zero curvature equations in order to generate bi-integrable couplings and associated Hamiltonian structures, through appropriate variational identities.

We will proceed with Sects. 2 through 6. In Sect. 2, we recall a soliton hierarchy presented in [32] for a matrix spectral problem in so(3, ℝ). In Sect. 3, we construct bi-integrable couplings from the results in Sect. 2 using an enlarged matrix loop algebra. We then use the corresponding variational identity to present the Hamiltonian structure of the bi-integrable coupling system in Sect. 4. In Sect. 5, infinitely many symmetries and conserved functionals are discussed. We finish the paper with a couple open questions.

2 A Soliton Hierarchy Associated with so(3, ℝ)

Let us recall the a soliton hierarchy [32] given by the spectral problem

$$\phi_x = U\phi\,, \quad U = U(u,\lambda) = \begin{bmatrix} 0 & q & \lambda \\ -q & 0 & -p \\ -\lambda & p & 0 \end{bmatrix} \in \bar{so}(3), \tag{5}$$

where

$$u = \begin{bmatrix} p \\ q \end{bmatrix}, \quad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix},$$

λ is a spectral parameter, $p = p(x,t)$, $q = q(x,t)$, and $\bar{so}(3)$ is the special matrix loop algebra, i.e.,

$$\bar{\mathfrak{g}} = \bar{so}(3) = \{A \in so(3) | \text{entries of } A \text{ are Laurent series in } \lambda\}\,. \tag{6}$$

Under the assumption that W is of the form

$$W = \begin{bmatrix} 0 & c & a \\ -c & 0 & -b \\ -a & b & 0 \end{bmatrix} = \sum_{i\ge 0} \begin{bmatrix} 0 & c_i & a_i \\ -c_i & 0 & -b_i \\ -a_i & b_i & 0 \end{bmatrix} \lambda^{-i} = \sum_{i\ge 0} W_i \lambda^{-i}, \tag{7}$$

then the stationary zero curvature equation,

$$W_x = [U, W], \tag{8}$$

determines the system of equations

$$\begin{cases} a_x = pc - qb, \\ b_x = -\lambda c + qa, \\ c_x = -pa + \lambda b. \end{cases} \tag{9}$$

After setting a, b, c to appropriate Laurent expansions, system (9) equivalently generates

$$\begin{cases} b_{i+1} = pa_i + c_{i,x}, \\ c_{i+1} = -b_{i,x} + qa_i, \qquad i \geq 0. \\ a_{i+1,x} = pc_{i+1} - qb_{i+1}, \end{cases} \tag{10}$$

Next, we set the initial conditions as $\{a_0 = -1,\ b_0 = 0 = c_0\}$ and take all constants of integration to be zero. We can present for $1 \leq i \leq 4$:

$$\begin{aligned} &a_1 = 0, \quad c_1 = -q, \quad b_1 = -p, \\ &a_2 = \frac{1}{2}(p^2 + q^2), \quad c_2 = p_x, \quad b_2 = -q_x, \\ &a_3 = pq_x - p_x q, \quad c_3 = q_{xx} + \frac{1}{2}p^2 q + \frac{1}{2}q^3, \quad b_3 = p_{xx} + \frac{1}{2}p^3 + \frac{1}{2}pq^2, \\ &a_4 = -\frac{3}{4}p^2q^2 - \frac{3}{8}p^4 + \frac{1}{2}p_x^2 - pp_{xx} - \frac{3}{8}q^4 + \frac{1}{2}q_x^2 - qq_{xx}, \\ &b_4 = q_{xxx} + \frac{1}{2}(3p^2 + 3q^2)q_x, \quad c_4 = -p_{xxx} - \frac{1}{2}(3p^2 + 3q^2)p_x. \end{aligned}$$

All functions $\{a_i, b_i, c_i | i \geq 0\}$ are differential polynomials of u with respect to x.

The zero curvature equations are

$$U_{t_m} - V_x^{[m]} + \left[U, V^{[m]}\right] = 0 \quad \text{with} \quad V^{[m]} = (\lambda^m W)_+, \tag{11}$$

where $m \geq 0$, and, therefore, provide a hierarchy of soliton equations, i.e.,

$$u_{t_m} = K_m = \begin{bmatrix} -c_{m+1} \\ b_{m+1} \end{bmatrix} = \Phi^m \begin{bmatrix} q \\ -p \end{bmatrix} = J \frac{\delta \mathcal{H}_m}{\delta u}, \tag{12}$$

where $m \geq 0$. The Hamiltonian operator J, the hereditary recursion operator Φ, and the Hamiltonian functions are defined as follows:

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \Phi = \begin{bmatrix} q\partial^{-1}p & \partial + q\partial^{-1}q \\ -\partial - p\partial^{-1}p & -p\partial^{-1}q \end{bmatrix}, \quad \mathcal{H}_m = \int -\frac{a_{m+2}}{m+1}\, \mathrm{d}x, \tag{13}$$

in which $m \geq 0$ and $\partial = \frac{\partial}{\partial x}$. The first nonlinear example is

$$u_{t_2} = K_2 = \begin{bmatrix} -q_{xx} - \frac{1}{2}p^2 q - \frac{1}{2}q^3 \\ p_{xx} + \frac{1}{2}p^3 + \frac{1}{2}pq^2 \end{bmatrix} = J \begin{bmatrix} p_{xx} + \frac{1}{2}p^3 + \frac{1}{2}pq^2 \\ q_{xx} + \frac{1}{2}p^2 q + \frac{1}{2}q^3 \end{bmatrix} = J\frac{\delta \mathcal{H}_2}{\delta u}. \quad (14)$$

3 Bi-Integrable Couplings

We construct Hamiltonian bi-integrable couplings for the soliton hierarchy by using a matrix loop Lie algebra. Define a triangular block matrix

$$M(A_1, A_2, A_3) = \begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & A_1 & \alpha A_2 \\ 0 & 0 & A_1 \end{bmatrix}. \quad (15)$$

It is known that block matrices of this form are closed under multiplication, i.e., constitute a Lie algebra [34]. The associated loop matrix Lie algebra $\tilde{\mathfrak{g}}(\lambda)$ is formed by all block matrices of the type

$$\tilde{\mathfrak{g}}(\lambda) = \{M(A_1, A_2, A_3) | M \text{ defined by (15), entries of A are Laurent series in } \lambda\}. \quad (16)$$

A spectral matrix is chosen from $\tilde{\mathfrak{g}}(\lambda)$ as

$$\bar{U} = \bar{U}(\bar{u}, \lambda) = M(U, U_1, U_2), \qquad \bar{u} = (p, q, r, s, v, w)^{\mathrm{T}}, \quad (17)$$

where U is defined as in (5) and the supplementary spectral matrices U_1 and U_2 are

$$U_1 = U_1(u_1) = \begin{bmatrix} 0 & s & 0 \\ -s & 0 & -r \\ 0 & r & 0 \end{bmatrix}, \qquad u_1 = \begin{bmatrix} r \\ s \end{bmatrix}, \quad (18)$$

$$U_2 = U_2(u_2) = \begin{bmatrix} 0 & w & 0 \\ -w & 0 & -v \\ 0 & v & 0 \end{bmatrix}, \qquad u_2 = \begin{bmatrix} v \\ w \end{bmatrix}. \quad (19)$$

In order to solve the enlarged stationary zero curvature equation,

$$\bar{W}_x = [\bar{U}, \bar{W}], \quad (20)$$

we take the solution to be of the following form:

$$\bar{W} = \bar{W}(\bar{u}, \lambda) = M(W, W_1, W_2) \in \tilde{\mathfrak{g}}(\lambda), \quad (21)$$

where W is defined by (7) and solves $W_x = [U, W]$, and W_1 and W_2 are assumed to be

$$W_1 = W_1(u, u_1, \lambda) = \begin{bmatrix} 0 & g & e \\ -g & 0 & -f \\ -e & f & 0 \end{bmatrix} \in \bar{so}(3), \quad (22)$$

and

$$W_2 = W_2(u, u_1, u_2, \lambda) = \begin{bmatrix} 0 & g' & e' \\ -g' & 0 & -f' \\ -e' & f' & 0 \end{bmatrix} \in \bar{so}(3). \tag{23}$$

Equation (20) is equivalent to satisfying the following matrix equations:

$$\begin{cases} W_x = [U, W], \\ W_{1,x} = [U, W_1] + [U_1, W], \\ W_{2,x} = [U, W_2] + [U_2, W] + \alpha[U_1, W_1]. \end{cases} \tag{24}$$

The second and third equations in (24) generate

$$\begin{cases} e_x = pg - qf + rc - sb, \\ f_x = -\lambda g + qe + sa, \\ g_x = -pe + \lambda f - ra, \end{cases} \tag{25}$$

and

$$\begin{cases} e'_x = -fs\alpha + gr\alpha - qf' + pg' - wb + vc, \\ f'_x = qe' - \lambda g' + wa + se\alpha, \\ g'_x = -pe' + \lambda f' - re\alpha - va, \end{cases} \tag{26}$$

respectively. Plugging into recursion relations (25) and (26) into the Laurent expansions,

$$\begin{cases} e = \sum_{i\geq 0} e_i\lambda^{-i}, \quad f = \sum_{i\geq 0} f_i\lambda^{-i}, \quad g = \sum_{i\geq 0} g_i\lambda^{-i}, \\ e' = \sum_{i\geq 0} e'_i\lambda^{-i}, \quad f' = \sum_{i\geq 0} f'_i\lambda^{-i}, \quad g' = \sum_{i\geq 0} g'_i\lambda^{-i}, \end{cases} \tag{27}$$

we have

$$\begin{cases} f_{i+1} = g_{i,x} + pe_i + ra_i, \\ g_{i+1} = -f_{i,x} + qe_i + sa_i, \\ e_{i+1,x} = pg_{i+1} - qf_{i+1} + rc_{i+1} - sb_{i+1}, \\ f'_{i+1} = g'_{i,x} + pe'_i + va_i + \alpha rc_i, \\ g'_{i+1} = -f'_{i,x} + qe'_i + wa_i + \alpha sc_i, \\ e'_{i+1,x} = pg'_{i+1} - qf'_{i+1} - \alpha sf_{i+1} + \alpha rg_{i+1} - wb_{i+1} + vc_{i+1}, \end{cases} \tag{28}$$

where $i \geq 0$. We take the initial data as $\{e_0 = -1,\ f_0 = g_0 = 0;\ e'_0 = -1,\ f'_0 = g'_0 = 0\}$ and suppose that the integration constants are zero. Then, recursion relation (28) uniquely generates $\{e_i, f_i, g_i, e'_i, f'_i, g'_i | i \geq 1\}$. We obtain

$$
\begin{cases}
e_1 = 0, \\
f_1 = -p - r, \\
g_1 = -q - s; \\
e_2 = \frac{1}{2}p^2 + \frac{1}{2}q^2 + rp + sq, \\
f_2 = -q_x - s_x, \\
g_2 = p_x + r_x; \\
e_3 = q_x p - q p_x - s p_x + r q_x + s_x p - r_x q, \\
f_3 = p_{xx} + \frac{1}{2}p^3 + \frac{1}{2}pq^2 + \frac{3}{2}rp^2 + psq + \frac{1}{2}rq^2 + r_{xx}, \\
g_3 = q_{xx} + \frac{1}{2}q^3 + \frac{1}{2}qp^2 + \frac{3}{2}sq^2 + qrp + \frac{1}{2}sp^2 + s_{xx}; \\
e_4 = (-p - r)p_{xx} + (-q - s)q_{xx} - pr_{xx} - qsxx\frac{1}{2}p_x^2 + p_x r_x + \frac{1}{2}q_x^2 + q_x s_x \\
\quad -\frac{3}{8}(p^2 + q^2)(p^2 + 4pr + q(q + 4s)), \\
f_4 = q_{xxx} + s_{xxx} + \frac{1}{2}(3p^2 + 6pr + 3q^2 + 6qs)q_x + \frac{1}{2}(3p^2 + 3q^2)s_x, \\
g_4 = -p_{xxx} - r_{xxx} + \frac{1}{2}(-3p^2 - 6pr - 3q^2 - 6qs)p_x + \frac{1}{2}(-3p^2 - 3q^2)r_x;
\end{cases}
$$

and

$$
\begin{cases}
e_1' = 0, \\
f_1' = -p - \alpha r - v, \\
g_1' = -q - \alpha s - w; \\
e_2' = \frac{1}{2}p^2 + \frac{1}{2}q^2 + \alpha rp + \alpha sq + vp + wq + \frac{1}{2}\alpha s^2 + \frac{1}{2}\alpha r^2, \\
f_2' = -q_x - \alpha s_x - w_x, \\
g_2' = p_x + \alpha r_x + v_x; \\
e_3' = q_x p - q p_x - \alpha s p_x + \alpha r q_x + \alpha s_x p - r_x q - w p_x + v q_x + w_x p - \alpha q r_x \\
\quad - v_x q + \alpha s_x r - \alpha s r_x, \\
f_3' = p_{xx} + \frac{1}{2}p^3 + \frac{1}{2}pq^2 + \alpha\frac{3}{2}rp^2 + \alpha psq + \alpha\frac{1}{2}rq^2 + r_{xx} + \alpha\frac{3}{2}pr^2 + \alpha rsq \\
\quad + pqw + \frac{3}{2}vp^2 + \frac{1}{2}vq^2 + \frac{1}{2}\alpha ps^2 + \alpha r_{xx} + v_{xx}, \\
g_3' = q_{xx} + \frac{1}{2}q^3 + \frac{1}{2}qp^2 + \alpha\frac{3}{2}sq^2 + \alpha qrp + \alpha\frac{1}{2}sp^2 + s_{xx} + \alpha\frac{3}{2}qs^2 + \alpha srp \\
\quad + pvq + \frac{3}{2}wq^2 + \frac{1}{2}wp^2 + \frac{1}{2}\alpha pr^2 + \alpha s_{xx} + w_{xx}; \\
e_4' = (-\alpha r - p - v)p_{xx} + (-\alpha s - q - w)q_{xx} - \alpha(p + r)r_{xx} - \alpha(q + s)s_{xx} \\
\quad - v_{xx}p - w_{xx}q + \frac{1}{2}p_x^2 + (\alpha r_x + v_x)p_x \\
\quad + \frac{1}{2}q_x^2 + (\alpha s_x + w_x)q_x + \frac{1}{2}\alpha r_x^2 + \frac{1}{2}\alpha s_x^2 - \frac{3}{8}p^4 + \frac{3}{2}(-\alpha r - v)p^3 \\
\quad + \frac{1}{8}(-6q^2 + (-12\alpha s - 12w)q - 18\alpha r^2 - 6\alpha s^2)p^2 \\
\quad - \frac{3}{2}q((\alpha r + v)q + 2\alpha rs)p - \frac{3}{2}q^2\left(\frac{1}{4}q^2 + (\alpha s + w)q + \frac{1}{2}\alpha(r^2 + 3s^2)\right), \\
f_4' = q_{xxx} + \alpha s_{xxx} + w_{xxx} + \frac{1}{2}(3p^2 + (6\alpha r + 6v)p + 3q^2 + (6\alpha s + 6w)q \\
\quad + 3\alpha r^2 + 3\alpha s^2)q_x + \frac{1}{2}(3\alpha p^2 + 6\alpha pr + 3\alpha q^2 + 6\alpha qs)s_x \\
\quad + \frac{1}{2}(3p^2 + 3q^2)w_x, \\
g_4' = -p_{xxx} - \alpha r_{xxx} - v_{xxx} + \frac{1}{2}(-3p^2 - (6\alpha r + 6v)p - 3q^2 - (6\alpha s + 6w)q \\
\quad - 3\alpha r^2 - 3\alpha s^2)p_x + \frac{1}{2}(-3\alpha p^2 - 6\alpha pr - 3\alpha q^2 - 6\alpha qs)r_x \\
\quad - \frac{1}{2}(3p^2 + 3q^2)v_x.
\end{cases}
$$

These functions are differential polynomials in the variables p, q, r, s, v, and w.

Similar to [35], for each integer $m \geq 0$, we further introduce an enlarged Lax matrix

$$\bar{V}^{[m]} = (\lambda^m \bar{W})_+ = M\left(V^{[m]}, V_1^{[m]}, V_2^{[m]}\right) \in \tilde{\mathfrak{g}}(\lambda), \tag{29}$$

where $V^{[m]}$ is defined by (11) and $V_i^{[m]} = (\lambda^m W_i)_+, i = 1, 2$. The enlarged zero curvature equation,

$$\bar{U}_{t_m} - \bar{V}_x^{[m]} + \left[\bar{U}, \bar{V}^{[m]}\right] = 0, \tag{30}$$

gives the following matrix equations:

$$\begin{cases} U_{1,t_m} - V_{1,x}^{[m]} + \left[U, V_1^{[m]}\right] + \left[U_1, V^{[m]}\right] = 0, \\ U_{2,t_m} - V_{2,x}^{[m]} + \left[U, V_2^{[m]}\right] + \left[U_2, V^{[m]}\right] + \alpha\left[U_1, V_1^{[m]}\right] = 0, \end{cases} \tag{31}$$

along with the system in (11). The above equations then present the additional systems

$$\bar{v}_{t_m} = S_m = S_m(\bar{v}) = \begin{bmatrix} S_{1,m}(u, u_1) \\ S_{2,m}(u, u_1, u_2) \end{bmatrix}, \quad m \geq 0, \tag{32}$$

where $\bar{v} = (r, s, v, w)^{\mathrm{T}}$ and

$$S_{1,m}(u, u_1) = \begin{bmatrix} -g_{m+1} \\ f_{m+1} \end{bmatrix},$$

and

$$S_{2,m}(u, u_1, u_2) = \begin{bmatrix} -g'_{m+1} \\ f'_{m+1} \end{bmatrix}.$$

Then the enlarged zero curvature equation generates a hierarchy of bi-integrable couplings,

$$\bar{u}_{t_m} = \begin{bmatrix} p \\ q \\ r \\ s \\ v \\ w \end{bmatrix}_{t_m} = \begin{bmatrix} -c_{m+1} \\ b_{m+1} \\ -g_{m+1} \\ f_{m+1} \\ -g'_{m+1} \\ f'_{m+1} \end{bmatrix} = \bar{K}_m(\bar{u}), \quad m \geq 0, \tag{33}$$

for soliton hierarchy (12).

In particular, when $m = 2$, we have $u_{t_2} = \bar{K}_2$, i.e.,

$$\begin{bmatrix} p \\ q \\ r \\ s \\ v \\ \\ w \\ \\ \end{bmatrix}_{t_2} = \begin{bmatrix} -q_{xx} - \frac{1}{2}p^2q - \frac{1}{2}q^3 \\ p_{xx} + \frac{1}{2}p^3 + \frac{1}{2}pq^2 \\ -q_{xx} - \frac{1}{2}q^3 - \frac{1}{2}qp^2 - \frac{3}{2}sq^2 - qrp - \frac{1}{2}sp^2 - s_{xx} \\ p_{xx} + \frac{1}{2}p^3 + \frac{1}{2}pq^2 + \frac{3}{2}rp^2 + psq + \frac{1}{2}rq^2 + r_{xx} \\ -q_{xx} - \frac{1}{2}q^3 - \frac{1}{2}qp^2 - \alpha\frac{3}{2}sq^2 - \alpha qrp - \alpha\frac{1}{2}sp^2 - s_{xx} - \alpha\frac{3}{2}qs^2 - \alpha srp \\ -pvq + \frac{3}{2}wq^2 - \frac{1}{2}wp^2 - \frac{1}{2}\alpha ps^2 - \alpha s_{xx} - w_{xx} \\ p_{xx} + \frac{1}{2}p^3 + \frac{1}{2}pq^2 + \alpha\frac{3}{2}rp^2 + \alpha psq + \alpha\frac{1}{2}rq^2 + r_{xx} + \alpha\frac{3}{2}pr^2 + \alpha rsq \\ +pqw + \frac{3}{2}vp^2 + \frac{1}{2}vq^2 + \frac{1}{2}\alpha ps^2 + \alpha r_{xx} + v_{xx} \end{bmatrix}. \tag{34}$$

4 Hamiltonian Structures

We have a systematic approach for generating Hamiltonian structures for the bi-integrable coupling in (33) using the variational identity over the enlarged matrix loop algebra $\tilde{\mathfrak{g}}(\lambda)$ [13,18]. The variational identity is as follows:

$$\frac{\delta}{\delta\bar{u}}\int\langle\bar{W},\bar{U}_\lambda\rangle\mathrm{d}x = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\langle\bar{W},\bar{U}_{\bar{u}}\rangle, \quad \gamma = \text{constant}. \tag{35}$$

As seen in [35], there is a convenient method to constructing a symmetric and ad-invariant bilinear form on $\tilde{\mathfrak{g}}(\lambda)$ by rewriting the semidirect sum $\tilde{\mathfrak{g}}(\lambda)$ into a vector form. First, we define a mapping

$$\sigma : \tilde{\mathfrak{g}}(\lambda) \mapsto \mathbb{R}^9, A \mapsto (a_1, \ldots, a_9)^{\mathrm{T}}, \tag{36}$$

where

$$A = M(A_1, A_2, A_3) \in \tilde{\mathfrak{g}}(\lambda), \quad A_i = \begin{bmatrix} 0 & a_{3i} & a_{3i-2} \\ -a_{3i} & 0 & -a_{3i-1} \\ -a_{3i-2} & a_{3i-1} & 0 \end{bmatrix}, \quad 1 \le i \le 3. \tag{37}$$

The map σ induces a Lie algebra structure on $\mathbb{R}^9$ isomorphic to the enlarged matrix loop algebra $\tilde{\mathfrak{g}}(\lambda)$. Thus, the corresponding Lie bracket $[\cdot,\cdot]$ on $\mathbb{R}^9$ is generated by letting

$$[a, b]^{\mathrm{T}} = a^{\mathrm{T}} R(b), \tag{38}$$

where $a = (a_1, \ldots, a_9)^{\mathrm{T}}, b = (b_1, \ldots, b_9)^{\mathrm{T}} \in \mathbb{R}^9$ and

$$R(b) = M(R_1, R_2, R_3), \tag{39}$$

with

$$R_i = \begin{bmatrix} 0 & -b_{3i} & b_{3i-1} \\ b_{3i} & 0 & -b_{3i-2} \\ -b_{3i-1} & b_{3i-2} & 0 \end{bmatrix}, \quad 1 \le i \le 3. \tag{40}$$

There is an Lie isomorphism, σ, between the Lie algebra $(\mathbb{R}^9, [\cdot,\cdot])$ with the enlarged matrix loop algebra $\tilde{\mathfrak{g}}(\lambda)$.

We may find a bilinear form on $\mathbb{R}^9$ by

$$\langle a, b\rangle = a^{\mathrm{T}} F b, \tag{41}$$

where F is a constant matrix and the symmetric property of $\langle\cdot,\cdot\rangle$ requires that

$$F^{\mathrm{T}} = F. \tag{42}$$

The symmetric condition along with the ad-invariance property

$$\langle a, [b, c]\rangle = \langle [a, b], c\rangle,$$

provides the condition

$$F(R(b))^{\mathrm{T}} = -R(b)F, \quad b \in \mathbb{R}^9. \tag{43}$$

Upon solving the derived system of equations from (43) for an arbitrary vector $b \in \mathbb{R}^9$, we find

$$F = \begin{bmatrix} \eta_1 & \eta_2 & \eta_3 \\ \eta_2 & \alpha\eta_3 & 0 \\ \eta_3 & 0 & 0 \end{bmatrix} \otimes F_0, \tag{44}$$

where

$$F_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{45}$$

and η_i, $1 \leq i \leq 3$, are arbitrary constants. Thus, the bilinear form on the semidirect sum $\tilde{\mathfrak{g}}(\lambda)$ of the two Lie subalgebras $\tilde{g}$ and $\tilde{g}_c$ is defined as

$$\begin{aligned}\langle A, B\rangle_{\tilde{\mathfrak{g}}(\lambda)} &= \langle \sigma(A), \sigma(B)\rangle_{\mathbb{R}^9} \\ &= (a_1, \ldots, a_9)F(b_1, \ldots, b_9)^{\mathrm{T}} \\ &= (a_1b_1 + a_2b_2 + a_3b_3)\eta_1 + (a_1b_4 + a_2b_5 + a_3b_6 + a_4b_1 + a_5b_2 \\ &\quad + a_6b_3)\eta_2 + (\alpha a_4b_4 + \alpha a_5b_5 + \alpha a_6b_6 + a_1b_7 + a_2b_8 + a_3b_9 \\ &\quad + a_7b_1 + a_8b_2 + a_9b_3)\eta_3, \end{aligned} \tag{46}$$

where A and B are two matrices in $\tilde{\mathfrak{g}}(\lambda)$ presented by

$$\begin{cases} A = \sigma^{-1}((a_1, \ldots, a_9)^{\mathrm{T}}) \in \tilde{\mathfrak{g}}(\lambda), \\ B = \sigma^{-1}((b_1, \ldots, b_9)^{\mathrm{T}}) \in \tilde{\mathfrak{g}}(\lambda). \end{cases} \tag{47}$$

Bilinear form (46) is symmetric and ad-invariant due to the isomorphism σ. A bilinear form, defined by (46), is non-degenerate iff the determinant of F is not zero, i.e.,

$$\det(F) = -\eta_3^9\alpha^3 \neq 0. \tag{48}$$

Therefore, we choose $\eta_3 \neq 0$ to obtain a non-degenerate, symmetric, and ad-invariant bilinear form over the enlarged matrix loop algebra $\tilde{\mathfrak{g}}(\lambda)$.

Now, we compute

$$\langle \bar{W}, \bar{U}_\lambda\rangle_{\tilde{\mathfrak{g}}(\lambda)} = a\eta_1 + e\eta_2 + e'\eta_3 \tag{49}$$

and

$$\langle \bar{W}, \bar{U}_{\bar{u}}\rangle_{\tilde{\mathfrak{g}}(\lambda)} = \begin{bmatrix} b\eta_1 + f\eta_2 + f'\eta_3 \\ c\eta_1 + g\eta_2 + g'\eta_3 \\ b\eta_2 + \alpha f\eta_3 \\ c\eta_2 + \alpha g\eta_3 \\ b\eta_3 \\ c\eta_3 \end{bmatrix}. \tag{50}$$

In addition, the formula $\gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\mathrm{tr}(W^2)|$ [19] yields that the constant $\gamma = 0$, and thus, the corresponding variational identity is

$$\frac{\delta}{\delta\bar{u}}\int \frac{-a_{m+1}\eta_1 - e_{m+1}\eta_2 - e'_{m+1}\eta_3}{m}\mathrm{d}x = \begin{bmatrix} b_m\eta_1 + f_m\eta_2 + f'_m\eta_3 \\ c_m\eta_1 + g_m\eta_2 + g'_m\eta_3 \\ b_m\eta_2 + \alpha f_m\eta_3 \\ c_m\eta_2 + \alpha g_m\eta_3 \\ b_m\eta_3 \\ c_m\eta_3 \end{bmatrix}, \quad m \geq 1. \tag{51}$$

We consequently obtain a Hamiltonian structure for hierarchy (33) of bi-integrable couplings,

$$\bar{u}_{t_m} = \bar{J}\frac{\delta\bar{\mathcal{H}}_m}{\delta\bar{u}}, \quad m \geq 0, \tag{52}$$

with the Hamiltonian functionals,

$$\bar{\mathcal{H}}_m = \int \frac{-a_{m+2}\eta_1 - e_{m+2}\eta_2 - e'_{m+2}\eta_3}{m+1}\mathrm{d}x, \tag{53}$$

and the Hamiltonian operator,

$$\bar{J} = \begin{bmatrix} 0 & \eta_1 & 0 & \eta_2 & 0 & \eta_3 \\ -\eta_1 & 0 & -\eta_2 & 0 & -\eta_3 & 0 \\ 0 & \eta_2 & 0 & \alpha\eta_3 & 0 & 0 \\ -\eta_2 & 0 & -\alpha\eta_3 & 0 & 0 & 0 \\ 0 & \eta_3 & 0 & 0 & 0 & 0 \\ -\eta_3 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{-1}, \tag{54}$$

and note that $\det(\bar{J}) \neq 0$. In particular, when $m = 2$, the Hamiltonian functional is

$$\bar{\mathcal{H}}_2 = \int \frac{1}{3}\left(-a_4\eta_1 - e_4\eta_2 - e'_4\eta_3\right)\mathrm{d}x, \tag{55}$$

where

$$\begin{aligned} -a_4\eta_1 - e_4\eta_2 - e'_4\eta_3 = {} & ((\eta_1+\eta_2+\eta_3)p + (\alpha\eta_3+\eta_2)r + \eta_3 v)\, p_{xx} \\ & + ((\eta_1+\eta_2+\eta_3)q + (\alpha\eta_3+\eta_2)s\eta_3 w)\, q_{xx} \\ & + ((\alpha\eta_3+\eta_2)p + \eta_3\alpha r)\, r_{xx}\, ((\alpha\eta_3+\eta_2)q + \eta_3\alpha s)\, s_{xx} \\ & + \eta_3 p v_{xx}\eta_3 q w_{xx} - \frac{1}{2}(\eta_1+\eta_2+\eta_3)p_x^2 \\ & + ((-\alpha\eta_3-\eta_2)r_x - \eta_3 v_x)\, p_x - (\eta_1+\eta_2+\eta_3)q_x^2 \\ & + ((-\alpha\eta_3-\eta_2)s_x - \eta_3 w_x)\, q_x - \frac{1}{2}\eta_3\alpha r_x^2 - \frac{1}{2}\eta_3\alpha s_x^2 \\ & + \frac{3}{8}(\eta_1+\eta_2+\eta_3)p^4\frac{3}{2}\left((\alpha\eta_3+\eta_2)r + \eta_3 v\right) p^3 \\ & + \frac{1}{8}\Bigg(6\,(\eta_1+\eta_2+\eta_3)\,q^2 + ((12\alpha\eta_3+12\eta_2)s + 12\eta_3 w)\, q \\ & + 18\eta_3\left(r^2+\frac{1}{3}s^2\right)\alpha\Bigg)p^2 + \frac{3}{2}(((\alpha\eta_3+\eta_2)r + \eta_3 v)q \\ & + 2\eta_3\alpha r s)pq + \frac{3}{8}q^2((\eta_1+\eta_2+\eta_3)q^2 + ((4\alpha\eta_3 \\ & + 4\eta_2)s + 4\eta_3 w)q + 2\alpha\eta_3(r^2+3s^2)). \end{aligned} \tag{56}$$

5 Symmetries and Conserved Functionals

We may solve the recursion relation of symmetries

$$\bar{K}_m = \bar{\Phi}\bar{K}_{m-1}, \quad m \geq 0, \tag{57}$$

for a recursion operator, $\bar{\Phi}$, to obtain

$$\bar{\Phi} = \begin{bmatrix} \Phi & 0 & 0 \\ \Phi_1 & \Phi & 0 \\ \Phi_2 & \alpha\Phi_1 & \Phi \end{bmatrix}, \tag{58}$$

where Φ is given by (13) and

$$\Phi_1 = \begin{bmatrix} q\partial^{-1}r + s\partial^{-1}p & q\partial^{-1}s + s\partial^{-1}q \\ -p\partial^{-1}r - r\partial^{-1}p & -p\partial^{-1}s - r\partial^{-1}q \end{bmatrix}, \tag{59}$$

and

$$\Phi_2 = \begin{bmatrix} q\partial^{-1}v + w\partial^{-1}p + \alpha s\partial^{-1}r & q\partial^{-1}w + w\partial^{-1}q + \alpha s\partial^{-1}s \\ -p\partial^{-1}v - v\partial^{-1}p - \alpha r\partial^{-1}r & -p\partial^{-1}w - v\partial^{-1}q - \alpha r\partial^{-1}s \end{bmatrix}. \tag{60}$$

It can be shown by a symbolic computation that $\bar{\Phi}$ is a hereditary operator [36,37]. Therefore,

$$\bar{\Phi}'(\bar{u})[\bar{\Phi}\bar{T}_1]\bar{T}_2 - \bar{\Phi}\bar{\Phi}'(\bar{u})[\bar{T}_1]\bar{T}_2$$

is symmetric with respect to $\bar{T}_1$ and $\bar{T}_2$, and the two operators $\bar{J}$ and $\bar{M} = \bar{\Phi}\bar{J}$ make a Hamiltonian pair [38], i.e., $\bar{J}$, $\bar{M}$, and $\bar{J} + \bar{M}$ are all Hamiltonian operators. Thus, the hierarchy (33) of bi-integrable couplings possesses a bi-Hamiltonian structure [38,39] and is Liouville integrable. It follows that there are infinitely many symmetries and conserved functionals:

$$[\bar{K}_m, \bar{K}_n] = 0, \quad m, n \geq 0, \tag{61}$$

and

$$\{\bar{\mathcal{H}}_m, \bar{\mathcal{H}}_n\}_{\bar{J}} = \{\bar{\mathcal{H}}_m, \bar{\mathcal{H}}_n\}_{\bar{M}} = 0, \quad m, n \geq 0. \tag{62}$$

6 Concluding Remarks

We have obtained a new class of bi-integrable couplings (33) for the soliton hierarchy (12) using on non-semisimple Lie algebra (16). We showed the resulting hierarchy of bi-integrable couplings possesses a bi-Hamiltonian structure and is Liouville integrable. It remains an open question how to generate a Hamiltonian structure for matrix loop algebra (15) when $\alpha = 0$ as the bilinear form presented in Sect. 4 is degenerate.

Some enlarged matrix loop algebras do not possess any non-degenerate, symmetric, and ad-invariant bilinear forms required in the variational identity. In the following example of a bi-integrable coupling,

$$\begin{cases} u_t = K(u) \\ v_t = K'(u)[v] \\ w_t = K'(u)[w]. \end{cases} \tag{63}$$

where $K'(u)$ denotes the Gateaux derivative, is there any Hamiltonian structure for this specific bi-integrable coupling?

References

1. Ablowitz, M.J., Clarkson, P.A.: Solitons, Nonlinear Evolution Equations and Inverse Scattering. Cambridge University Press, Cambridge (1991)
2. Xu, X.X.: An integrable coupling hierarchy of the MkdV_integrable systems, its Hamiltonian structure and corresponding nonisospectral integrable hierarchy. Appl. Math. Comput. **216**(1), 344–353 (2010)
3. Wang, X.R., Zhang, X.E., Zhao, P.Y.: Binary nonlinearization for AKNS-KN coupling system. Abstr. Appl. Anal. 2014 (2014) (Article ID 253102), 12 pages, https://doi.org/10.1155/2014/253102
4. Zhao, Q.L., Wang, X.Z.: The integrable coupling system of a 3 × 3 discrete matrix spectral problem. Appl. Math. Comps. **216**(3), 730–743 (2010)
5. Dong, H.H., Zhao, K., Yang, H.W., Li, Y.Q.: Generalised (2 + 1)-dimensional super MKdV hierarchy for integrable systems in soliton theory. East Asian J. Appl. Math. **5**(3), 256–272 (2015)
6. Dong, H.H., Guo, B.Y., Yin, B.S.: Generalized fractional supertrace identity for Hamiltonian structure of NLS-MKdV hierarchy with self-consistent sources. Anal. Math. Phys. **6**(2), 199–209 (2016)
7. Ma, W.X., Xu, X.X., Zhang, Y.F.: Semi-direct sums of Lie algebras and continuous integrable couplings. Phys. Lett. A **351**, 125–130 (2006)
8. Ma, W.X., Xu, X.X., Zhang, Y.F.: Semidirect sums of Lie algebras and discrete integrable couplings. J. Math. Phys. **47**, 053501 (2006)
9. Ma, W.X., Chen, M.: Hamiltonian and quasi-Hamiltonian structures associated with semidirect sums of Lie algebras. J. Phys. A: Math. Gen. **39**, 10787–10801 (2006)
10. Ma, W.X.: A discrete variational identity on semi-direct sums of Lie algebras. J. Phys. A: Math. Theor. **40**, 15055–15069 (2007)
11. Xia, T.C., Chen, H.X., Chen, D.Y.: A new Lax integrable hierarchy, N Hamiltonian structure and its integrable coupling system. Chaos Solitons Fractals **23**, 451–458 (2005)
12. Yu, E.Y., Zhang, H.Q.: Hamiltonian structure of the integrable couplings for the multicomponent Dirac hierarchy. App. Math. Comput. **197**, 828–835 (2008)
13. Zhang, Y.F., Feng, B.I.: A few Lie algebras and their applications for generating integrable hierarchies of evolution types. Commun. Nonlinear Sci. Numer. Simul. **16**, 3045–3061 (2011)
14. Manukure, S., Ma, W.X.: Bi-integrable couplings of a new soliton hierarchy associated with a non-semisimple Lie algebra. App. Math. Comp. **245**, 4452 (2014)
15. Ma, W.X.: Loop algebras and bi-integrable couplings. Chin. Ann. Math. **33**, 207–224 (2012)
16. Yu, S., Yao, Y., Shen, S., Ma, W.X.: Bi-integrable couplings of a Kaup–Newell type soliton hierarchy and their bi-Hamiltonian structures. Commun. Nonlinear Sci. Numer. Simulat. **23**, 366–377 (2015)
17. Meng, J.H.: (2012) Bi-integrable and tri-integrable couplings and their Hamiltonian structures. Ph.D. dissertation, University of South Florida. http://scholarcommons.usf.edu/etd/4371. Accessed 2 Nov 2013
18. Ma, W.X.: Variational identities and applications to Hamiltonian structures of soliton equations. Nonlinear Anal. **71**, e1716–e1726 (2009)
19. Tu, G.Z.: On Liouville integrability of zero-curvature equations and the Yang hierarchy. J. Phys. A: Math. Gen. **22**, 2375–2392 (1989)
20. Tu, G.Z.: The trace identity, a powerful tool for constructing the Hamiltonian structure of integrable systems. J. Math. Phys. **30**, 330–338 (1989)

21. Ma, W.X.: Variational identities and Hamiltonian structures, In: Ma, W.X., Hu, X.B., Liu, Q.P. (eds.) Nonlinear and modern mathematical physics, vol. 1212, pp. 1–27, AIP conference proceedings. American Institute of Physics, Melville, NY (2010)
22. Ma, W.X., Zhang, Y.F.: Component-trace identities for Hamiltonian structures. App. Anal. **89**, 457–472 (2010)
23. Blaszak, M., Szablikowski, B.M., Silindir, B.: Construction and separability of nonlinear soliton integrable couplings. Appl. Math. Comput. **219**, 1866–1873 (2012)
24. Ma, W.X., Meng, J.H., Zhang, H.Q.: Tri-integrable couplings by matrix loop algebras. Int. J. Nonlinear Sci. Numer. Simul. **14**, 377–388 (2013)
25. Ma, W.X., Fuchssteiner, B.: Integrable theory of the perturbation equations. Choas Solitons Fractals **7**, 1227–1250 (1996)
26. Ma, W.X.: Integrable couplings of soliton equations by perturbations I—a general theory and application to the KdV hierarchy. Methods Appl. Anal. **7**, 21–55 (2000)
27. Sakovich, S.Y.: On integrability of a (2 + 1)-dimensional perturbed KdV equation. J. Nonlinear Math. Phys. **5**, 230–233 (1998)
28. Sakovich, S.Y.: Coupled KdV equations of Hirota–Satsuma type. J. Nonlinear Math. Phys. **6**, 255–262 (1999)
29. Ma, W.X.: Nonlinear continuous integrable Hamiltonian couplings. Appl. Math. Comput. **217**, 7238–7244 (2011)
30. He, B., Chen, L., Cao, Y.: Bi-integrable couplings and tri-integrable couplings of the modified Ablowitz–Kaup–Newell–Segur hierarchy with self-consistent sources. J. Math. Phys. **56**, 013502 (2015)
31. Tang, Y.N., Wang, L., Ma, W.X.: Integrable couplings, bi-integrable couplings and their Hamiltonian structures of the Giachetti–Johnson soliton hierarchy. Math. Methods Appl. Sci. **38**, 2305–2315 (2015)
32. Ma, W.X.: A soliton hierarchy associated with so(3, ℝ). Appl. Math. Comput. **220**, 117–122 (2013)
33. Lax, P.D.: Integrals of nonlinear equations of evolution and solitary waves. Commun. Pure Appl. Math. **21**, 467–490 (1968)
34. Ma, W.X., Zhang, H.Q., Meng, J.H.: A block matrix loop algebra and bi-integrable couplings of the Dirac equations. East Asian J. Appl. Math. **3**, 171–189 (2013)
35. Ma, W.X., Zhu, Z.N.: Constructing nonlinear discrete integrable Hamiltonian couplings. Comput. Math. Appl. **60**, 2601–2608 (2010)
36. Fuchssteiner, B.: Application of hereditary symmetries to nonlinear evolution equations. Nonlinear Anal. **3**, 849–862 (1979)
37. Fuchssteiner, B., Fokas, A.S.: Symplectic structures, their Bäcklund transformations and hereditary symmetries. Phys. D **4**, 47–66 (1981)
38. Magri, F.: A simple model of the integrable Hamiltonian equation. J. Math. Phys. **19**, 1156–1162 (1978)
39. Olver, P.J.: Applications of Lie Groups to Differential Equations, Graduate Texts in Mathematics, vol. 107. Springer, New York (1986)