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Lump solutions to a $(2+1)$-dimensional extended KP equation

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ABSTRACT

With the aid of a computer algebra system, we present lump solutions to a $(2+1)$-dimensional extended Kadomtsev–Petviashvili equation (eKP) and give the sufficient and necessary conditions to guarantee analyticity and rational localization of the solutions. We plot a few solutions for some specific values of the free parameters involved and finally derive one of the lump solutions of the Kadomtsev–Petviashvili (KP) equations from the lump solutions of the eKP equation.

1. Introduction

Integrable equations are known to possess many different kinds of solutions such as solitons, positons, complexitons and lump solutions. While there are many methods to find some of these solutions with the most famous being the Inverse Scattering method, the Hirota bilinear method [1] remains a very powerful tool in finding solutions to integrable equations. It is perhaps the fastest method for solving integrable equations. For many years, solitons (nondispersive travelling wave solutions) have been at the forefront of the study of integrable systems. However, in recent years, other solutions like lump solutions have attracted a lot of attention in the mathematical physics community. Lump solutions are rational function solutions, localized in all directions in the space [2]. Like solitons, they also have many important applications in nonlinear dynamics. For example, they can be used to describe nonlinear patterns in plasma [3], nonlinear optic media [3], etc. Over the years, lump solutions have been constructed and studied for many integrable equations such as the Kadomtsev–Petviashvili equation [4,5], the $(2+1)$-dimensional BKP equation [6], the $(2+1)$-dimensional Ito equation [7], the $(2+1)$-dimensional Sawada–Kotera equation [8] and the Ishimori-II equation [9].

The Kadomtsev–Petviashvili (KP) equation, discovered in 1970 by Kadomtsev and Petviashvili [10], is a two-dimensional extension of the KdV equation. Like the KdV equation and other nonlinear equations such as the nonlinear Schrödinger equation and the sine-Gordon equation, the KP equation possesses soliton solutions via the inverse scattering transform [11], and remains one of the most widely studied integrable equations in $(2+1)$-dimensions. It can be used to describe various physical phenomena in areas such as water waves, nonlinear optics and plasma physics. For example, in the study of water

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waves, the KP equation appears in the description of a tsunami wave travelling in the inhomogeneous zone on the bottom of the ocean [12], and in plasma physics, it appears in the study of nonlinear ion acoustic waves in magnetized dusty plasma [13,14]. The considerable interest in the KP equation over the last couple of decades has led to the construction and study of many extensions of this equation. For instance, in [15], a completely integrable extension of the KP equation is introduced and its multi-soliton solutions are derived. Another extension is also introduced in [16] for the study of solitary waves in plasmas with multi-temperature electrons. Again, another extension of the KP equation also appears in [17].

In this paper, we would like to find lump solutions for an extended Kadomtsev–Petviashvili (eKP) equation by means of the Hirota bilinear method. The eKP equation to be considered is

$$(u_t - 6uu_x + u_{xxx})_x - u_{yy} + \alpha u_{tt} + \beta u_{ty} = 0, \tag{1.1}$$

where α and β are non-zero constants. More precisely, this equation is an extension of the KPI equation. It is important to remark that its KPII extended version appears in [18] as a nonintegrable example to show the applicability of Hirota's method to nonintegrable systems. As noted in [18], the eKPII equation is not integrable in the sense that it does not have a three-soliton solution unless $\alpha = \beta^2/12$ in which case it is transformable to the KP equation. Thus, the introduction of the additional terms αu_{tt} and βu_{ty} renders the extended equation non-integrable in contrast to other extended KP equations where the additional terms do not "kill" the integrability of the equation (see e.g. [15]). It is also worth mentioning that our interest in investigating the existence of lump solutions for the extended KPI equation stems from the fact that the KPI equation is known to have lump solutions while the KPII equation does not.

The eKP equation (1.1) can be transformed into the Hirota bilinear equation

$$\begin{aligned} &P(D_t, D_x, D_y)f \cdot f \\ &= (D_tD_x + D_x^4 - D_y^2 + \alpha D_t^2 + \beta D_tD_y)f \cdot f \\ &= f_{tx}f - f_tf_x + f_{xxxx}f - 4f_xf_{xxx} + 3f_{xx}^2 - f_{yy}f + f_y^2 + \alpha f_{tt}f - \alpha f_t^2 + \beta f_{ty}f - \beta f_tf_y = 0, \end{aligned} \tag{1.2}$$

under the transformation

$$u = 2(\ln f)_{xx}, \tag{1.3}$$

where D_t, D_x and D_y are Hirota bilinear derivatives defined by

$$D_x^n f \cdot g = \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)^n f(x_1)g(x_2)|_{x_2=x_1=x}. \tag{1.4}$$

2. Quadratic function solutions

Let us first consider positive quadratic solutions f to the bilinear equation (1.2) in the form

$$f = f_1^2 + f_2^2 + a_7, \quad a_7 > 0. \tag{2.5}$$

with

$$f_1(x, y, t) = a_1x + a_2y + a_3t + c_1, \;\; f_2(x, y, t) = a_4x + a_5y + a_6t + c_2, \tag{2.6}$$

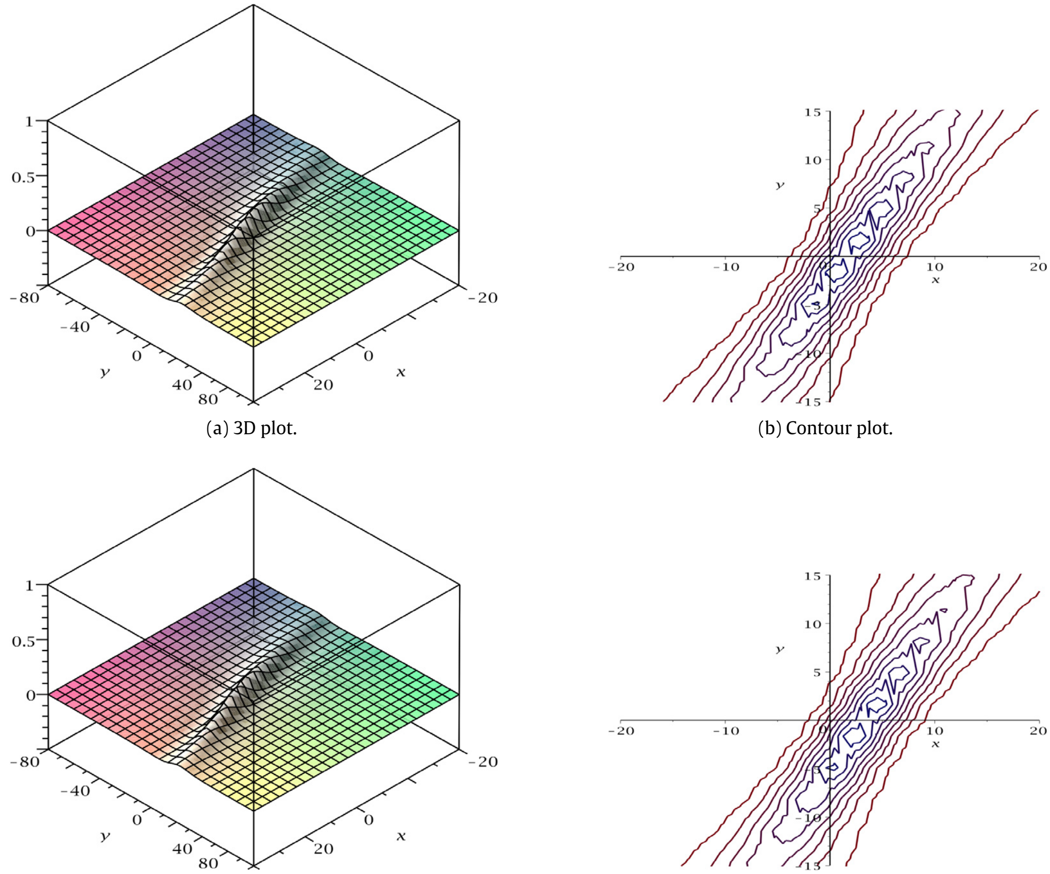
where a_i, $1 \le i \le 7$ and c_1, c_2 are real constants to be determined.

Thus, we obtain from (1.2) with the aid of a computer algebra system, the following equations:

$$a_1 = \frac{a_3(a_2^2 - a_5^2) + 2a_2a_5a_6 - \beta(a_3^2 + a_6^2)a_2 - \alpha(a_3^2 + a_6^2)a_3}{a_3^2 + a_6^2}, \tag{2.7}$$

$$a_4 = \frac{a_6(a_5^2 - a_2^2) + 2a_2a_3a_5 - \beta(a_3^2 + a_6^2)a_5 - \alpha(a_3^2 + a_6^2)a_6}{a_3^2 + a_6^2}, \tag{2.8}$$

$$\begin{aligned} a_7 = &\frac{3}{(a_2a_6 - a_3a_5)^2(a_3^2 + a_6^2)^3}\Big[\Big(\beta(a_3^2 + a_6^2)a_2 + \alpha(a_3^2 + a_6^2)a_3 + a_3(a_5^2 - a_2^2) - 2a_2a_5a_6\Big)^4 \\ &+ 2\Big(\beta(a_3^2 + a_6^2)a_5 + \alpha(a_3^2 + a_6^2)a_6 + a_6(a_2^2 - a_5^2) - 2a_2a_3a_5\Big)^2\Big(\beta(a_3^2 + a_6^2)a_2 + \alpha(a_3^2 + a_6^2)a_3 \\ &+ a_3(a_5^2 - a_2^2) - 2a_2a_5a_6\Big)^2 + \Big(\beta(a_3^2 + a_6^2)a_5 + \alpha(a_3^2 + a_6^2)a_6 + a_6(a_2^2 - a_5^2) - 2a_2a_3a_5\Big)^4\Big]. \end{aligned} \tag{2.9}$$



(a) 3D plot. (b) Contour plot.

(c) 3D plot. (d) Contour plot.

Fig. 1. (a) & (b) Plots of a lump solution for $a_2 = 1, a_3 = 2, a_5 = 0, a_6 = 1, c_1 = c_2 = 0, \alpha = \beta = 0.5$ and $t = 1$. (c) & (d) Plots of a lump solution for $a_2 = 1, a_3 = 2, a_5 = 0, a_6 = 1, c_1 = c_2 = 0, \alpha = \beta = 0.5$ and $t = 2$.

The remaining parameters including c_1, c_2 are all arbitrary with a_2, a_3, a_5, and a_6 satisfying the condition $a_2a_6 - a_3a_5 \neq 0$. This implies $a_3^2 + a_6^2 \neq 0$. For f to be well-defined, the functions f_1 and f_2 must be linearly independent, i.e. $a_1a_5 - a_2a_4 = \frac{\left(a_2a_6-a_3a_5\right)\left(\alpha(a_3^2+a_6^2)+a_2^2+a_5^2\right)}{(a_3^2+a_6^2)} \neq 0$, yielding the above two conditions and the additional condition $a_2^2 + a_5^2 \neq 0$. Note that this additional condition guarantees that the remaining terms in (2.9) are nonzero. (*For if*, $a_5^2 = -a_2^2$ *and* $a_2a_6 = a_3a_5$, *then* $a_6(a_2^2 - a_5^2) - 2a_2a_3a_5 = 2a_2^2a_6 - 2a_2^2a_6 = 0$.) Thus, $a_7 > 0$. Therefore f yields the following solution to the eKP equation:

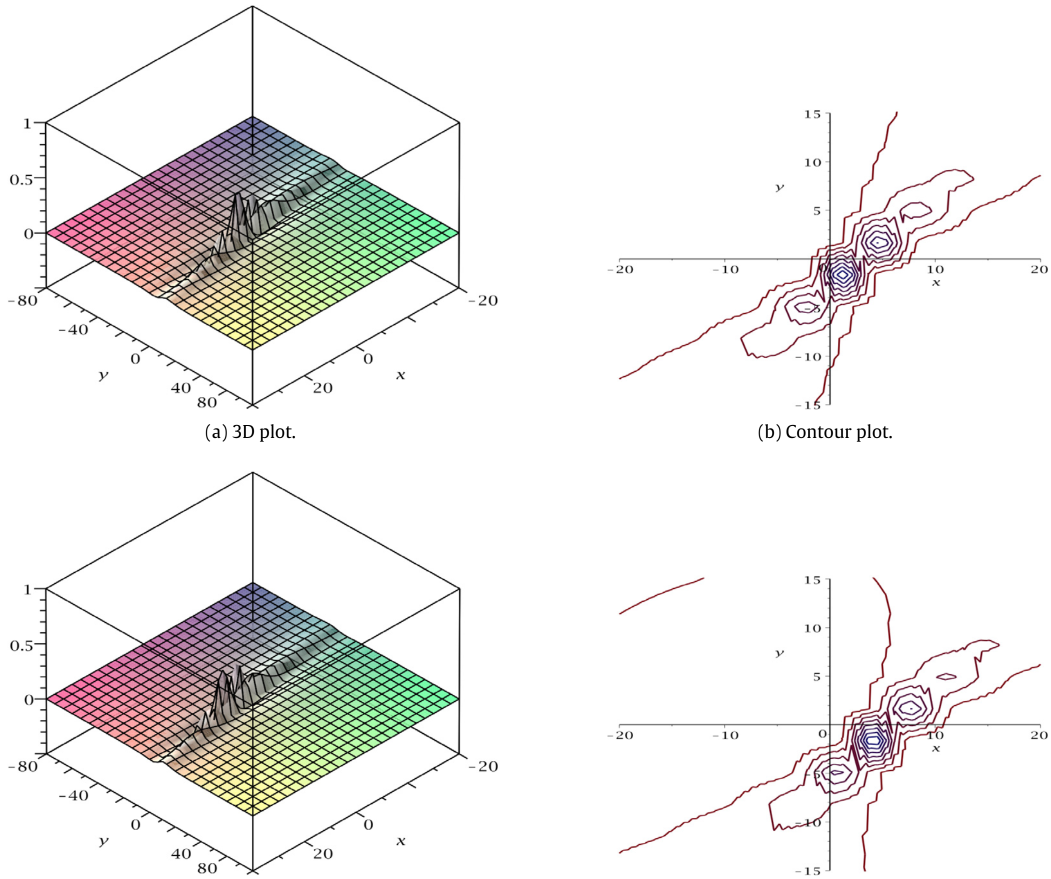
$$u = \frac{4f(a_1^2 + a_4^2) - 8(f_1a_1 + f_2a_4)^2}{f^2}, \tag{2.10}$$

where,

$$f_1 = \frac{a_3(a_2^2 - a_5^2) + 2a_2a_5a_6 - \beta(a_3^2 + a_6^2)a_2 - \alpha(a_3^2 + a_6^2)a_3}{a_3^2 + a_6^2}x + a_2y + a_3t + c_1, \tag{2.11}$$

$$f_2 = \frac{a_6(a_5^2 - a_2^2) + 2a_2a_3a_5 - \beta(a_3^2 + a_6^2)a_5 - \alpha(a_3^2 + a_6^2)a_6}{a_3^2 + a_6^2}x + a_5y + a_6t + c_2. \tag{2.12}$$

The above function, u is analytic which is guaranteed by the fact that $a_1a_5 - a_2a_4 \neq 0$. This same condition also guarantees that u decays in all directions (i.e. $u \to 0$ as $x^2 + y^2 \to \infty$). Thus the condition $a_1a_5 - a_2a_4 \neq 0$ is a necessary and sufficient condition for u to be analytic and rationally localized. It follows that u is a lump solution to the eKP equation.



(a) 3D plot. (b) Contour plot.

(c) 3D plot. (d) Contour plot.

Fig. 2. (a) & (b) Plots of a lump solution for $a_2 = 1, a_3 = 2, a_5 = 0, a_6 = 1, c_1 = c_2 = 0, \alpha = 0.25, \beta = 0.5$ and $t = 1$. (c) & (d) Plots of a lump solution for $a_2 = 1, a_3 = 2, a_5 = 0, a_6 = 1, c_1 = c_2 = 0, \alpha = 0.25, \beta = 0.5$ and $t = 2$.

3. Examples

(i) $c_1 = c_2 = 0$ and $\alpha = \beta = 0.5$.
Choosing the following parameters:

$$a_2 = 1, a_3 = 2, a_5 = 0, a_6 = 1, \tag{3.13}$$

we have

$$f(x, t, y) = (2t - 1.1x + y)^2 + (t - 0.7x)^2 + 43.35. \tag{3.14}$$

For $t = 1$, we obtain the lump solution

$$u = \frac{261.5 + 39.44x - 23.84y - 11.56x^2 + 14.96xy - 2.88y^2}{(48.35 - 5.8x + 4.y + 1.7x^2 - 2.2xy + y^2)^2}. \tag{3.15}$$

The 3D and contour plots of u are shown in Fig. 1(a) and (b).

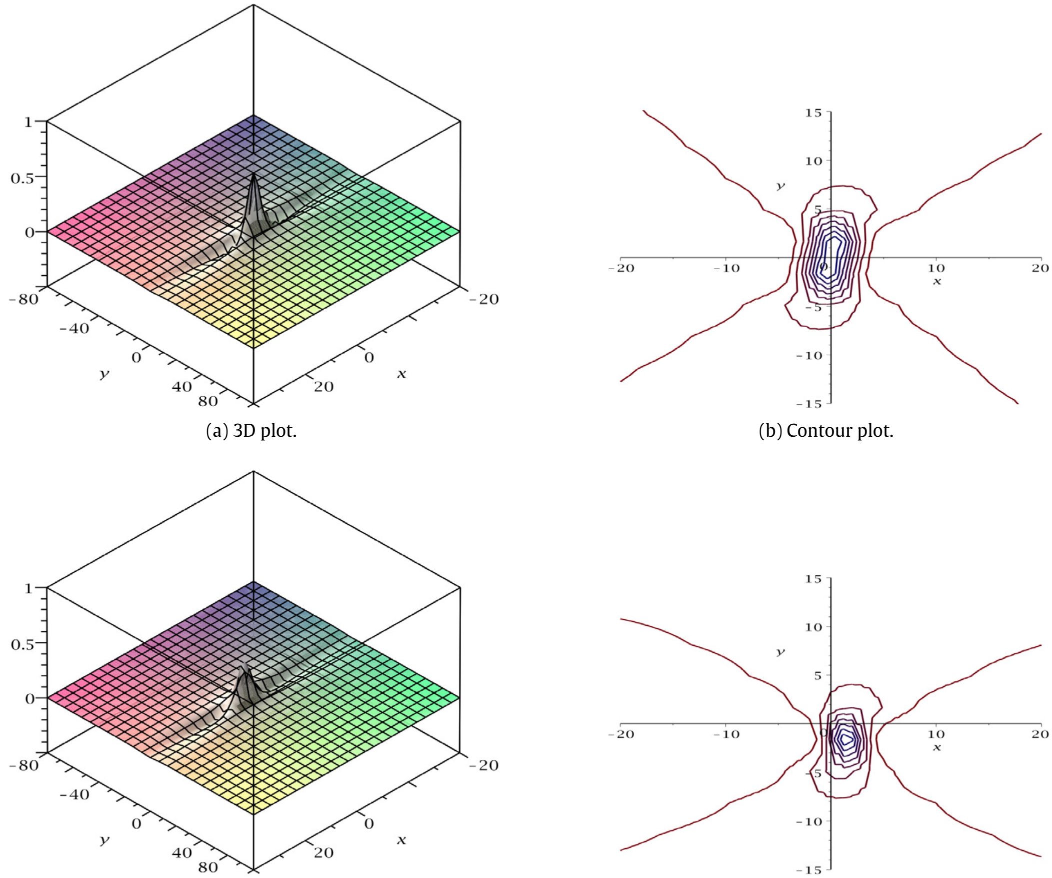
For $t = 2$, we obtain the lump solution

$$u = \frac{161.66 + 78.88x - 47.68y - 11.56x^2 + 14.96xy - 2.88y^2}{(63.35 - 11.6x + 8y + 1.7x^2 - 2.2xy + y^2)^2}. \tag{3.16}$$

The 3D and contour plots of u are shown in Fig. 1(c) and (d).

(ii) $c_1 = c_2 = 0$, $\alpha = 0.25$ and $\beta = 0.5$.
Choosing the same parameters in (3.13), we have

$$f(x, t, y) := (2t - 5.1x + y)^2 + (t - 2.7x)^2 + 16633.35. \tag{3.17}$$



(a) 3D plot. (b) Contour plot.

(c) 3D plot. (d) Contour plot.

Fig. 3. (a) & (b) Plots of a lump solution for $a_2 = 1, a_3 = 2, a_5 = 0, a_6 = 1, c_1 = -2, c_2 = -1, \alpha = 0.5, \beta = -0.5, t = 1$. (c) & (d) Plots of a lump solution for $a_2 = 1, a_3 = 2, a_5 = 0, a_6 = 1, c_1 = -2, c_2 = -1, \alpha = 0.5, \beta = -0.5, t = 2$.

For $t = 1$, we obtain the lump solution

$$u = \frac{0.14871094 + 7.425x - 6.84y - 1.265625x^2 + 2.7xy - 0.63y^2}{(9.74609375 - 3.3x + 4y + 0.5625x^2 - 1.2xy + y^2)^2}. \tag{3.18}$$

The 3D and contour plots of u are shown in Fig. 2(a) and (b).

For $t = 2$, we obtain the lump solution

$$u = \frac{-31.44128906 + 14.85x - 13.68y - 1.265625x^2 + 2.7xy - 0.63y^2}{(24.74609375 - 6.6x + 8y + 0.5625x^2 - 1.2xy + y^2)^2}. \tag{3.19}$$

The 3D and contour plots of u are shown in Fig. 2(c) and (d).

(iii) $c_1 = -2$, $c_2 = -1$, $\alpha = 0.5$ and $\beta = -0.5$.

Again, choosing the same parameters in (3.13), we obtain,

$$f(x, y, t) = (2t - 0.1x + y - 2)^2 + (t - 0.7x - 1)^2 + 3.75. \tag{3.20}$$

For $t = 1$, we have the lump solution

$$u = \frac{-x^2 + 0.4xy + 1.92y^2 + 7.5}{(0.5x^2 - 0.2xy + y^2 + 3.75)^2}. \tag{3.21}$$

The 3D and contour plots of u are shown in Fig. 3(a) and (b).

And for $t = 2$, we have the lump solution

$$u = \frac{11.02 + 3.6x + 6.56y - x^2 + 0.4xy + 1.92y^2}{(8.75 - 1.8x + 4y + 0.5x^2 - 0.2xy + y^2)^2}. \quad (3.22)$$

The 3D and contour plots of u are shown in Fig. 3(c) and (d).

4. Concluding remarks

We have presented lump solutions to an extended KP equation by means of the Hirota bilinear formulation. The solutions have been depicted for various values of the parameters α and β and two different values of t. In Figs. 1 and 2, we have chosen the same values for the constants c_1 and c_2 but different values for the parameters α and β in an effort to demonstrate the effect of the parameters on the evolution of the lump profile. Fig. 1 shows the evolution of the lump profile with a small amplitude for the same values of α and β. Fig. 2 shows that different values of the parameters α and β give rise to multiple clustered lumps with a higher amplitude. Finally, Fig. 3 depicts the evolution of the lump profile for different values of the parameters and the constants c_1 and c_2.

We remark that the eKP equation (1.1) is a reduction of a generalized KP equation introduced in [2] when the coefficients c_1 and c_2 of u_{xx} and u_{xy}, respectively, of the generalized KP equation [2] are chosen to be zero. However, our solution (2.10)–(2.12) is different from the solution in [2]. One may see that unlike the solution in [2], our solution does not satisfy the relations $c_3\nu_{1,1} + c_4\nu_{1,2} + c_5\nu_{1,3} + c_6\nu_{1,4} = 0$ and $c_3\nu_{2,1} + c_4\nu_{2,2} + c_5\nu_{2,3} + c_6\nu_{2,4} = 0$ which are a consequence of the conditions $c_1 = c_2 = 0$ (see [2] for details).

It is easily observed that when $\alpha = \beta = 0$ the eKP equation reduces to the KP equation and thus lump solutions of the KP equation can be obtained from the lump solutions of the eKP equation when $\alpha = \beta = 0$. For example, the choice of the parameters $a_2 = -1, a_3 = -2, a_5 = -1, a_6 = 1, c_1 = c_2 = 0$ and $\alpha = \beta = 0$ leads to the positive quadratic function,

$$f(x, y, t) = \left(-2t + \frac{2}{5}x - y\right)^2 + \left(t - \frac{4}{5}x - y\right)^2 + \frac{16}{15}$$

which in turn yields the lump solution,

$$u = -\frac{48(21t^2 - 48tx - 78ty + 12x^2 + 12xy - 24y^2 - 16)}{(75t^2 - 48tx + 30ty + 12x^2 + 12xy + 30y^2 + 16)^2}.$$

This is exactly one of the lump solutions presented for the KP equation in [4].

There are two interesting problems on lump solutions for further research. One is to consider higher order positive polynomial solutions to Hirota bilinear equations and generalized bilinear equations, and the other is to classify bilinear equations which possess specific classes of lump solutions. We plan to develop more explanations for those two problems in future projects.

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References

[1] R. Hirota, The Direct Method in Soliton Theory, Cambridge University Press, Cambridge, 2004.
[2] W.X. Ma, Y. Zhou, Lump solutions to nonlinear partial differential equations via Hirota bilinear forms, J. Diff. Eqn. 264 (2018) 2633–2659.
[3] I.S. Aranson, A. Pikovsky, N.F. Rulkov, L.S. Tsimring (Eds.), Advances in Dynamics, Patterns, Cognition: Challenges in Complexity, Springer, 2017.
[4] W.X. Ma, Lump solutions to the Kadomtsev–Petviashvili equation, Phys. Lett. A 379 (36) (2015) 1975–1978.
[5] J.Y. Yang, W.X. Ma, Abundant interaction solutions of the KP equation, Nonlinear Dynam. 89 (2017) 1539–1544.
[6] J.B. Zhang, W.X. Ma, Mixed lump-kink solutions to the BKP equation, Comput. Math. Appl. 74 (2017) 591–596.
[7] J.Y. Yang, W.X. Ma, Z.Y. Qin, Lump and lump-soliton solutions to the (2 + 1)-dimensional Ito equation, Anal. Math. Phys. (2017).
[8] H.Q. Zhang, W.X. Ma, Lump solutions to the (2 + 1)-dimensional Sawada-Kotera equation, Nonlinear Dynam. 87 (4) (2017) 2305–2310.
[9] K. Imai, K. Nozaki, Lump solutions of the Ishimori-II equation, Progr. Theoret. Phys. 96 (3) (1996) 521–526.
[10] B.B. Kadomtsev, V.I. Petviashvili, On the stability of solitary waves in weakly dispersing media, Sov. Phys. Dokl. 15 (1970).
[11] A.A. Minzoni, N.F. Smyth, Evolution of lump solutions for the KP equation, Wave Motion 24 (1996) 291–305.
[12] V.E. Zakharov, Shock waves propagated on solitons on the surface of a fluid, Radiophys. Quantum Electronics 29 (1986) 813–817.
[13] U.K. Samanta, A. Saha, P. Chatterjee, Phys. Plasmas 20 (2013) 022111.
[14] A. Saha, N. Pal, P. Chatterjee, Bifurcation and quasiperiodic behaviors of ion acoustic waves in magnetoplasmas with nonthermal electrons featuring tsallis distribution, Braz. J. Phys. 45 (2015) 325–333.
[15] A.M. Wazwaz, Extended KP equations and extended system of KP equations: multiple-soliton solutions, Can. J. Phys. 89 (2011) 739–743.
[16] G.C. Das, J. Sarma, Evolution of solitary waves in multicomponent plasmas, Chaos Solitons Fractals 9 (1998) 901–911.
[17] H. Tsuji, M. Oikawa, Oblique interaction of solitons in an extended Kadomtsev–Petviashvili equation, J. Phys. Soc. Japan 76 (2007) 084401–084408.
[18] A. Peckan, The Hirota Direct Method (Masters thesis), Bilkent University, 2005.