



An integrable generalization of the D-Kaup–Newell soliton hierarchy and its bi-Hamiltonian reduced hierarchy



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ABSTRACT

We present a new spectral problem, a generalization of the D-Kaup–Newell spectral problem, associated with the Lie algebra $sl(2, \mathbb{R})$. Zero curvature equations furnish the soliton hierarchy. The trace identity produces the Hamiltonian structure for the hierarchy and shows its Liouville integrability. Lastly, a reduction of the spectral problem is shown to have a different soliton hierarchy with a bi-Hamiltonian structure. The major motivation of this paper is to present spectral problems that generate two soliton hierarchies with infinitely many conservation laws and high-order symmetries.

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1. Introduction

The study of soliton equations has been of considerable importance to the understanding of nonlinear phenomenon over the past few decades. In recent years, soliton theory has enriched the understanding of the nature of integrability in partial and ordinary differential equations (see [1]); one way is through the existence of infinitely many conservation laws and symmetries. Constructed from spectral problems associated with matrix Lie algebras, systems of solitons equations often give rise to soliton hierarchies [2–4]. Frequently, these hierarchies possess infinitely many symmetries and conserved functionals. Some hierarchies of this particular type include the Ablowitz–Kaup–Newell–Segur [5], the Kaup–Newell [6], the D-Kaup–Newell [7], the KdV [8], and the Dirac hierarchies [9]. This paper presents two spectral problems that generate different soliton hierarchies; both hierarchies have infinitely many conservation laws and high-order symmetries implying Liouville integrability.

There is a general scheme for soliton hierarchy construction [10,11]. We begin with a matrix spectral problem associated with a matrix loop algebra. Under the assumption of a solution to the stationary zero curvature equation, a series of Lax matrices is introduced. The Lax matrices give rise to the temporal spectral problems. Each temporal spectral problem paired with the original spectral problem form a member in a series of compatibility conditions known as the zero curvature equa-

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tions. After solving the zero curvature equations, we generate the hierarchy of soliton equations. The final step is finding the Hamiltonian structure for the hierarchy using the trace identity [10,11]. This produces a hierarchy of Hamiltonian equations.

We must begin by introducing a matrix loop algebra. The two spectral matrices in this paper are associated with $\mathfrak{sl}(2, \mathbb{R})$, a three-dimensional special linear Lie algebra consisting of trace-free 2×2 matrices. The basis for the simple Lie algebra is

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad (1)$$

with commutator properties

$$[e_1, e_2] = 2e_2, \quad [e_1, e_3] = -2e_3, \quad [e_2, e_3] = e_1. \quad (2)$$

To generate the hierarchy, we use the following matrix loop algebra:

$$\tilde{\mathfrak{sl}}(2, \mathbb{R}) = \left\{ \sum_{i \geq 0} A_i \lambda^{n-i} | A_i \in \mathfrak{sl}(2, \mathbb{R}), i \geq 0, n \in \mathbb{Z} \right\}. \quad (3)$$

In particular, the matrix loop algebra $\tilde{\mathfrak{sl}}(2, \mathbb{R})$ contains elements of the form $\lambda^m e_1 + \lambda^l e_2 + \lambda^p e_3$ with arbitrary integers m, l, p . Many well-known soliton hierarchies are generated from the matrix loop algebra $\mathfrak{sl}(2, \mathbb{R})$ [5–11].

In this paper, we will introduce a new spectral matrix and explain why it generalizes the D-Kaup–Newell spectral matrix. We then generate its soliton hierarchy. Next, we apply the trace identity to engender the Hamiltonian structure and discuss why the hierarchy is Liouville integrable, i.e., the hierarchy has infinitely many commuting symmetries and conserved functionals. We present a reduction of the spectral matrix to produce a completely different soliton hierarchy which is shown to have bi-Hamiltonian structure. Lastly, we discuss a few ideas for further research associated with the Lie algebra $\mathfrak{so}(3, \mathbb{R})$.

2. A generalized D-Kaup–Newell spectral problem

Let us introduce a spectral matrix:

$$U = U(u, \lambda) = (\lambda^2 - r)e_1 + (\lambda p + s)e_2 + (\lambda q + v)e_3 = \begin{bmatrix} \lambda^2 - r & \lambda p + s \\ \lambda q + v & -\lambda^2 + r \end{bmatrix}, \quad (4)$$

and consider the following isospectral problem:

$$\phi_x = U\phi = \begin{bmatrix} \lambda^2 - r & \lambda p + s \\ \lambda q + v & -\lambda^2 + r \end{bmatrix} \phi, \quad U \in \tilde{\mathfrak{sl}}(2, \mathbb{R}), \quad u = \begin{bmatrix} p \\ q \\ r \\ s \\ v \end{bmatrix}, \quad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}, \quad (5)$$

where p, q, r, s , and v are potentials.

The D-Kaup–Newell spectral problem is known [7] to be

$$\phi_x = U\phi = \begin{bmatrix} \lambda^2 + r & \lambda p \\ \lambda q & -\lambda^2 - r \end{bmatrix} \phi, \quad U \in \tilde{\mathfrak{sl}}(2, \mathbb{R}), \quad u = \begin{bmatrix} p \\ q \\ r \end{bmatrix}, \quad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}, \quad (6)$$

which depends on three potentials: p, q , and r . The new spectral problem (5) is a generalization of the D-Kaup–Newell spectral problem adding two new potentials s and v . Previously, the cases $r = \alpha$ and $r = \alpha pq$, where α is a constant, have been shown to generate integrable hierarchies [12,13] for the D-Kaup–Newell spectral problem (6).

3. The soliton hierarchy

We assume a solution to the stationary zero curvature equation, $W_x = [U, W]$, to be of the form

$$W = ae_1 + be_2 + ce_3 = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \in \tilde{\mathfrak{sl}}(2, \mathbb{R}), \quad (7)$$

and we get the equations

$$\begin{cases} a_x = -qb\lambda + pc\lambda - vb + sc, \\ b_x = -2pa\lambda + 2b\lambda^2 - 2sa - 2rb, \\ c_x = 2qa\lambda - 2c\lambda^2 + 2va + 2rc. \end{cases} \quad (8)$$

Next, we assume that a , b , and c have Laurent expansions

$$a = \sum_{i \geq 0} a_i \lambda^{-i}, \quad b = \sum_{i \geq 0} b_i \lambda^{-i}, \quad c = \sum_{i \geq 0} c_i \lambda^{-i}, \quad (9)$$

which gives us the following recursive relations:

$$\begin{cases} b_{i+1} = \frac{b_{i-1,x}}{2} + pa_i + sa_{i-1} + rb_{i-1}, \\ c_{i+1} = -\frac{c_{i-1,x}}{2} + qa_i + va_{i-1} + rc_{i-1}, & i \geq 0, \\ a_{i+1,x} = -q \frac{b_{i,x}}{2} - p \frac{c_{i,x}}{2} + (pv - qs)a_i - qrb_i + prc_i + sc_{i+1} - vb_{i+1}. \end{cases} \quad (10)$$

Now, we take the initial values of

$$a_0 = \alpha, \quad b_0 = c_0 = a_1 = 0, \quad b_1 = \alpha p, \quad c_1 = \alpha q, \quad (11)$$

and impose the conditions for integration

$$a_i|_{u=0} = b_i|_{u=0} = c_i|_{u=0} = 0, \quad i \geq 1, \quad (12)$$

to determine the sequence of $\{a_i, b_i, c_i | i \geq 1\}$ uniquely. For $i = 2, 3$, we have the following: $b_2 = \alpha s$, $c_2 = \alpha v$, $a_2 = -\alpha \frac{1}{2} pq$, $b_3 = \alpha \frac{1}{2} (-p^2 q + 2pr + p_x)$, $c_3 = -\alpha \frac{1}{2} (q^2 p - 2qr + q_x)$, $a_3 = -\alpha \frac{1}{2} (pv + qs)$.

We can see the localness of $\{a_i, b_i, c_i | 0 \leq i \leq 3\}$. This result can be proven for all $i \geq 0$ as follows:

Proposition 3.1. Let $\{a_0, b_0, c_0, a_1, b_1, c_1\}$ be given by Eq. (11). Then all functions $\{a_i, b_i, c_i | i \geq 2\}$ determined by Eq. (10) with the conditions in (12) are differential polynomials in u with respect to x , and thus, are local.

Proof. We compute from the stationary zero curvature equation, $W_x = [U, W]$,

$$\frac{d}{dx} \text{tr}(W^2) = 2 \text{tr}(WW_x) = 2 \text{tr}(W[U, W]) = 2(\text{tr}(W^2 U) - \text{tr}(W^2 U)) = 0, \quad (13)$$

and seeing that the $\text{tr}(W^2) = 2(a^2 + bc)$, we have

$$a^2 + bc = (a^2 + bc)|_{u=0} = \alpha^2, \quad (14)$$

following from the initial data (11). Now, we use (9), the Laurent expansions of a, b, c , to give

$$a_i = \frac{\alpha}{2} - \frac{1}{2\alpha} \sum_{k+l=i, k, l \geq 1} a_k a_l - \frac{1}{2\alpha} \sum_{k+l=i, k, l \geq 0} b_k c_l, \quad i \geq 1. \quad (15)$$

Finally, based on the recursion relations (10) and (15), we use mathematical induction to see that all functions $\{a_i, b_i, c_i, i \geq 0\}$ are differential polynomials in u with respect to x , and therefore, are local. This completes the proof. $\square$

Now, we need to solve the zero curvature equations

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0, \quad m \geq 0, \quad (16)$$

which is equivalent to (5) together with the temporal problems

$$\phi_{t_m} = V^{[m]} \phi = V^{[m]}(u, \lambda) \phi, \quad m \geq 0. \quad (17)$$

In order to solve these, we introduce a series of Lax operators

$$V^{[m]} = ((\lambda^m W)_+ + \Delta_m) \in \tilde{\text{sl}}(2, \mathbb{R}), \quad (18)$$

where $(\lambda^m W)_+$ denotes the polynomial part of $\lambda^m W$ and Δ_m is a modification term. Solving (16), we may choose $\Delta_m = 0$ and generate a hierarchy of soliton equations

$$u_{t_m} = K_m = \begin{bmatrix} 2b_{m+1} \\ -2c_{m+1} \\ qb_{m+1} - pc_{m+1} \\ -2pa_{m+1} + 2b_{m+2} \\ 2qa_{m+1} - 2c_{m+2} \end{bmatrix} = \Phi \begin{bmatrix} 2b_m \\ -2c_m \\ qb_m - pc_m \\ -2pa_m + 2b_{m+1} \\ 2qa_m - 2c_{m+1} \end{bmatrix}, \quad m \geq 0. \quad (19)$$

Φ is a recursion operator determined by (10) and given by

$$\Phi = \begin{bmatrix} -p\partial^{-1}v & -p\partial^{-1}s & 0 & 1-p\partial^{-1}q & -p\partial^{-1}p \\ q\partial^{-1}v & q\partial^{-1}s & 0 & q\partial^{-1}q & 1+q\partial^{-1}p \\ 0 & 0 & 0 & \frac{q}{2} & \frac{p}{2} \\ \frac{1}{2}\partial + r - s\partial^{-1}v & -s\partial^{-1}s & 0 & -s\partial^{-1}q & -s\partial^{-1}p \\ v\partial^{-1}v & -\frac{1}{2}\partial + r + v\partial^{-1}s & 0 & v\partial^{-1}q & v\partial^{-1}p \end{bmatrix} \quad (20)$$

with $\partial = \frac{\partial}{\partial x}$ and ∂^{-1} as the inverse operator of ∂ .

Let's look at some of the reductions of this system. First, when $m = 6$, we may let $r, p, q = 0, v = -1, s = u$ and $\alpha = -4$ to find the KdV equation [8]

$$u_t = -6uu_x - u_{xxx}. \quad (21)$$

Also for $m = 6$, we let $r, p, q = 0, v = -u, s = u$ and $\alpha = -4$ to find the mKdV [14,15]

$$u_t = -6u^2u_x - u_{xxx}. \quad (22)$$

We may see the NLSE [6] by letting $r, p, q = 0, v = -\bar{u}, s = u$ and $\alpha = -2i$ when $m = 4$

$$iu_t = 2|u|^2u + u_{xx}. \quad (23)$$

Lastly, when $m = 4$, we can see the Gerdjikov-Ivanov (G-I) equations [16,17]

$$\begin{cases} q_t = \frac{1}{2}q^3r^2 - q^2r_x - q_{xx} \\ r_t = -\frac{1}{2}r^3q^2 - r^2q_x + r_{xx}, \end{cases} \quad (24)$$

if we let $r = -2qr, p = q, q = r, s = v = 0$ and $\alpha = -2$. The G-I equation is the third derivative nonlinear Schrödinger equation (DNLSIII) [17], or equivalently,

$$iq_t = -\frac{1}{2}q^3\bar{q}^2 + iq^2\bar{q}_x - q_{xx}. \quad (25)$$

4. Hamiltonian structure

The Hamiltonian structure for the hierarchy (19) is found using the trace identity [10,11]

$$\frac{\delta}{\delta u} \int \text{tr} \left(\frac{\partial U}{\partial \lambda} W \right) dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \text{tr} \left(\frac{\partial U}{\partial u} W \right), \quad \gamma = -\frac{\lambda}{2} \frac{d}{d\lambda} \ln |\text{tr}(W^2)|. \quad (26)$$

We compute

$$\begin{cases} \frac{\partial U}{\partial p} = \begin{bmatrix} 0 & \lambda \\ 0 & 0 \end{bmatrix}, \frac{\partial U}{\partial q} = \begin{bmatrix} 0 & 0 \\ \lambda & 0 \end{bmatrix}, \frac{\partial U}{\partial r} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \\ \frac{\partial U}{\partial s} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \frac{\partial U}{\partial v} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \frac{\partial U}{\partial \lambda} = \begin{bmatrix} 2\lambda & p \\ q & -2\lambda \end{bmatrix}. \end{cases} \quad (27)$$

The following traces are computed:

$$\begin{cases} \text{tr} \left(W \frac{\partial U}{\partial \lambda} \right) = 4a\lambda + bq + cp, \text{tr} \left(W \frac{\partial U}{\partial p} \right) = c\lambda, \text{tr} \left(W \frac{\partial U}{\partial q} \right) = b\lambda, \\ \text{tr} \left(W \frac{\partial U}{\partial r} \right) = -2a, \text{tr} \left(W \frac{\partial U}{\partial s} \right) = c, \text{tr} \left(W \frac{\partial U}{\partial v} \right) = b, \end{cases} \quad (28)$$

and using the trace identity (26), we have

$$\frac{\delta}{\delta u} \int (4a\lambda + bq + cp) dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma [c\lambda, b\lambda, -2a, c, b]^T. \quad (29)$$

After balancing the coefficients of all powers of λ ,

$$\frac{\delta}{\delta u} \int (4a_{m+2} + b_{m+1}q + c_{m+1}p) dx = (\gamma - m)[c_{m+1}, b_{m+1}, -2a_m, c_m, b_m]^T, \quad m \geq 0.$$

Considering the case where $m = 1$, we see $\gamma = 0$, and we have the following:

$$\frac{\delta}{\delta u} \int -\frac{4a_{m+2} + b_{m+1}q + c_{m+1}p}{m} dx = [c_{m+1}, b_{m+1}, -2a_m, c_m, b_m]^T, \quad m \geq 1. \quad (30)$$

Thus, we arrive at the Hamiltonian structure for the generalized D-Kaup–Newell soliton hierarchy (19)

$$u_{t_m} = K_m = \begin{bmatrix} 2b_{m+1} \\ -2c_{m+1} \\ -a_{m,x} - vb_m + sc_m \\ b_{m,x} + 2sa_m + 2rb_m \\ c_{m,x} - 2va_m - 2rc_m \end{bmatrix} = J \begin{bmatrix} c_{m+1} \\ b_{m+1} \\ -2a_m \\ c_m \\ b_m \end{bmatrix} = J \frac{\delta \mathcal{H}_m}{\delta u}, \quad m \geq 0, \quad (31)$$

with the Hamiltonian operator

$$J = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\partial & s & -v \\ 0 & 0 & -s & 0 & \partial + 2r \\ 0 & 0 & v & \partial - 2r & 0 \end{bmatrix}, \quad (32)$$

and the Hamiltonian functionals

$$\mathcal{H}_0 = \int (-2\alpha r + \alpha pq) dx, \quad \mathcal{H}_m = \int -\frac{4a_{m+2} + b_{m+1}q + c_{m+1}p}{m} dx, \quad m \geq 1, \quad (33)$$

where $\mathcal{H}_0$ can be found directly from the vector $[c_1, b_1, -2a_0, c_0, b_0]^T$.

We can easily show that the operator J is a Hamiltonian operator as it is skew-symmetric and satisfies the Jacobi condition

$$\int K^T J'(u) [JS] T dx + \text{cycle}(K, S, T) = 0, \quad (34)$$

for all vector fields K, S , and T , where $J'(u)[K]$ denotes the Gateaux derivative of J with respect to u in the direction of K . The proof is similar and even simpler than the one presented in [18].

The above functionals (33) correspond to common conservation laws for each soliton system in the soliton hierarchy (19). Differential polynomial conservation laws can be generated either from some Riccati equation obtained from the underlying matrix spectral problem [19–21] or directly by a computer algebra system [22].

Based on the Hamiltonian structure (33), we can say that the soliton hierarchy (19) is integrable in the Liouville sense:

$$[K_k, K_l] = K'_k(u)[K_l] - K'_l(u)[K_k] = 0, \quad k, l \geq 0, \quad (35)$$

and

$$\{\mathcal{H}_k, \mathcal{H}_l\}_J = \int \left(\frac{\delta \mathcal{H}_k}{\delta u} \right)^T J \frac{\delta \mathcal{H}_l}{\delta u} dx = 0, \quad k, l \geq 0. \quad (36)$$

The commuting relations of the symmetries and conserved functionals are also consequences of the Virasoro algebra of Lax operators. For further reference on the algebraic structures of Lax operators and zero curvature equations, see [23–25].

5. A bi-Hamiltonian reduced integrable hierarchy

Let's introduce a reduction of the spectral problem (4):

$$\phi_x = U(u, \lambda) \phi = \begin{bmatrix} \lambda^2 - \tilde{r} & \lambda p + s \\ \lambda q + v & -\lambda^2 + \tilde{r} \end{bmatrix} \phi, \quad U \in \tilde{\mathfrak{sl}}(2, \mathbb{R}), \quad u = \begin{bmatrix} p \\ q \\ s \\ v \end{bmatrix}, \quad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}, \quad (37)$$

where $\tilde{r} = \frac{1}{2}pq$. We would like to construct its associated integrable hierarchy possessing a bi-Hamiltonian structure. We will see that this spectral problem will generate a different soliton hierarchy than (19).

Assume a solution to stationary zero curvature equation to be of the same form

$$W = ae_1 + be_2 + ce_3 = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \in \tilde{\mathfrak{sl}}(2, \mathbb{R}), \quad (38)$$

and we produce

$$\begin{cases} a_x = -qb\lambda + pc\lambda - vb + sc, \\ b_x = -2pa\lambda + 2b\lambda^2 - 2sa - pqb, \\ c_x = 2qa\lambda - 2c\lambda^2 + 2vb + pqc. \end{cases} \quad (39)$$

Taking the same Laurent expansions as before (9), the recursion relations are

$$\begin{cases} b_{i+1} = \frac{b_{i-1,x}}{2} + pa_i + sa_{i-1} + \frac{1}{2}pq b_{i-1}, \\ c_{i+1} = -\frac{c_{i-1,x}}{2} + qa_i + va_{i-1} + \frac{1}{2}pq c_{i-1}, \\ a_{i+1,x} = -q \frac{b_{i,x}}{2} - p \frac{c_{i,x}}{2} + (pv - qs)a_i - \frac{1}{2}pq^2 b_i + \frac{1}{2}p^2 q c_i + sc_{i+1} - vb_{i+1}. \end{cases} \quad i \geq 0. \quad (40)$$

We note that we have a different set of recursion relations than those found in (10). In order to determine the sequence of $\{a_i, b_i, c_i | i \geq 1\}$ uniquely, we take the same initial values of

$$a_0 = \alpha, \quad b_0 = c_0 = a_1 = 0, \quad b_1 = \alpha p, \quad c_1 = \alpha q, \quad (41)$$

and conditions for integration

$$a_i|_{u=0} = b_i|_{u=0} = c_i|_{u=0} = 0, \quad i \geq 1. \quad (42)$$

Again, for all $i \geq 0$, $\{a_i, b_i, c_i\}$ are differential polynomials in u with respect to x . The proof is almost exactly the same as (3.1) but we use the recursion relations (40) instead of (10).

After solving the zero curvature Eq. (16) and taking the modification terms Δ_m to be zero, we generate a completely new soliton hierarchy

$$u_{t_m} = K_m = \begin{bmatrix} 2b_{m+1} \\ -2c_{m+1} \\ -2pa_{m+1} + 2b_{m+2} \\ 2qa_{m+1} - 2c_{m+2} \end{bmatrix} = \Phi \begin{bmatrix} 2b_m \\ -2c_m \\ -2pa_m + 2b_{m+1} \\ 2qa_m - 2c_{m+1} \end{bmatrix}, \quad m \geq 0, \quad (43)$$

with recursion operator

$$\Phi = \begin{bmatrix} -p\partial^{-1}\nu & -p\partial^{-1}s & 1 - p\partial^{-1}q & -p\partial^{-1}p \\ q\partial^{-1}\nu & q\partial^{-1}s & q\partial^{-1}q & 1 + q\partial^{-1}p \\ \frac{1}{2}\partial + \frac{1}{2}pq - s\partial^{-1}\nu & -s\partial^{-1}s & -s\partial^{-1}q & -s\partial^{-1}p \\ \nu\partial^{-1}\nu & -\frac{1}{2}\partial + \frac{1}{2}pq + \nu\partial^{-1}s & \nu\partial^{-1}q & \nu\partial^{-1}p \end{bmatrix}. \quad (44)$$

We may construct a Hamiltonian structure from the trace identity [10,11] for the above soliton hierarchy (43). We figure out

$$\frac{\partial U}{\partial p} = \begin{bmatrix} -\frac{1}{2}q & \lambda \\ 0 & \frac{1}{2}q \end{bmatrix}, \quad \frac{\partial U}{\partial q} = \begin{bmatrix} -\frac{1}{2}p & 0 \\ \lambda & \frac{1}{2}p \end{bmatrix}, \quad (45)$$

and note $\frac{\partial U}{\partial s}$, $\frac{\partial U}{\partial v}$, and $\frac{\partial U}{\partial \lambda}$ are as the same as in (27). The new traces are found:

$$\text{tr}\left(W \frac{\partial U}{\partial p}\right) = -aq + c\lambda, \quad \text{tr}\left(W \frac{\partial U}{\partial q}\right) = -ap + b\lambda, \quad (46)$$

while $\text{tr}(W \frac{\partial U}{\partial \lambda})$, $\text{tr}(W \frac{\partial U}{\partial s})$, and $\text{tr}(W \frac{\partial U}{\partial v})$ are as before in (28).

Plugging these into the trace identity (26), we have

$$\frac{\delta}{\delta u} \int (4a\lambda + bq + cp)dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma [-aq + c\lambda, -ap + b\lambda, c, b]^T. \quad (47)$$

After balancing the coefficients of all powers of λ and then considering the case where $m = 1$, we see $\gamma = 0$. We then have the following for $m \geq 1$:

$$\frac{\delta}{\delta u} \int -\frac{4a_{m+2} + b_{m+1}q + c_{m+1}p}{m} dx = [-a_m q + c_{m+1}, -a_m p + b_{m+1}, c_m, b_m]^T. \quad (48)$$

Therefore, we formulate the Hamiltonian structure for the reduced soliton hierarchy (43)

$$u_{t_m} = K_m = \begin{bmatrix} 2b_{m+1} \\ -2c_{m+1} \\ -2pa_{m+1} + 2b_{m+2} \\ 2qa_{m+1} - 2c_{m+2} \end{bmatrix} = J \begin{bmatrix} -a_{m+1}q + c_{m+2} \\ -a_{m+1}p + b_{m+2} \\ c_{m+1} \\ b_{m+1} \end{bmatrix} = J \frac{\delta \mathcal{H}_{m+1}}{\delta u}, \quad m \geq 0, \quad (49)$$

with the Hamiltonian operator

$$J = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix}, \quad (50)$$

and the Hamiltonian functionals

$$\mathcal{H}_m = \int -\frac{4a_{m+2} + b_{m+1}q + c_{m+1}p}{m} dx, \quad m \geq 1. \quad (51)$$

It can be computed that all members in the soliton hierarchy (43) are bi-Hamiltonian (see [26–29] for bi-Hamiltonian theory), i.e.,

$$u_{t_m} = K_m = J \frac{\delta \mathcal{H}_{m+1}}{\delta u} = M \frac{\delta \mathcal{H}_m}{\delta u}, \quad m \geq 1, \quad (52)$$

where the second Hamiltonian operator is

$$M = \Phi J = \begin{bmatrix} 2p\partial^{-1}p & 2 - 2p\partial^{-1}q & 2p\partial^{-1}s & -2p\partial^{-1}\nu \\ -2 - 2q\partial^{-1}p & 2q\partial^{-1}q & -2q\partial^{-1}s & 2q\partial^{-1}\nu \\ 2s\partial^{-1}p & -2s\partial^{-1}q & 2s\partial^{-1}s & \partial + pq - 2s\partial^{-1}\nu \\ -2\nu\partial^{-1}p & 2\nu\partial^{-1}q & \partial - pq - 2\nu\partial^{-1}s & 2\nu\partial^{-1}\nu \end{bmatrix}. \quad (53)$$

This means that J and M constitute a Hamiltonian pair, or, J , M , and $N = \alpha J + \beta M$, for any $\alpha, \beta \in \mathbb{Z}$, are all Hamiltonian operators. As a direct result of the bi-Hamiltonian structure (52), we can say that the soliton hierarchy (43) is integrable in the Liouville sense:

$$\begin{cases} \{\mathcal{H}_k, \mathcal{H}_l\}_M = \int \left(\frac{\delta \mathcal{H}_k}{\delta u} \right)^T M \frac{\delta \mathcal{H}_l}{\delta u} dx = 0, \\ \{\mathcal{H}_k, \mathcal{H}_l\}_J = \int \left(\frac{\delta \mathcal{H}_k}{\delta u} \right)^T J \frac{\delta \mathcal{H}_l}{\delta u} dx = 0, \end{cases} \quad (54)$$

and

$$[K_k, K_l] = K'_k(u)[K_l] - K'_l(u)[K_k] = 0, \quad k, l \geq 0. \quad (55)$$

6. Concluding remarks

Starting with the matrix loop algebra $\tilde{\mathfrak{sl}}(2, \mathbb{R})$, we introduced a generalized D-Kaup–Newell spectral problem by adding two potentials s and ν . Each system of equations in the soliton hierarchy is Hamiltonian, and so, Liouville integrable. Then we presented a reduction to the new spectral problem generating a different soliton hierarchy with a bi-Hamiltonian structure.

We would like to note that the soliton hierarchy (19) is different from the hierarchies presented in [5–9]. The AKNS, Kaup–Newell, and D-Kaup–Newell hierarchies are found from the following spectral matrices, respectively:

$$U = \begin{bmatrix} \lambda & p \\ q & -\lambda \end{bmatrix}, U = \begin{bmatrix} \lambda^2 & \lambda p \\ \lambda q & -\lambda^2 \end{bmatrix}, U = \begin{bmatrix} \lambda^2 + r & \lambda p \\ \lambda q & -\lambda^2 - r \end{bmatrix}. \quad (56)$$

Recall, the spectral matrix for the soliton hierarchy (19) is

$$U = \begin{bmatrix} \lambda^2 - r & \lambda p + s \\ \lambda q + \nu & -\lambda^2 + r \end{bmatrix}. \quad (57)$$

It is clear that (57) is a generalization of the Kaup–Newell and D-Kaup–Newell spectral matrices. We will note that AKNS hierarchy [5] may be found from (57) by letting $p = q = r = 0$.

Recently, the matrix Lie loop algebra $\tilde{\mathfrak{so}}(3, \mathbb{R})$ defined by

$$\tilde{\mathfrak{so}}(3, \mathbb{R}) = \left\{ \sum_{i \geq 0} A_i \lambda^{n-i} | A_i \in \mathfrak{so}(3, \mathbb{R}), i \geq 0, n \in \mathbb{Z} \right\} \quad (58)$$

with basis elements

$$e_1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (59)$$

and commutator relations

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2, \quad (60)$$

have been studied [12,30–33]. Mostly, spectral matrices with two dependent variables have been discussed:

$$U(u, \lambda) = \lambda e_1 + p e_2 + q e_3,$$

$$U(u, \lambda) = \lambda^2 e_1 + \lambda p e_2 + \lambda q e_3,$$

$$U(u, \lambda) = \lambda e_1 + \lambda p e_2 + \lambda q e_3,$$

$$U(u, \lambda) = \lambda q e_1 + (\lambda^2 + \lambda p) e_2 + (-\lambda^2 + \lambda p) e_3.$$

A spectral matrix with three dependent variables has been analyzed [32] of the following form:

$$U(u, \lambda) = (\lambda^2 + r) e_1 + \lambda p e_2 + \lambda q e_3.$$

We hope to see more soliton hierarchies using three or more dependent variables. For instance, what kind of beautiful structure is possible for a spectral matrix similar to (57) associated with $\mathfrak{so}(3, \mathbb{R})$?

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