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Integrability, bilinearization, exact traveling wave solutions, lump and lump-multi-kink solutions of a (3 + 1)-dimensional negative-order KdV–Calogero–Bogoyavlenskii–Schiff equation

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Abstract In this article, we consider a (3 + 1)-dimensional negative-order KdV–CBS equation which represents interactions of long wave propagation dynamics with remarkable applications in the field of fluid mechanics and quantum mechanics. We investigate the integrability aspect of the considered model in the framework of Hirota bilinear differential calculus, construct infinitely many conservations laws and formulate a Lax pair. At first, we introduce the concept of Bell polynomial theory and utilize it to obtain the Hirota bilinear form. We introduce a two-field condition to determine the bilinear Bäcklund transformation. We use the Cole–Hopf transformation in bilinear Bäcklund transformation and linearize it to obtain the Lax pair formulation. The existence of infinitely many conservation laws has been checked through the Bell polynomial theory. Moreover, we derive one-kink, two-kink and three-kink soliton solution from the Hirota bilinear form. We have successfully investigated the existence of traveling wave solution for the (3 + 1)-dimensional negative-order KDV–CBS equation and the conditions for the existence of the solution are reported. The traveling wave solutions are extracted in the form of incomplete elliptic integral of second kind and Jacobi elliptic function. Particularly, the use of long wave limit yields kink soliton solutions. Furthermore, we exhibit necessary and sufficient condition for extracting lump solutions of (3 + 1)-dimensional nonlinear evolution equations, which have few particular types of Hirota bilinear form. The lump solutions are exploited by means of well-known test function in the Hirota bilinear form. This method reduces the number of algebraic equations to solve in deriving lump solutions of variety of NLLEs in comparison with the previously available methods in literature. Finally, two new forms of test functions are chosen and lump-multi-kink solutions have been determined.

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1 Introduction

Nonlinear evolution equations (NEEs) play a pivotal role across numerous fields of science and engineering as they capture real-life phenomena observed in disciplines like fluid mechanics [1], oceanography [2], optical fibers, solid-state physics [3], geochemistry, plasma physics [4], nonlinear optics [5] and wave propagation in shallow water [6]. In recent decades, researchers have been increasingly interested in finding exact solutions to these NEEs. The integrability of NEEs has emerged as a prominent research topic, as it guarantees the existence of exact solutions. Several sophisticated approaches are available to test the integrability of nonlinear evolution equations, including the inverse scattering method [7], Hirota bilinear method [8], Darboux transformation method [9,10], Painlevè analysis technique [11], Lie symmetry analysis [12], bilinear Bäcklund transformation method [13], bilinear neural network method (BNN) [14–16] and bilinear residual network method (RNN) [17]. BNN and RNN method can be considered as general symbolic computing method which is used to determine exact analytical solution of a nonlinear partial differential equation. These methods give more accurate analytic solution than other existing methods. Painlevè analysis technique is one of the most complicated but efficient methods for investigating integrability characteristic of nonlinear partial differential equations [18,19]. Salah et al [20] employed discrete singular convolution algorithm and obtained exact solution of five dimensional Fokas equation. While there is no definitive definition of integrability for NEEs, indicators such as infinite conservation laws, bilinear Bäcklund transformation, Hirota bilinear form and Lax pair can characterize their integrability. Among these techniques, the Hirota bilinear method is the simplest approach to determine the integrability of NEEs. This method involves a dependent variable transformation that converts a NEE into Hirota bilinear form. After obtaining Hirota bilinear form, soliton solutions for the NEE can be easily derived. Gilson et al. [21], Lambert and Springael [22,23] developed a revolutionary method to derive the Hirota bilinear form of NEEs using Bell polynomial theory. Using this method, one can directly deduce the bilinear Bäcklund transformation and Lax pair from the Hirota bilinear form. E. Fan [24] extended this method to derive infinite conservation laws directly from the bilinear Bäcklund transformation. Recently, many researchers have employed these methods to assess the integrability of various types of NEEs. For instance, Xu and Wazwaz [25] examined the integrability of a new $(n+1)$-dimensional generalized Kadomtsev–Petviashvili equation, obtaining the Lax pair, Bäcklund transformation, infinite conservation laws and deriving breather solutions, N-soliton solutions, lump solutions and breather-soliton mixed solutions. Mandal et al. [26] demonstrated that the generalized $(2+1)$-dimensional Hirota bilinear equation is completely integrable, deriving one, two and three-soliton solutions, as well as invariant solutions. Raut et al. [27] investigated the integrability of the non-autonomous Kadomtsev–Petviashvili equation and calculated various types of periodic solutions.

On the other hand, utilization of traveling wave solutions has become ubiquitous across various fields of study. Numerous well-established methods have been employed to obtain exact solutions for nonlinear partial differential equations (NLPDEs). These methods include the Jacobi elliptic function method [28–30], F-expansion method [31,32], Bäcklund transformation method [33], (G'/G)-expansion method [34], extended tanh method [35,36], new auxiliary equation approach [37] and the exp-function method [38], among others. The exact traveling wave solutions provide better physical insight of the dynamical behavior or the propagation dynamics of the concerned model. Exact solutions of NEEs, particularly solitons, lumps, breathers and rogue waves, have garnered increasing attention from researchers due to their intriguing dynamical properties. As waves propagate through nonlinear media, their amplitudes and widths undergo fluctuations. However, in certain scenarios, the interplay between nonlinearity and dispersion can lead to the formation of permanent and localized waves known as solitons. There are only a few mathematical physics techniques available for identifying solitons in nonlinear dispersive models, with two notable examples being the Hirota bilinear method [8] and the inverse scattering transform technique [7]. The Riemann–Hilbert approach is considered as one of the most powerful method in solving integrable models, especially for constructing soliton solutions. A large variety of local and nonlocal integrable equations have been investigated, and their soliton solutions are constructed via Riemann–Hilbert approach in [39–42]. Lump waves (or rogue waves) can be described within the framework of integrable models, and they can explain a wide range of nonlinear phenomena [43,44]. These waves exhibit localization in all spatial directions and are determined using rational function methods [45]. General rational function solutions are possessed by different integrable equations such as the KdV equation, Toda lattice equation and the Boussinesq equation and are obtained using the concept of Wronskian and Casoratian determinant technique [46–48]. The concept of rational functions was also applied to develop a powerful approach to accurate traveling wave solutions [49]. Generating positive quadratic function solutions in case of Hirota bilinear equations constitutes an important role in developing lump waves. Once the quadratic function solutions are confirmed to be positive, logarithmic derivative transformations are employed to generate lump waves [50]. Another approach to obtaining lump solutions involves applying the long wave limit to the soliton solution derived from the Hirota bilinear form [51]. Chen and Lü [52] present a necessary and sufficient condition for calculating lump solutions for nonlinear partial differential equations, particularly those with specific types of Hirota bilinear forms. They also demonstrate that if a lump solution is derived, a lump-multi-kink solution can also be obtained. Chen et al. also derive interaction between lump and multistrip solutions of (2+1)-dimensional nonlinear models [53–55].

We consider non-dimensional classical KdV equation [56] in the form

$$u_t + 6uu_x + u_{xxx} = 0\,, \tag{1}$$

where x represents spatial coordinate and t denotes temporal and $u(x, t)$ represents the water wave velocity in the shallow water surface. The KdV equation demonstrates dynamical systems governed by weak quadratic nonlinearity and weak dispersion, e.g., in the study of long, negligible amplitudes, surface gravity waves generated in shallow water, stratified internal waves, plasma physics, ion-acoustic wave, lattice dynamics, etc. It is well known that the KdV equation is completely integrable in terms of inverse scattering transform, Painlevè analysis technique, etc.

We consider the (2 + 1)-dimensional Calogero–Bogoyavlenskii–Schiff (CBS) equation [57]

$$u_t + 4uu_y + 2u_x\partial_x^{-1}u_y + u_{xxy} = 0\,, \tag{2}$$

which narrates the nonlinear interaction among Riemann propagating wave along y-axis along with long propagating wave along x-axis. Using the recursion operator

$$\phi = \partial_x^2 + 4u + 2u_x\partial_x^{-1}\,, \tag{3}$$

one can obtain the KdV equation and CBS equation as follows

$$\begin{aligned} u_t + \phi(u_x) &= \partial_x^2(u_x) + 4uu_x + 2u_x\partial_x^{-1}(u_x) \\ &= u_t + u_{xxx} + 6uu_x = 0\,, \end{aligned} \tag{4a}$$

$$\begin{aligned} u_t + \phi(u_y) &= \partial_x^2(u_y) + 4uu_y + 2u_x\partial_x^{-1}(u_y) \\ &= u_t + u_{xxy} + 4uu_y + 2u_x\partial_x^{-1}(u_y) = 0\,. \end{aligned} \tag{4b}$$

Extending Olver's work in [59], Verosky [58] developed a new way to find a sequence of equations having increasingly negative orders in negative direction. According to Verosky [58], the evolution equations have following hierarchy

$$u_t + \phi(v_x) = 0\,, \tag{5}$$

$$u_t + \phi(v_y) = 0\,, \tag{6}$$

in the derivation of the KdV and CBS equations respectively can be alternatively used in the negative-order hierarchy in the form

$$u_t + \phi^{-1}(u_x) = 0\,, \tag{7}$$

$$u_t + \phi^{-1}(u_y) = 0\,, \tag{8}$$

for the KdV and CBS equations, respectively. Wazwaz implemented the negative-order hierarchy Eqs. (7) and (8) and obtained the integrable negative-order KdV equation and integrable negative-order CBS equation as below

$$u_{xxxt} + 4u_xu_{xt} + 2u_{xx}u_t + u_{xx} = 0 \tag{9}$$

and

$$u_{xxxt} + 4u_xu_{xt} + 2u_{xx}u_t + u_{xy} = 0, \tag{10}$$

respectively. In article [59–61], Wazwaz proved that the above two equations pass the Painlevè integrability test. Also, the integrability of the equations is proved by means of consistent Riccati expansion method. The truncated Painlevè expansion and simplified Hirota's methods are utilized to verify the integrability characteristics. Abundant solutions having different physical structure, e.g., multiple soliton solutions, kink solutions, multiple complex soliton solutions and singular solutions, are derived in explicit forms.

Recently, Wazwaz [62] derived a new (3 + 1)-dimensional negative-order KDV–CBS equation by combining Eqs. (9) and (10) as

$$u_{xt} + u_{xxxy} + 4u_x u_{xy} + 2u_{xx} u_y + \lambda u_{xx} + \mu u_{xy} + \nu u_{xz} = 0, \quad (11)$$

which represents interactions of long wave propagation dynamics with remarkable applications in the field of fluid mechanics and quantum mechanics. When $\mu = 0$, $\nu = 0$ Eq. (11) converts to a negative-order KdV equation, and when $\lambda = 0$, $\nu = 0$ Eq. (11) converts to a negative-order CBS equation. It is also examined that Eq. (11) passes the Painlevè integrability test without any constraint on compatibility condition. Additionally, Wazwaz et al. [63] obtained the Hirota bilinear form, periodic wave, lump wave, rogue wave and their interaction solution of the same model. Maria [64] checked the integrability of Eq. (11) by Lie symmetry analysis and obtained various kind of traveling wave solutions. Singh and Saha [65] obtained an integrable version of Eq. (11) by Painlevè analysis and also obtained several types of analytic solutions like exponential solution and rational function solution using auto-Bäcklund transformation. We have noticed that several other aspects of integrability for the above-mentioned model are not studied yet. The concept of Bell polynomial theory can be executed to obtain the Hirota bilinear form, existence of infinite conservation laws, Lax pair formulation and Bäcklund transformation which can be studied in view of the concerned nonlinear evolution equation.

Main contexts of this paper are organized as follows. In Sect. 2, basic introduction of multi-dimensional Bell polynomial and the connection between Hirota bilinear form and Bell polynomial are given. In Sect. 3, we derive the Hirota bilinear form, Bäcklund transformation and Lax pair formulation of Eq. (11). In Sect. 4, infinitely many conservation laws of Eq. (11) are constructed. In Sect. 5, one-, two- and three-kink soliton solutions of the above equation are derived. In Sect. 6, we explore the traveling wave solution in terms of Jacobi elliptic function and obtain their long wave limits. In Sect. 7, we construct lump and lump-multi-kink solution of the above-mentioned equation. Finally, in Sect. 8, we draw few conclusion of our work.

2 Multi-dimensional Bell polynomials

In this section, we briefly demonstrate the underlying principles and expressions of Bell polynomials [21,22]. Let ϕ be a C^∞ function of t; then one-dimensional Bell polynomial [21] is defined as

$$Y_{nt}(\phi) = Y_n(\phi_1, \phi_2, ..., \phi_{nt}) = e^{-\phi}\partial_t{}^n e^{\phi}, \ \ n = 1, 2, 3, \dots . \quad (12)$$

The following are few one-dimensional Bell polynomials that can be obtained from the above statement.

$$Y_t = \phi_t, \ \ Y_{2t} = \phi_{2t} + \phi_t{}^2, \quad Y_{3t} = \phi_{3t} + 3\phi_t\phi_{2t} + \phi_t{}^3, \dots . \quad (13)$$

We use the formula

$$Y_{nt}(\phi) = \sum \frac{n!}{a_1!a_2!...a_n!}\left(\frac{\phi_t}{1!}\right)^{a_1}\left(\frac{\phi_{2t}}{2!}\right)^{a_2}...\left(\frac{\phi_{nt}}{n!}\right)^{a_n}, \quad (14)$$

where the sum run over all partitions of $n = a_1 + 2a_2 + ... + na_n$ and obtain the above expressions Eq. (13). By assuming that $\phi = \phi(t_1, t_2, ..., t_s)$ as a C^∞ multi-variable function, we can expand the dimension of the Bell polynomial and then the multi-dimensional Bell polynomial can be described as follows

$$Y_{n_1t_1,...,n_st_s}(\phi) \equiv Y_{n_1,...,n_s}(\phi_{m_1t_1,...,m_st_s}) = e^{-\phi}\partial_{t_1}^{n_1}...\partial_{t_s}^{n_s}\, e^{\phi}, \quad (15)$$

where $\phi_{m_1t_1,...,m_st_s} = \partial_{t_1}^{m_1}...\partial_{t_s}^{m_s}\phi, m_i = 0, 1, ..., n_i$ and $i = 1, 2, ..., s$. Here $Y_{n_1t_1,...,n_st_s}(\phi)$ denotes the multi-variable Bell polynomial with respect to $\phi_{m_1t_1,...,m_st_s}$. Particularly, if we choose ϕ as a function of t and z, then corresponding few lowest order two-dimensional Bell polynomials can be determined as follows

$$Y_{2t(\phi)} = \phi_{2t} + \phi_t^2, \quad Y_{3t(\phi)} = \phi_{3t} + 3\phi_{2t}\phi_t + \phi_t^3, \quad (16)$$

$$Y_{t,z} = \phi_{t,z} + \phi_t\phi_z, \quad Y_{2t,z}(\phi) = \phi_{2t,z} + \phi_{2t}\phi_z + 2\phi_{t,z}\phi_t + \phi_t^2\phi_z, ... \quad (17)$$

According to the aforesaid Bell polynomials Eq. (15), multi-dimensional binary Bell polynomials can be described as follows

$$\mathscr{Y}_{n_1t_1,...,n_st_s}(\mathscr{h},\mathscr{g}) = Y_{n_1t_1,...,n_st_s}(\phi), \tag{18}$$

where

$$\phi_{m_1t_1,...,m_st_s} = \begin{cases} \mathscr{h}_{m_1t_1,...,m_st_s}, & m_1+...+m_s \text{ is odd}, \\ \mathscr{g}_{m_1t_1,...,m_st_s}, & m_1+...+m_s \text{ is even}. \end{cases} \tag{19}$$

The following are a few one-dimensional Bell polynomials that can be obtained from the above statement:

$$\mathscr{Y}_t(\mathscr{h}) = \mathscr{h}_t, \quad \mathscr{Y}_{2t}(\mathscr{h},\mathscr{g}) = \mathscr{g}_{2t} + \mathscr{h}_t^2,$$
$$\mathscr{Y}_{t,z}(\mathscr{h},\mathscr{g}) = \mathscr{g}_{t,z} + \mathscr{h}_t\mathscr{h}_z,$$
$$\mathscr{Y}_{2t,z}(\mathscr{h},\mathscr{g}) = \mathscr{h}_{2t,z} + \mathscr{g}_{2t}\mathscr{h}_z + 2\mathscr{g}_{t,z}\mathscr{h}_t + \mathscr{h}_t^2\mathscr{h}_z,$$
$$\mathscr{Y}_{3t} = \mathscr{h}_{3t} + 3\mathscr{g}_{2t}\mathscr{h}_t + \mathscr{h}_t^3, ... \tag{20}$$

With the help of the identity

$$(\phi\psi)^{-1}D_{t_1}^{n_1}...D_{t_s}^{n_s}\phi.\psi = \mathscr{Y}_{n_1t_1,...,n_st_s}(\mathscr{h} = \ln\phi/\psi, \mathscr{g} = \ln\phi\psi), \tag{21}$$

we can link the standard Hirota bilinear expression $D_{t_1}^{n_1}...D_{t_s}^{n_s}\phi.\psi$ and binary Bell polynomials, where the D-operator is presented by Hirota [8] as

$$D_{t_1}^{n_1}...D_{t_s}^{n_s}\phi.\psi = (\partial_{t_1} - \partial_{t_1'})^{n_1}...(\partial_{t_s} - \partial_{t_s'})^{n_s}\phi (t_1,...,t_s).\psi(t_1',...t_s')|_{t_1'=t_1,...,t_s'=t_s}.$$

In case when $\phi = \psi$, the identity Eq. (21) becomes

$$(\phi)^{-2}D_{t_1}^{n_1}...D_{t_s}^{n_s}\phi.\phi = \mathscr{Y}_{n_1t_1,...,n_st_s}(\mathscr{h}=0, \mathscr{g} = 2\ln\phi) = \begin{cases} 0, & n_1+...+n_s \text{ is odd}, \\ \mathscr{P}_{n_1t_1,...,n_st_s}(p), & n_1+...+n_s \text{ is even}. \end{cases} \tag{22}$$

where $\mathscr{P}$-polynomials are the even-ordered $\mathscr{Y}$-polynomials and first few of them are as below

$$\mathscr{P}_{2t}(\mathscr{q}) = \mathscr{q}_{2t}, \quad \mathscr{P}_{t,z}(\mathscr{q}) = \mathscr{q}_{t,z},$$
$$\mathscr{P}_{3t,z}(\mathscr{q}) = \mathscr{q}_{3t,z} + 3\mathscr{q}_{t,z}\mathscr{q}_{2t},$$
$$\mathscr{P}_{4t}(\mathscr{q}) = \mathscr{q}_{4t} + 3\mathscr{q}_{2t}^2. \tag{23}$$

The binary Bell polynomial $\mathscr{Y}_{n_1t_1,...,n_st_s}(\mathscr{h},\mathscr{g})$ can be written as a linear combination of $\mathscr{P}$-polynomials and Bell polynomials $Y_{n_1t_1,...,n_st_s}(\mathscr{h})$ as

$$\begin{aligned}(\phi\psi)^{-1}D_{t_1}^{n_1}...D_{t_s}^{n_s}\phi.\psi &= \mathscr{Y}_{n_1t_1,...,n_st_s}(\mathscr{h},\mathscr{g}), \\ &\quad \text{where } \mathscr{h} = \ln\phi/\psi \text{ and } \mathscr{g} = \ln\phi\psi \\ &= \mathscr{Y}_{n_1t_1,...,n_st_s}(\mathscr{h},\mathscr{h}+p), \\ &\quad \text{where } \mathscr{h} = \ln\phi/\psi \text{ and } \mathscr{q} = 2\ln\psi \\ &= \sum_{m_1=0}^{n_1}...\sum_{m_s=0}^{n_s}\prod_{i=1}^{s}\binom{n_i}{m_i}\mathscr{P}_{m_1t_1,...,m_st_s}(p) \\ &\quad Y_{(n_1-m_1)t_1,...,(n_s-m_s)t_s}(\mathscr{h}). \end{aligned} \tag{24}$$

Using Hopf–Cole transformation $\mathscr{h} = \ln\psi$, binary Bell polynomial can be expressed in the following form

$$Y_{n_1t_1,...,n_st_s}(\mathscr{h} = \ln\psi) = \frac{\psi_{n_1t_1,...,n_st_s}}{\psi}, \tag{25}$$

through which Eq. (24) can be reexpressed as

$$(\phi\psi)^{-1}D_{t_1}^{n_1}...D_{t_s}^{n_s}\phi.\psi = \psi^{-1}\sum_{m_1=0}^{n_1}...\sum_{m_s=0}^{n_s}\prod_{i=1}^{s}\binom{n_i}{m_i}\mathscr{P}_{m_1t_1,...,m_st_s}(\mathscr{q}) \psi_{(n_1-m_1)t_1,...,(n_s-m_s)t_s}. \tag{26}$$

The identity Eq. (26) offers the most straightforward method for developing the related Lax pair of the appropriate nonlinear evolution equation. The above-mentioned concepts of Bell polynomial theory will be utilized further to determine bilinear form, bilinear Bäcklund transformation and the Lax pair formulation of Eq. (11).

3 Bilinear form, bilinear Bäcklund transformation and Lax pair

We consider the transformation $u = q_x$ in Eq. (11) and integrate it which yield

$$E(q) = q_{xt} + q_{3xy} + 3q_{2x}q_{xy} + \lambda q_{2x} + \mu q_{xy} + \nu q_{xz} = 0. \tag{27}$$

Additionally, we choose $q = 2\log f$ and using the connection between $\mathscr{P}$-polynomial and Hirota bilinear form, we obtain the Hirota bilinear form of Eq. (11) as

$$(D_xD_t + D_x^3D_y + \lambda D_x^2 + \mu D_xD_y + \nu D_xD_z)f.f = 0. \tag{28}$$

In order to construct bilinear Bäcklund transformation, we consider $q' = 2\log f'$ as an another solution of Eq. (11). Then the corresponding two-field condition can be written as

$$E(q') - E(q) = (q'-q)_{xt} + (q'-q)_{3xy} + 3q'_{2x}q'_{xy} - 3q_{2x}q_{xy} + \lambda(q'-q)_{2x} + \mu(q'-q)_{xy} + \nu(q'-q)_{xz} = 0. \tag{29}$$

Furthermore, we introduce two new variables $W = \log ff'$, $V = \log\frac{f'}{f}$ and consider $\lambda = \mu = \nu = 1$ and rewrite Eq. (29) as

$$E(q') - E(q) = 2\big[V_{xt} + V_{3xy} + 3(W_{2x}.V_{xy}$$

$$+V_{2x}W_{xy}) + V_{2x} + V_{xy} + V_{xz}\big] = 0,$$
$$= 2\frac{\partial}{\partial y}[\mathscr{Y}_{3x}(V,W) + \mathscr{Y}_x(V)] + 2\frac{\partial}{\partial x}[\mathscr{Y}_t(V) + \mathscr{Y}_z(V)] + 6\,\text{Wronskian}[\mathscr{Y}_{xy}(V,W) + \frac{1}{3}, \mathscr{Y}_x(V)] = 0\,. \tag{30}$$

We Decouple the above two-field condition and derive three $\mathscr{Y}$-polynomials as follows

$$\mathscr{Y}_{xy}(V,W) + \frac{1}{3} - \delta\mathscr{Y}_x(V) = 0\,, \tag{31a}$$
$$\mathscr{Y}_{3x}(V,W) + \mathscr{Y}_x(V) - \beta = 0\,, \tag{31b}$$
$$\mathscr{Y}_t(V) + \mathscr{Y}_z(V) - \gamma = 0\,. \tag{31c}$$

Finally, with the help of Eq. (21), we have rewritten Eq. (31) in the bilinear form as

$$(f'\cdot f)^{-1}\left[D_xD_y + \frac{1}{3} - \delta D_x\right](f'\cdot f) = 0\,, \tag{32a}$$
$$(f'\cdot f)^{-1}[D_x^3 + D_x - \beta](f'\cdot f) = \beta\,, \tag{32b}$$
$$(f'\cdot f)^{-1}[D_t + D_z - \gamma](f'\cdot f) = 0\,, \tag{32c}$$

which is bilinear Bäcklund transformation of Eq. (11).

Using the Cole–Hopf transformation $V = \log\phi$ and linearizing the above Bell polynomial system Eq. (31), we formulate the Lax pair of Eq. (11) as

$$\phi_{xy} + q_{xy}\phi + \frac{1}{3}\phi - \delta\phi_x = 0\,, \tag{33a}$$
$$\phi_{3x} + (1 + 3q_{2x})\phi_x - \beta\phi = 0\,, \tag{33b}$$
$$\phi_t + \phi_z - \gamma\phi = 0\,. \tag{33c}$$

4 Infinitely many conservation laws

To construct the conservation laws of the Eq. (11), we have rewritten the two-field condition as

$$W_{xy} + V_xV_y + \frac{1}{3} - \delta V_x = 0\,, \tag{34}$$
$$\frac{\partial}{\partial t}[\mathscr{Y}_x(V)] + \frac{\partial}{\partial x}[\mathscr{Y}_y(V)] + \frac{\partial}{\partial y}[\mathscr{Y}_{3x}(V,W)] + \frac{\partial}{\partial z}[\mathscr{Y}_x(V)] = 0\,. \tag{35}$$

We introduce a new potential function $\zeta = \frac{q'_x - q_x}{2}$ and obtain

$$V = \frac{q'-q}{2},\ W = \frac{q'+q}{2},\ V_x = \zeta,$$
$$V_y = \partial_x^{-1}(\zeta_y),\ W_x = q_x + \zeta. \tag{36}$$

Substituting Eq. (36) into Eq. (35) yields a Riccati type equation as

$$q_{xy} + \zeta_y + \zeta\partial_x^{-1}(\zeta_y) + \frac{1}{3} - \delta\zeta = 0\,. \tag{37}$$

We consider the form of ζ as

$$\zeta = \sum_{n=1}^{\infty} J_n(q, q_x, q_{2x}, ...)\delta^{-1}\,. \tag{38}$$

Finally, we substitute Eq. (38) into Eq. (37) and equate all the like powers of δ and we obtain the recursions relations for the conserved densities as below

$$J_1 = q_{xy} + \frac{1}{3} = u_y + \frac{1}{3}\,, \tag{39a}$$
$$J_2 = J_{1,y} = u_{yy}\,, \tag{39b}$$
$$J_3 = J_{2,y} + J_1\partial_x^{-1}J_{1,y} = u_{3y} + (u_y + \frac{1}{3})(\partial_x^{-1})u_{yy}\,, \tag{39c}$$
$$J_4 = J_{3,y} + (J_1\partial_x^{-1}J_{2,y} + J_2\partial_x^{-1}J_{1,y})\,, \tag{39d}$$
$$\cdots\cdots$$
$$J_n = J_{n-1,y} + \sum_{i=1}^{n-1} J_i(\partial_x^{-1}J_{n-1-i,y})\,. \tag{39e}$$

Substitution of Eq. (36) into Eq. (35) provides a divergence-type equation

$$\zeta_t + \frac{\partial}{\partial y}\left[\zeta_{2x} + 3\zeta q_{2x} + 3\zeta\zeta_x + \zeta^3\right] + \zeta_z + \frac{\partial}{\partial x}\left[\partial_x^{-1}\zeta_y\right] = 0\,. \tag{40}$$

Moreover, we substitute Eq. (38) into Eq. (40) and have

$$\sum_{n=1}^{\infty} J_{n,t}\delta^{-n} + \partial y\left[\sum_{n=1}^{\infty} J_{n,2x}\delta^{-n} + 3\left(\sum_{n=1}^{\infty} J_n\delta^{-n}\right)\left(q_{2x} + \sum_{n=1}^{\infty} J_{n,x}\delta^{-n}\right) + \left(\sum_{n=1}^{\infty} J_n\delta^{-n}\right)^3\right] + \partial x\left[\partial_x^{-1}\sum_{n=1}^{\infty} J_{n,y}\delta^{-n}\right] + \sum_{n=1}^{\infty} J_{n,z}\delta^{-n} = 0\,. \tag{41}$$

The conservation laws of Eq. (11) can be found from Eq. (41) as

$$\mathscr{A}_{n,t} + \mathscr{B}_{n,y} + \mathscr{C}_{n,x} + \mathscr{D}_{n,z} = 0,\ \ n = 1, 2, 3, ..., \tag{42}$$

where,

$$\mathscr{B}_1 = J_{1,2x} + 3q_{2x}J_1 = J_{1,2x} + 3u_xJ_1\,, \tag{43a}$$

$$\mathscr{B}_2 = J_{2,2x} + 3(J_2 u_x + J_1 J_{1,x}), \tag{43b}$$

$$\mathscr{B}_3 = J_{3,2x} + 3(J_3 u_x + J_1 J_{2,x} + J_2 J_{1,x} + J_1^3), \tag{43c}$$

$$\cdots\cdots$$

$$\mathscr{B}_n = J_{n,2x} + 3\sum_{i+j=n} J_i I_{j,x} + 3J_n u_x + \sum_{i+j+k=n} J_i J_j J_k, \tag{43d}$$

$\mathscr{C}_n = \partial_x^{-1} J_{n,y}$ and $\mathscr{D}_n = J_n$, $n = 1, 2, 3, \ldots$. The values of J_n are given by the recursion relations Eq. (39).

5 Soliton solutions

5.1 One-soliton solution

To construct one-soliton solution of negative-order KDV–CBS equation Eq. (11), we choose the form of f as

$$f = 1 + e^{\varsigma_1}, \tag{44}$$

where $\varsigma_1 = k_1(x + p_1 y + q_1 z + w_1 t) + \varsigma_1^0$ and $k_1, p_1, q_1, \varsigma_1^0$ are arbitrary constants. Then we substitute Eq. (44) into Eq. (28) and equate all the exponential function to zero and obtain dispersion relation as below

$$w_1 = -(1 + p_1 + q_1 + k_1^2 p_1). \tag{45}$$

Finally, substitution of Eq. (44) along with Eq. (45) into Eq. (28) yields one-soliton solution of Eq. (11) as

$$u = \frac{2k_1 \exp\left[k_1\left(-t\left(k_1^2 p_1 + p_1 + q_1 + 1\right) + p_1 y + q_1 z + x\right)\right]}{\exp\left[k_1\left(-t\left(k_1^2 p_1 + p_1 + q_1 + 1\right) + p_1 y + q_1 z + x\right)\right] + 1}. \tag{46}$$

Particularly, when the parameters takes the values $k_1 = 3$, $p_1 = 2$, $q_1 = 1$ and $\varsigma_1^0 = 0$ and extract the one-kink soliton solution which is shown in Fig. 1a.

5.2 Two-soliton solution

We construct two-soliton solution of Eq. (11) by assuming the form of f as

$$f = 1 + e^{\varsigma_1} + e^{\varsigma_2} + A_{12}\, e^{\varsigma_1 + \varsigma_2}, \tag{47}$$

where $\varsigma_i = k_i(x + p_i y + q_i z + w_i t) + \varsigma_i^0$, $(i = 1, 2)$ and $k_i, p_i, q_i, \varsigma_i^0$, $(i = 1, 2)$ are arbitrary constants. We substitute Eq. (47) into Eq. (28) and derive the dispersion relation and A_{12} as

$$w_i = -(1 + p_1 + q_1 + k_1^2 p_1), \ (i = 1, 2), \tag{48a}$$

$$A_{12} = -\frac{(k_1 - k_2)[(k_2 p_1 - k_1 p_2) + 2(k_1 p_1 - k_2 p_2)]}{(k_1 + k_2)[(k_2 p_1 + k_1 p_2) + 2(k_1 p_1 + k_2 p_2)]}. \tag{48b}$$

Further, substituting Eq. (47) with Eq. (48) into Eq. (28), we obtain the two-soliton solution of Eq. (11) as

$$u = \frac{\partial_x(1 + e^{\varsigma_1} + e^{\varsigma_2} + A_{12}\, e^{\varsigma_1 + \varsigma_2})}{1 + e^{\varsigma_1} + e^{\varsigma_2} + A_{12}\, e^{\varsigma_1 + \varsigma_2}}. \tag{49}$$

For particular parametric values $k_1 = 1, k_2 = 2, p_1 = 3, p_2 = 1, q_1 = 1, q_2 = 1$ along with $\varsigma_i^0 = 0$, $(i = 1, 2)$, we find two-kink soliton solution, which is shown in Fig. 1b.

5.3 Three-soliton solution

We construct three-soliton solution of Eq. (11) by assuming the form of f as

$$f = 1 + e^{\varsigma_1} + e^{\varsigma_2} + e^{\varsigma_3} + A_{12}\, e^{\varsigma_1 + \varsigma_2} + A_{23}\, e^{\varsigma_2 + \varsigma_3} + A_{13}\, e^{\varsigma_1 + \varsigma_3} + A_{123}\, e^{\varsigma_1 + \varsigma_2 + \varsigma_3}, \tag{50}$$

where $\varsigma_i = k_i(x + p_i y + q_i z + w_i t) + \varsigma_i^0$, $(i = 1, 2, 3)$ and $k_i, p_i, q_i, \varsigma_i^0$, $(i = 1, 2, 3)$ are arbitrary constants. We substitute Eq. (50) into Eq. (28) and derive the dispersion relation and A_{ij} as follows

$$w_i = -(1 + p_i + q_i + k_i^2 p_i), \ (i = 1, 2, 3), \tag{51a}$$

$$A_{ij} = -\frac{(k_i - k_j)[(k_j p_i - k_i p_j) + 2(k_i p_i - k_j p_j)]}{(k_i + k_j)[(k_j p_i + k_i p_j) + 2(k_i p_i + k_j p_j)]}, \tag{51b}$$

$$A_{123} = A_{12} A_{23} A_{13}. \tag{51c}$$

Further, substituting Eq. (50) with Eq. (51) into Eq. (28), we obtain the three-soliton solution of Eq. (11) as

$$u = \frac{\partial_x(1 + e^{\varsigma_1} + e^{\varsigma_2} + e^{\varsigma_3} + A_{12}\, e^{\varsigma_1+\varsigma_2} + A_{23}\, e^{\varsigma_2+\varsigma_3} + A_{13}\, e^{\varsigma_1+\varsigma_3} + A_{123}\, e^{\varsigma_1+\varsigma_2+\varsigma_3})}{1 + e^{\varsigma_1} + e^{\varsigma_2} + e^{\varsigma_3} + A_{12}\, e^{\varsigma_1+\varsigma_2} + A_{23}\, e^{\varsigma_2+\varsigma_3} + A_{13}\, e^{\varsigma_1+\varsigma_3} + A_{123}\, e^{\varsigma_1+\varsigma_2+\varsigma_3}} . \tag{52}$$

Taking the parametric values as $k_1 = -3, k_2 = -1, k_3 = -2, p_1 = -2, p_2 = 4, p_3 = 3, q_1 = 1, q_2 = 5, q_3 = -1$ and $\varsigma_i^0 = 0,\ (i = 1, 2, 3)$, we have obtained the three-kink soliton solution, which is shown in Fig. 1c.

6 Traveling wave solutions

In [64], the conservation laws are obtained for the negative-order KDV–CBS equation in (3+1)-dimension and is used to determine a triple reduction to an ordinary differential equation of second order, which yields traveling wave solution and soliton solutions. The solution are also obtained using the modified simple equation method portrait via 3-D plot for particular parametric values.

Here we consider the traveling wave transformation in the form

$$u(x, y, z, t) = \Re(\chi), \quad \chi = x + by + cz - at , \tag{53}$$

where a, b and c are arbitrary constants and χ is a traveling wave transformation.

Substituting Eq. (53) into Eq. (11), we obtain the fourth-order nonlinear ordinary differential equation as

$$b\Re'''' + (6b\Re' + \lambda + b\mu + c\nu - a)\Re'' = 0 . \tag{54}$$

Integrating both sides of Eq. (54), we have

$$b\Re''' + 3\Re'^2 + (\lambda + b\mu + c\nu - a)\Re' = C_1 . \tag{55}$$

Multiplying both sides of this resulting integrals by $2\Re''$ and integrating again, we derive a second-order ODE as

$$\Re''^2 + 2\Re'^3 + \left(\mu + \frac{c\nu - a + \lambda}{b}\right)\Re'^2 - \frac{2C_2}{b}\Re' + \frac{2C_1}{b} = 0 , \tag{56}$$

where C_1 and C_2 are integration constants.

We consider the transformation $\Re'(\chi) = \Im(\chi)$ and then Eq. (56) reduce to

$$(\Im')^2 + 2\Im^3 + \left(\mu + \frac{c\nu - a + \lambda}{b}\right)\Im^2 - \frac{2C_2}{b}\Im + \frac{2C_1}{b} = 0 . \tag{57}$$

Equation (57) can be reduced in standard form as

$$(\Im')^2 = -2\Im^3 + A\Im^2 + B\Im + C , \tag{58}$$

where

$$A = -\left(\mu + \frac{c\nu - a + \lambda}{b}\right), \quad B = \frac{2C_2}{b}, \quad C = -\frac{2C_1}{b}$$

Another transformation $f(\chi) = 2\Im(\chi)$ yields

$$\left(\frac{df}{d\chi}\right)^2 = -f^3 + \alpha f^2 + \beta f + \gamma , \tag{59}$$

where $\alpha = A,\ \beta = \frac{4C_2}{b}$ and $\gamma = -\frac{8C_1}{b}$
Assuming $P(f) = -f^3 + \alpha f^2 + \beta f + \gamma$ be the polynomial such that

$$-f^3 + \alpha f^2 + \beta f + \gamma = -(f - f_1)(f - f_2)(f - f_3) , \tag{60}$$

where f_1, f_2, f_3 are the roots of $P(f) = 0$.

From Eq. (60), we obtain

$$f_1 + f_2 + f_3 = \alpha , \tag{61}$$

$$f_1 f_2 + f_1 f_3 + f_2 f_3 = -\beta , \tag{62}$$

$$f_1 f_2 f_3 = \gamma . \tag{63}$$

To construct the solution of Eq. (59) via Jacobi elliptic function, we choose a new variable transformation

$$z^2 = \frac{f - f_3}{f_2 - f_3} . \tag{64}$$

Then Eq. (59) ensures

$$\frac{dz}{d\xi} = \eta\sqrt{(1 - z^2)(1 - \zeta^2 z^2)} . \tag{65}$$

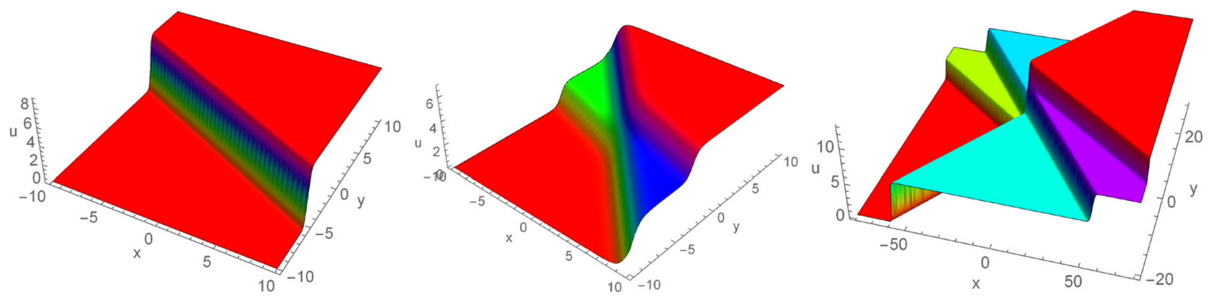
Hence, applying the transformation in Eq. (64) the solution of Eq. (59) is obtained as

$$f = f_2 - (f_2 - f_3)cn^2(\eta(\chi - \chi_0), \zeta) , \tag{66}$$

where $\mathrm{cn}(\chi, \zeta)$ is the Jacobi elliptic function and the roots f_1, f_2 and f_3 are real with ordered relation $f_1 < f_2 < f_3$.

The obtained values of f_1, f_2 and f_3 of Eq. (60) are

$$f_1 = \frac{\alpha}{3} + \frac{4\eta^2}{3}(\zeta^2 - 2) , \tag{67}$$



(a) (b) (c)

Fig. 1 Evolution of **a** one-kink soliton Eq. (46), **b** two-kink soliton Eq. (49) and **c** three-kink soliton Eq. (52)

$$f_2 = \frac{\alpha}{3} + \frac{4\eta^2}{3}(1 - 2\zeta^2)\,, \tag{68}$$

$$f_3 = \frac{\alpha}{3} + \frac{4\eta^2}{3}(1 + \zeta^2)\,. \tag{69}$$

The parameters η and ζ in this solution are given by

$$\eta = \frac{1}{2}\sqrt{f_3 - f_1},\ \zeta = \sqrt{\frac{f_3 - f_2}{f_3 - f_1}}\,. \tag{70}$$

Using Eqs. (67), (69) and (70), we obtain the parameter η as

$$\eta = \frac{1}{2}\left(\frac{\alpha^2 + 3\beta}{\zeta^4 - \zeta^2 + 1}\right)^{\frac{1}{4}}\,. \tag{71}$$

Hence, the exact solution of Eq. (58) can be obtained by using Eq. (66) with $(0 < \zeta < 1)$ as

$$\Im(\chi) = \frac{f_2}{2} - \frac{(f_2 - f_3)}{2}\text{cn}^2\{\eta(\chi - \chi_0), \zeta\}\,. \tag{72}$$

Case 1: Periodic wave solution via Jacobi elliptic function $\text{cn}(\chi, \zeta)$ for the condition $\gamma \neq 0$

The transformation $\Re'(\chi) = \Im(\chi)$ leads to a family of bounded periodic wave solutions of Eq. (11) as

$$u(x, y, z, t) = \Phi\chi + \Psi(\zeta^2 - 1)\chi + E[\text{sn}\{\eta(\chi - \chi_0), \zeta\}, \zeta] \tag{73}$$

where $E(\chi, \zeta)$ is the incomplete elliptic integral of second kind with $0 < \zeta < 1$ as f_1,f_2 and f_3 are in order $f_1 < f_2 < f_3$. The parameters $\Phi = \frac{f_2}{2}$ and $\Psi = -\frac{1}{2}\frac{(f_2 - f_3)}{\eta\zeta^2}$ are given by

$$\Phi = \frac{1}{6}\{\alpha + 4\eta^2(1 - 2\zeta^2)\}\ \text{ and}$$

$$\Psi = \left(\frac{\alpha^2 + 3\beta}{\zeta^4 - \zeta^2 + 1}\right)^{\frac{1}{4}}\,. \tag{74}$$

Case 2: The periodic solution via Jacobi elliptic function $\text{dn}(\chi, \zeta)$ for the condition $\gamma \neq 0$

In this case, the bounded periodic wave solution of Eq. (11) is given by

$$u(x, y, z, t) = \Theta\chi + \Omega E[\text{sn}\{\eta(\chi - \chi_1), \zeta\}, \zeta]\,, \tag{75}$$

where the parameters $\Theta = \frac{f_1}{2}$ and $\Omega = \frac{f_3 - f_1}{2\eta}$ are as follows

$$\Theta = \frac{1}{6}\{\alpha + 4\eta^2(\zeta^2 - 2)\},\quad \Omega = \left(\frac{\alpha^2 + 3\beta}{\zeta^4 - \zeta^2 + 1}\right)^{\frac{1}{4}} \tag{76}$$

Case 3: Exact traveling wave solution for the condition $f_1 = 0$

When $f_1 = 0\,(\gamma = 0)$, the exact solution Eq. (75) reduces to

$$u(x, y, z, t) = \Pi\ E[\text{sn}\{\eta(\chi - \chi_1), \zeta\}, \zeta]\,, \tag{77}$$

where the parameters $\Pi = \frac{f_3}{2\eta}$ and η are as below

$$\Pi = \frac{1}{6\eta}(\alpha + 8\eta^2),\quad \eta = \sqrt{\frac{\alpha}{4(2 - \zeta^2)}}\,. \tag{78}$$

It is to be noted that the necessary condition for the existence of the above solutions are $\alpha > 0$ and $\alpha^2 + 3\beta > 0$. Particularly, when $f_1 = 0$, the solution Eq. (73) does not change, but due to the changes in the values of Θ and Ω in Eq. (76), the solution Eq. (75) changes. For different parametric values, the exact solutions are depicted in Figs. 2 and 3.

Case 4: Exact traveling wave solution for the condition $f_1 = 0,\ f_3 = 0$

In case when $f_1 = 0$ and $f_3 = 0$, from Eqs. (61), (62) and (63) we retrieve that $\beta = \gamma = 0$ and $\alpha = f_2$. A direct computation gives

$$u(x, y, z, t) = -\frac{2\kappa}{1 + e^{\kappa(\chi + C_3)}}\,, \tag{79}$$

where $\kappa = \sqrt{\alpha}$ and C_3 is the constant of integration. The condition $\alpha > 0$ is necessary for the existence

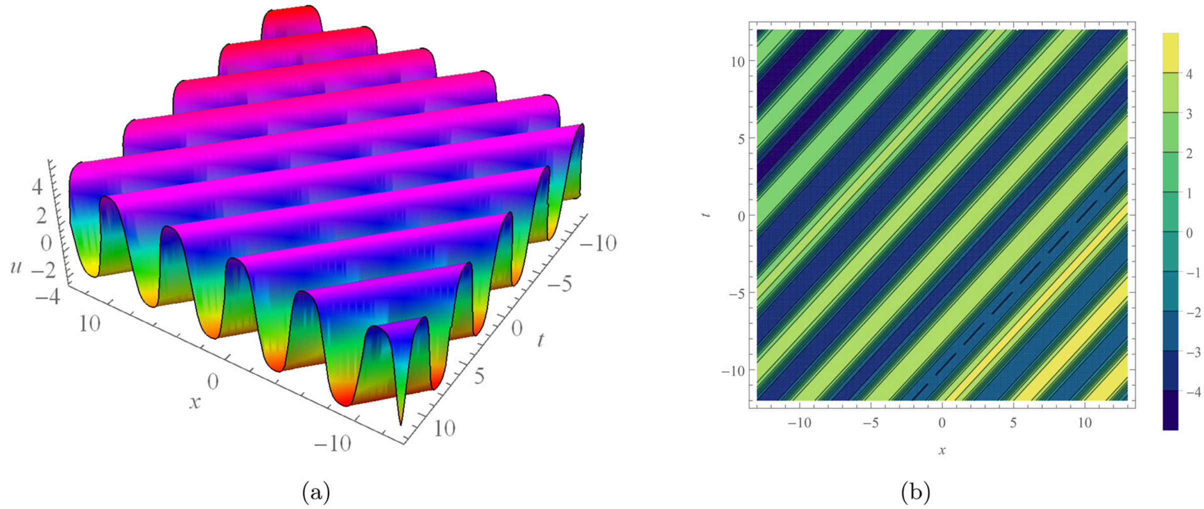


Fig. 2 Homoclinic breather solution of Eq. (11) given by Eq. (73), when $\mu = 3.2, c = 0.3, v = 10, a = -0.5, \lambda = 1, b = -0.1, \zeta = 0.9, C_2 = 0$ **a** 3-dimensional plot **b** Contour plot

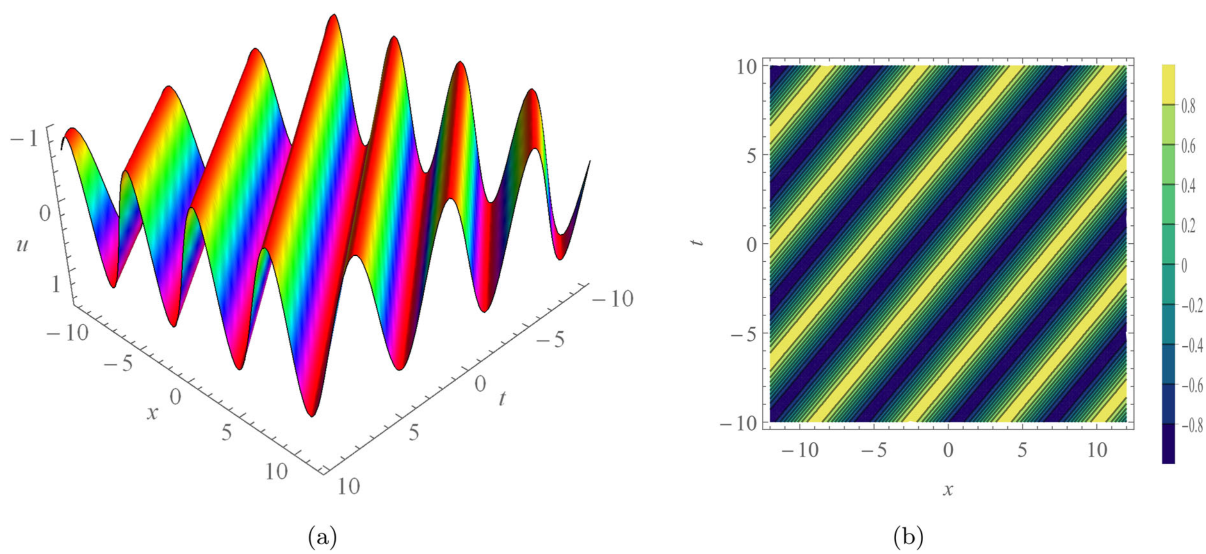


Fig. 3 Homoclinic breather solution of Eq. (11) given by Eq. (77), when $\mu = 1, c = 0.92, v = 0.89, a = 0.51, \lambda = 1, b = 2, \zeta = 0.65, C_2 = 1$ **a** 3-dimensional plot **b** Contour plot

of the solution. We choose the parametric values as $\mu = -1.9, c = -1, v = 0.89, a = 1, b = 2, \zeta = 0.4, C_3 = 0.7$ and $\lambda = 1$ and obtained one-kink soliton solution which is depicted in Fig. 4.

In this section, we consider the long wave limit $\zeta \to 1$ corresponding to the solitary wave solutions Eq. (73) and Eq. (75) in terms of Jacobi elliptic functions and obtain the trigonometric function solutions of Eq. (11).

In the limiting case $\zeta \to 1$, the solution in Eq. (73) with $\gamma \neq 0$ reduces to

$$u(x, y, z, t) = \Delta\chi + 2\eta_0[\tanh(\eta_0(\chi - \chi_0)) + \tanh(\eta_0\chi_0)], \tag{80}$$

where the parameters η_0 and Δ are

$$\eta_0 = \frac{1}{2}(\alpha^2 + 3\beta)^{\frac{1}{4}}, \ \Delta = \frac{1}{6}(\alpha - 4\eta_0^2).$$

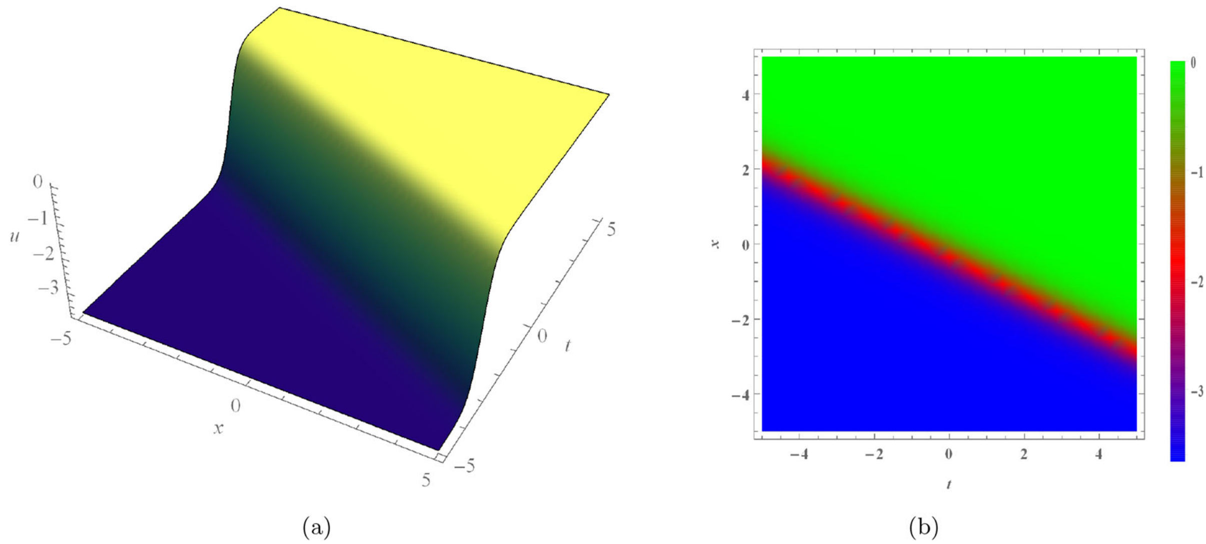


Fig. 4 Kink-type solution Eq. (79) of Eq. (11) when $\mu = -1.9, c = -1, v = 0.89, a = 1, b = 2, \zeta = 0.4, C_3 = 0.7, \lambda = 1$ **a** 3-dimensional plot **b** Density plot

In case when $\beta > 0$, we have $\alpha^2 + 3\beta > 0$ and $\Delta < 0$. The parametric condition $\Delta < 0$ in Eq. (80) leads to a kink wave solution.

In this particular case, the roots of Eq. (60) are

$$f_1 = \frac{\alpha}{3} - \frac{4}{3}\eta_0^2, \quad f_2 = \frac{\alpha}{3} - \frac{4}{3}\eta_0^2,$$
$$f_3 = \frac{1}{3}(\alpha + 8\eta_0^2)\,, \tag{81}$$

In the limiting case $\zeta \to 1$, the solution in Eq. (75) becomes

$$u(x, y, z, t) = \Gamma\chi - 2\eta_0[\tanh\{\eta_0(\chi - \chi_1)\} + \tanh(\eta_0\chi_1)]\,, \tag{82}$$

where

$$\Gamma = \frac{1}{6}(\alpha + 8\eta_0^2)\,. \tag{83}$$

It is to be mentioned here that in limiting wave case $\zeta \to 1$ and when the integration constant $\gamma \neq 0$, Eq. (81) reveals that $f_1 = f_2$ and f_3 become arbitrary. Moreover, $\gamma \neq 0$ yields $f_i \neq 0$ $(i = 1, 2, 3)$ which implies $\alpha + 8\eta_0^2 \neq 0$. Additionally, the parameter Γ in the solution Eq. (82) is negative only when $\alpha < 0$ otherwise $\Gamma \geq 0$. We also observe that the condition $\Gamma = 0$ refers to $\alpha < 0$ which holds for the solution in Eq. (82) leading to exact solution of Eq. (11) as kink-type solution as shown in Fig. 5.

7 Lump solutions and lump-multi-kink solutions

The Hirota bilinear operators D [8] can be defined as

$$(D_x^n D_y^m D_z^p D_t^r) f.f = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^m \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial z'}\right)^p \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^r f(x, y, z, t)\, f(x', y', z', t')|_{x=x', y=y', z=z', t=t'}\,, \tag{84}$$

where m, n, p and r are nonnegative integers.

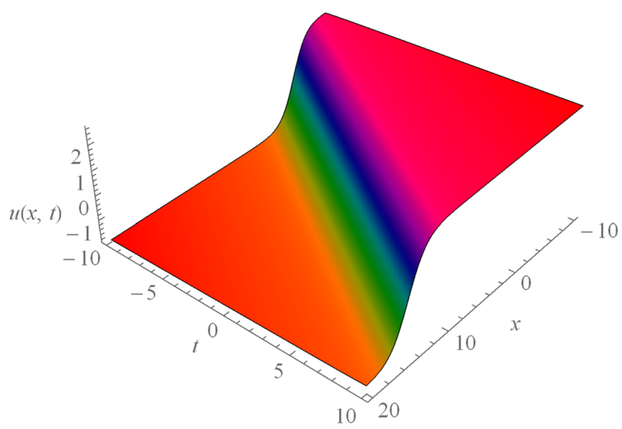
The Hirota bilinear form of a general (3 + 1)-dimensional nonlinear evolution equation can be considered as

$$G(D_x, D_y, D_z, D_t) f.f = 0. \tag{85}$$

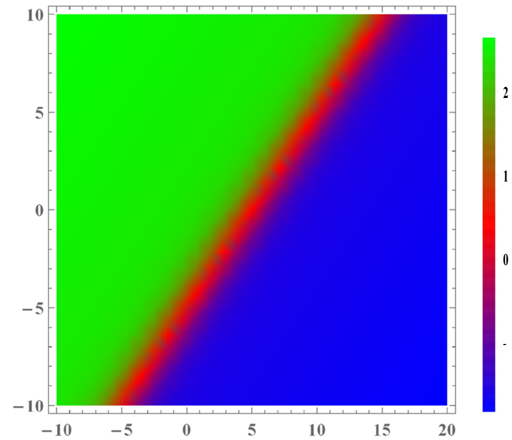
We choose a dependent variable transformation $u = \lambda(\log x)_x$ or $u = \lambda(\log x)_{2x}$, where λ is a constant and reexpresses Eq. (85)as a $(3+1)$-dimensional nonlinear evolution equation as

$$H(u, u_x, u_y, u_z, u_t, u_{xx}, u_{xy}, \cdots) = 0. \tag{86}$$

Here $u = u(x, y, z, t)$ and H is a polynomial of the dependent variable u and its different partial derivatives.



(a)



(b)

Fig. 5 Kink-type solution Eq. (82) of Eq. (11) when $\mu = 2, c = -1, v = 0, a = -1, b = -4, \zeta = 0.99, C_2 = 1, \lambda = 1$ **a** 3-dimensional plot **b** Density plot

7.1 Lump solution

To generate the lump solution of Eq. (11), we have considered the test function as

$$f = f_1^2 + f_2^2 + c_1\,, \tag{87}$$

where

$$f_1 = b_1 x + b_2 y + b_3 z + b_4 t + b_5\,,$$
$$f_2 = b_6 x + b_7 y + b_8 z + b_9 t + b_{10}\,.$$

The values of $b_m (m = 1, 2, ..., 10)$ and $c_1 > 0$ are constant and are to be calculated. The constants $b_m (m = 1, 2, 3, 6, 7, 8)$ satisfy the condition $(b_1, b_2, b_3) \nparallel (b_6, b_7, b_8)$.

Theorem 1 *The test function in Eq. (87) originates lump solutions of a* $(3+1)$*-dimensional nonlinear evolution equations having the particular type of Hirota bilinear form*

$$D_x F(D) f.f = 0, \tag{88}$$

where $F(D) = \lambda_1 D_x + \lambda_2 D_y + \lambda_3 D_z + \lambda_4 D_t + \lambda_5 D_x^3 + \lambda_6 D_y^3 + \lambda_7 D_z^3 + \lambda_8 D_t^3 + \lambda_9 D_x^2 D_y + \lambda_{10} D_x^2 D_z + \lambda_{11} D_x^2 D_t$ *and* $\lambda_i (i = 1,\ 2,\ \cdots 11)$ *are constants, under the transformation* $u = \lambda(logx)_x$ *or* $u = \lambda(logx)_{2x}$*, if and only if*

$$\lambda_1 f_x + \lambda_2 f_y + \lambda_3 f_z + \lambda_4 f_t = 0\,, \tag{89a}$$

$$\lambda_5 f_{2x}^2 + \lambda_6 f_{2y} f_{xy} + \lambda_7 f_{2z} f_{xz} + \lambda_8 f_{2t} f_{xt} + \lambda_9 f_{2x} f_{xy} + \lambda_{10} f_{2x} f_{xz} + \lambda_{11} f_{2x} f_{xt} = 0. \tag{89b}$$

Proof Substituting the test function Eq. (87) into Eq. (88), we have the following

$$f(\lambda_1 f_{2x} + \lambda_2 f_{xy} + \lambda_3 f_{xz} + \lambda_4 f_{xt}) - f_x(\lambda_1 f_x + \lambda_2 f_y + \lambda_3 f_z + \lambda_4 f_t) + 3(\lambda_5 f_{2x}^2 + \lambda_6 f_{2y} f_{xy} + \lambda_7 f_{2z} f_{xz} + \lambda_8 f_{2t} f_{xt} + \lambda_9 f_{2x} f_{xy} + \lambda_{10} f_{2x} f_{xz} + \lambda_{11} f_{2x} f_{xt}) = 0 \tag{90}$$

Comparing the coefficient of x^2 from both sides of Eq. (90), we obtain

$$\lambda_1 f_{2x} + \lambda_2 f_{xy} + \lambda_3 f_{xz} + \lambda_4 f_{xt} = 0\,. \tag{91}$$

Integrating Eq. (91) with respect to x, we have

$$\lambda_1 f_x + \lambda_2 f_y + \lambda_3 f_z + \lambda_4 f_t = \psi(y, z, t)\,, \tag{92}$$

where $\psi(y, z, t)$ is a function of y, z and t. Then Eq. (90) takes the form

$$3(\lambda_5 f_{2x}^2 + \lambda_6 f_{2y} f_{xy} + \lambda_7 f_{2z} f_{xz} + \lambda_8 f_{2t} f_{xt} + \lambda_9 f_{2x} f_{xy} + \lambda_{10} f_{2x} f_{xz} + \lambda_{11} f_{2x} f_{xt}) - f_x \psi(y, z, t) = 0. \tag{93}$$

Comparing the coefficient of x from both sides of Eq. (93), we derive $\psi(y, z, t) = 0$, that is,

$$\lambda_1 f_x + \lambda_2 f_y + \lambda_3 f_z + \lambda_4 f_t = 0\,, \tag{94}$$

Then Eq. (93) becomes

$$3(\lambda_5 \quad f_{2x}^2 + \lambda_6 f_{2y} f_{xy} + \lambda_7 f_{2z} f_{xz} + \lambda_8 f_{2t} f_{xt} + \lambda_9 f_{2x} f_{xy} + \lambda_{10} f_{2x} f_{xz} + \lambda_{11} f_{2x} f_{xt}) = 0. \tag{95}$$

The test function Eq. (87) generates lump solutions of (3 + 1)-dimensional nonlinear evolution equations, having the Hirota bilinear form Eq. (88) under the transformation $u = \lambda(\log x)_x$ or $u = \lambda(\log x)_{2x}$, if and only if Eqs. (94) and (95) are satisfied. □

Theorem 2 *The test function Eq. (87) originates lump solutions of* (3 + 1)*-dimensional nonlinear evolution equations having the particular type of Hirota bilinear form*

$$D_y F(D) f.f = 0, \tag{96}$$

under the transformation $u = \lambda(logx)_x$ *or* $u = \lambda(logx)_{2x}$ *if and only if*

$$\lambda_1 f_x + \lambda_2 f_y + \lambda_3 f_z + \lambda_4 f_t = 0\,, \tag{97a}$$

$$\begin{aligned}
&3(\lambda_5 f_{2x} f_{xy} + \lambda_6 f_{2y}^2 \\
&\quad + \lambda_7 f_{2z} f_{yz} + \lambda_8 f_{2t} + f_{yt}) \\
&\quad + \lambda_9 (f_{2x} f_{2y} + 2 f_{xy}^2) \\
&\quad + \lambda_{10} (f_{2x} f_{yz} + 2 f_{xy} f_{xz}) \\
&\quad + \lambda_{11} (f_{2x} f_{yt} + 2 f_{xy} f_{yt}) = 0.
\end{aligned} \tag{97b}$$

Theorem 3 *The test function Eq. (87) originates lump solutions of* (3 + 1)*-dimensional nonlinear evolution equations having the particular type of Hirota bilinear form*

$$D_z F(D) f.f = 0, \tag{98}$$

under the transformation $u = \lambda(logx)_x$ *or* $u = \lambda(logx)_{2x}$ *if and only if*

$$\lambda_1 f_x + \lambda_2 f_y + \lambda_3 f_z + \lambda_4 f_t = 0\,, \tag{99a}$$

$$\begin{aligned}
&3(\lambda_5 f_{2x} f_{xz} + \lambda_6 f_{2y} f_{yz} + \lambda_7 f_{2z}^2 \\
&\quad + \lambda_8 f_{2t} + f_{zt}) + \lambda_9 (f_{2x} f_{yz} \\
&\quad + 2 f_{xy} f_{xz}) + \lambda_{10} (f_{2x} f_{2z} + 2 f_{xz}^2) \\
&\quad + \lambda_{11} (f_{2x} f_{zt} + 2 f_{xz} f_{xt}) = 0.
\end{aligned} \tag{99b}$$

Theorem 4 *The test function Eq. (87) originates lump solutions of* (3 + 1)*-dimensional nonlinear evolution equations having the particular type of Hirota bilinear form*

$$D_t F(D) f.f = 0, \tag{100}$$

under the transformation $u = \lambda(logx)_x$ *or* $u = \lambda(logx)_{2x}$ *if and only if*

$$\lambda_1 f_x + \lambda_2 f_y + \lambda_3 f_z + \lambda_4 f_t = 0\,, \tag{101a}$$

$$\begin{aligned}
&3(\lambda_5 f_{2x} f_{xt} + \lambda_6 f_{2y} f_{yt} + \lambda_7 f_{2z} f_{zt} + \lambda_8 f_{tt}^2) \\
&\quad + \lambda_9 (f_{2x} f_{yt} + 2 f_{xy} f_{xt}) + \lambda_{10} (f_{2x} f_{tz} + 2 f_{xt} f_{xz}) \\
&\quad + \lambda_{11} (f_{2x} f_{2t} + 2 f_{xt}^2) = 0.
\end{aligned} \tag{101b}$$

Similar approach can be used to prove Theorems 2-4.

7.2 Application to the (3 + 1)-dimensional negative-order KDV–CBS equation

We rewrite the previously obtained Hirota bilinear form Eq. (28) of the (3 + 1)-dimensional negative-order KdV–Calogero–Bogoyavlenskii–Schiff Eq. (11) as below

$$D_x(D_t + D_x^2 D_y + \lambda D_x + \mu D_y + \nu D_z) f.f = 0 \tag{102}$$

Taking $\lambda = \mu = \nu = 1$ and applying **Theorem 1**, we obtain the lump solution of Eq. (11) as

$$u = -\frac{2}{f}\left\{-\frac{b_6 b_7}{b_2} f_1 + 2 b_6 f_2\right\}, \tag{103}$$

where

$$\begin{aligned}
f_1 &= -\frac{b_6 b_7}{b_2} x + b_2 y + \left(\frac{b_6 b_7}{b_2} - b_2 - b_4\right) z + b_4 t + b_5\,, \\
f_2 &= b_6 x + b_7 y + (-b_6 - b_7 - b_9) z + b_9 t + b_{10}\,, \\
f &= \left\{-\frac{b_6 b_7}{b_2} x + b_2 y + \left(\frac{b_6 b_7}{b_2} - b_2 - b_4\right) z + b_4 t + b_5\right\}^2 \\
&\quad + \{_6 x + b_7 y + (-b_6 - b_7 - b_9) z + b_9 t + b_{10}\}^2\,.
\end{aligned}$$

7.3 Lump-multi-kink solutions

For the lump-multi-kink solution, we have considered the two types of test functions as

$$(i)\ f = f_1^2 + f_2^2 + c_1 + \sum_{i=1}^{n} e^{\varsigma_i}\,, \tag{105}$$

$$(ii)\ f = f_1^2 + f_2^2 + c_1 + \sum_{i=1}^{n} \cosh \varsigma_i\,. \tag{106}$$

We substitute Eq. (105) into Eq. (28) and find the relation between the parameters as

$$\begin{aligned}
b_1 &= -\frac{b_6 b_7}{b_2},\ b_3 = \frac{b_6 b_7}{b_2} - b_2 - b_4, \\
b_8 &= -b_6 - b_7 - b_9,
\end{aligned}$$

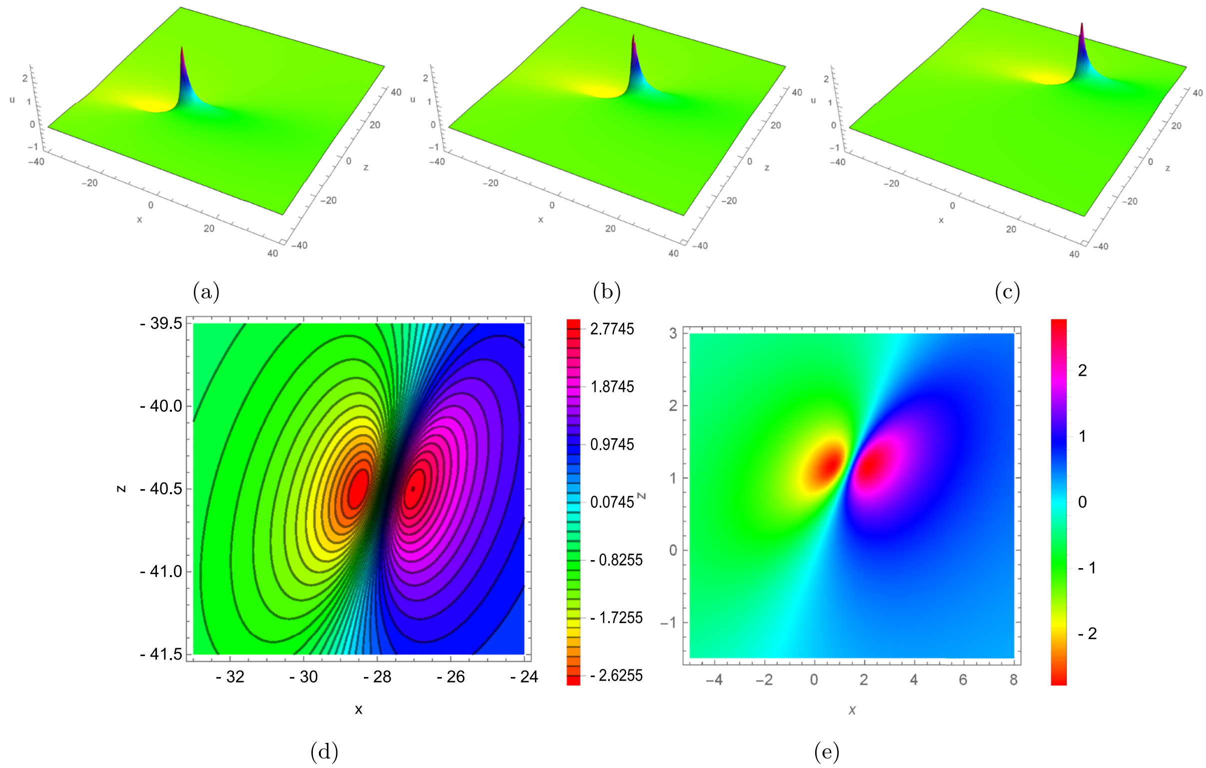


Fig. 6 The propagation of lump solution of Eq. (11) given by Eq. (103) at **a** $t=-150$, **b** $t=0$, **c** $t=150$, **d** contour plot, **e** density plot, when $b_2=3, b_4=1, b_5=2, b_6=2, b_7=2, b_9=1, b_{10}=1, c_1=3$.

$$w_i=-(1+p_i+q_i+k_i^2 p_i),\ \ (i=1,2,3)\,. \tag{107}$$

The lump-multi-kink solution of Eq. (11) is found as

$$u=-\frac{2}{f}\left\{-\frac{2b_6b_7}{b_2}f_1+2b_6f_2+k_i\sum_{i=1}^{n}e^{\varsigma_i}\right\}, \tag{108}$$

with

$$f_1=-\frac{b_6b_7}{b_2}x+b_2y+\left(\frac{b_6b_7}{b_2}-b_2-b_4\right)z+b_4t+b_5\,, \tag{109a}$$

$$f_2=b_6x+b_7y+(-b_6-b_7-b_9)z+b_9t+b_{10}\,, \tag{109b}$$

$$\begin{aligned}f=&\left\{-\frac{b_6b_7}{b_2}x+b_2y+\left(\frac{b_6b_7}{b_2}-b_2-b_4\right)z+b_4t+b_5\right\}^2\\&+\{b_6x+b_7y+(-b_6-b_7-b_9)z+b_9t+b_{10}\}^2\\&+c_1+\sum_{i=1}^{n}e^{\varsigma_i}\,,\end{aligned} \tag{109c}$$

where $\varsigma_i=k_i(x+p_iy+q_iz+w_it)+\varsigma_i^0,\ \ (i=1,2,3)$ and $c_1>0$ is an arbitrary constant.

Similarly, substituting Eq. (106) into Eq. (28), we obtain the same relation between the parameters as Eq. (107) and the lump-multi-kink solution of Eq. (11) is obtained as

$$u=-\frac{2}{f}\left\{-\frac{2b_6b_7}{b_2}f_1+2b_6f_2+k_i\sum_{i=1}^{n}\sinh\varsigma_i\right\}, \tag{110}$$

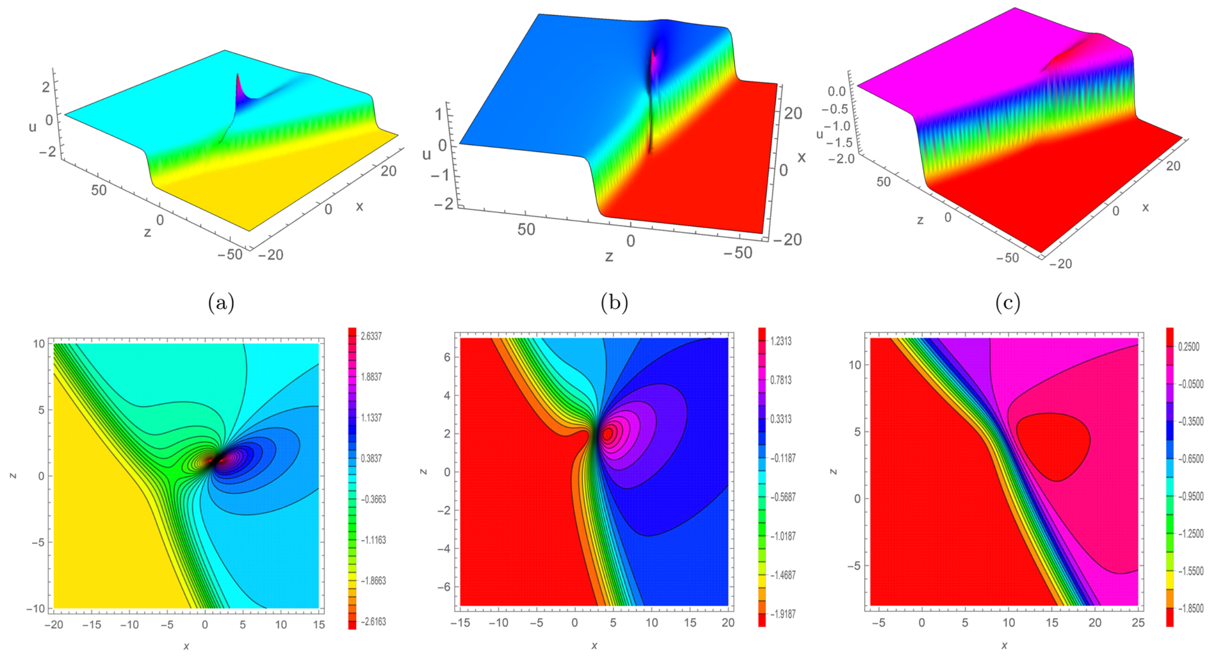
with

$$f_1=-\frac{b_6b_7}{b_2}x+b_2y+\left(\frac{b_6b_7}{b_2}-b_2-b_4\right)z+b_4t+b_5\,, \tag{111a}$$

$$f_2=b_6x+b_7y+(-b_6-b_7-b_9)z+b_9t+b_{10}\,, \tag{111b}$$

$$\begin{aligned}f=&\left\{-\frac{b_6b_7}{b_2}x+b_2y+\left(\frac{b_6b_7}{b_2}-b_2-b_4\right)z+b_4t+b_5\right\}^2\\&+\{b_6x+b_7y+(-b_6-b_7-b_9)z+b_9t+b_{10}\}^2\\&+c_1+\sum_{i=1}^{n}\cosh\varsigma_i\,,\end{aligned} \tag{111c}$$

where $\varsigma_i=k_i(x+p_iy+q_iz+w_it)+\varsigma_i^0,\ \ (i=1,2,3)$ and $c_1>0$ is an arbitrary constant.



(a) (b) (c)

(d) (e) (f)

Fig. 7 The propagation of lump-one-kink solution of Eq. (11) given by Eq. (112) at **a** $t = 0$, **b** $t = 2$, **c** $t = 5$, **d** contour plot at $t = 0$, **e** contour plot at $t = 2$, and **f** contour plot at $t = 5$, when $b_2 = 3, b_4 = 1, b_5 = 2, b_6 = 2, b_7 = 2, b_9 = 1, b_{10} = 1, c_1 = 3, k_1 = -1, p_1 = 1, and q_1 = 1$

7.4 Lump-multi-kink solutions using test function I

7.4.1 Lump-one-kink solution

We choose $n = 1$ in Eq. (108) and obtain lump-one-kink solution as

$$u = -\frac{2}{f}\left\{-\frac{2b_6b_7}{b_2}f_1 + 2b_6 f_2 + k_1 e^{\varsigma_1}\right\}, \tag{112}$$

where

$$f_1 = -\frac{b_6b_7}{b_2}x + b_2 y + \left(\frac{b_6b_7}{b_2} - b_2 - b_4\right)z + b_4 t + b_5, \tag{113a}$$

$$f_2 = b_6 x + b_7 y + (-b_6 - b_7 - b_9)z + b_9 t + b_{10}, \tag{113b}$$

$$\begin{aligned} f = &\left\{-\frac{b_6b_7}{b_2}x + b_2 y + \left(\frac{b_6b_7}{b_2} - b_2 - b_4\right)z + b_4 t + b_5\right\}^2 \\ &+ \{b_6 x + b_7 y + (-b_6 - b_7 - b_9)z \\ &+ b_9 t + b_{10}\}^2 + c_1 + e^{\varsigma_1}. \end{aligned} \tag{113c}$$

7.4.2 Lump-two-kink solution

We choose $n = 2$ in Eq. (108) and obtain lump-two-kink solution as

$$u = -\frac{2}{f}\left\{-\frac{2b_6b_7}{b_2}f_1 + 2b_6 f_2 + k_1 e^{\varsigma_1} + k_2 e^{\varsigma_2}\right\}, \tag{114}$$

where

$$f_1 = -\frac{b_6b_7}{b_2}x + b_2 y + \left(\frac{b_6b_7}{b_2} - b_2 - b_4\right)z + b_4 t + b_5, \tag{115a}$$

$$f_2 = b_6 x + b_7 y + (-b_6 - b_7 - b_9)z + b_9 t + b_{10}, \tag{115b}$$

$$\begin{aligned} f = &\left\{-\frac{b_6b_7}{b_2}x + b_2 y + \left(\frac{b_6b_7}{b_2} - b_2 - b_4\right)z + b_4 t + b_5\right\}^2 \\ &+ \{b_6 x + b_7 y + (-b_6 - b_7 - b_9)z + b_9 t + b_{10}\}^2 \\ &+ c_1 + e^{\varsigma_1} + e^{\varsigma_2}. \end{aligned} \tag{115c}$$

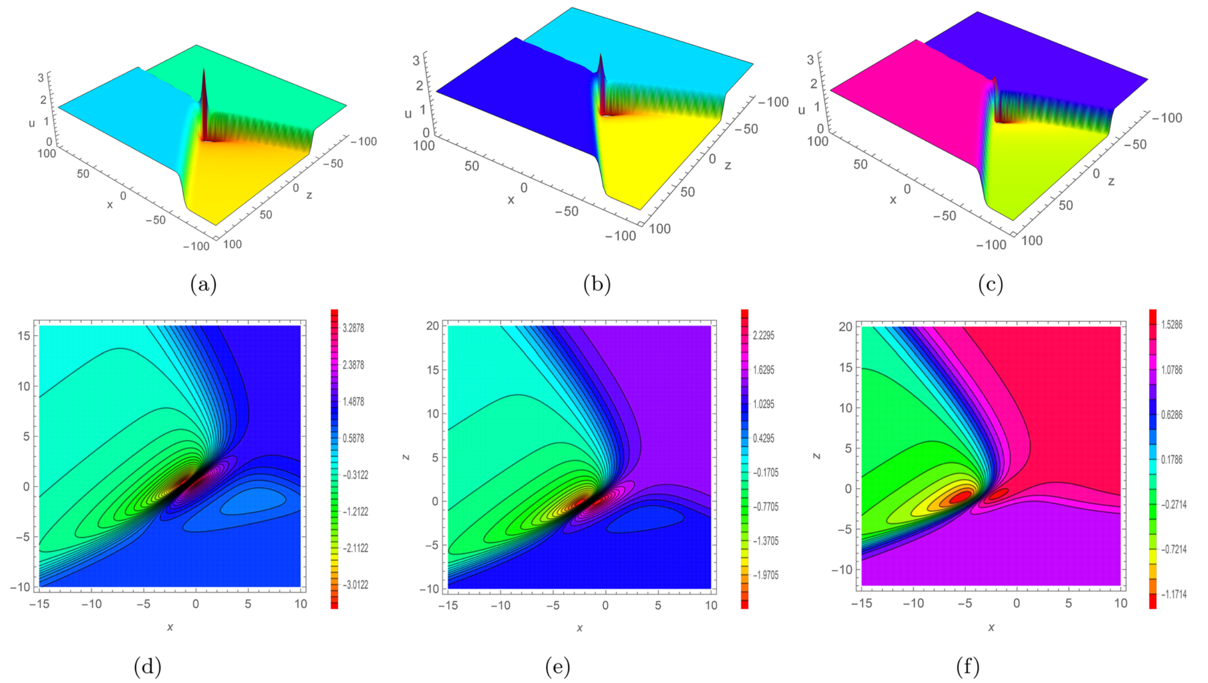


Fig. 8 The propagation of lump-two-kink solution of Eq. (11) given by Eq. (114) at **a** $t = 5$, **b** $t = 7$, **c** $t = 11$, **d** contour plot at $t = 5$, and **e** contour plot at $t = 7$, **f** contour plot at $t = 11$, when $b_2 = 1.5, b_4 = 1, b_5 = 3, b_6 = -3, b_7 = 4, b_9 = -1, b_{10} = 1, c_1 = 3, k_1 = 0.5, p_1 = 1, q_1 = -3, k_2 = 0.7, p_2 = -2, and q_2 = 0.8$

7.4.3 Lump-three-kink solution

We choose $n = 3$ in Eq. (108) and obtain lump-three-kink solution as

$$u = -\frac{2}{f}\left\{-\frac{2b_6b_7}{b_2}f_1 + 2b_6 f_2 + k_1 e^{\varsigma_1} + k_2 e^{\varsigma_2} + k_3 e^{\varsigma_3}\right\}, \tag{116}$$

where

$$f_1 = -\frac{b_6b_7}{b_2}x + b_2 y + \left(\frac{b_6b_7}{b_2} - b_2 - b_4\right) z + b_4 t + b_5, \tag{117a}$$

$$f_2 = b_6 x + b_7 y + (-b_6 - b_7 - b_9)z + b_9 t + b_{10}, \tag{117b}$$

$$f = \left\{-\frac{b_6b_7}{b_2}x + b_2 y + \left(\frac{b_6b_7}{b_2} - b_2 - b_4\right) z + b_4 t + b_5\right\}^2 + \{b_6 x + b_7 y + (-b_6 - b_7 - b_9)z + b_9 t + b_{10}\}^2 + c_1 + e^{\varsigma_1} + e^{\varsigma_2} + e^{\varsigma_3}. \tag{117c}$$

7.5 Lump-multi-kink solutions using test function II

7.5.1 Lump-one-kink solution

We choose $n = 1$ in Eq. (110) and obtain lump-one-kink solution as

$$u = -\frac{2}{f}\left\{-\frac{2b_6b_7}{b_2}f_1 + 2b_6 f_2 + k_1 \sinh \varsigma_1\right\}, \tag{118}$$

where

$$f_1 = -\frac{b_6b_7}{b_2}x + b_2 y + \left(\frac{b_6b_7}{b_2} - b_2 - b_4\right) z + b_4 t + b_5, \tag{119a}$$

$$f_2 = b_6 x + b_7 y + (-b_6 - b_7 - b_9)z + b_9 t + b_{10}, \tag{119b}$$

$$f = \left\{-\frac{b_6b_7}{b_2}x + b_2 y + \left(\frac{b_6b_7}{b_2} - b_2 - b_4\right) z + b_4 t + b_5\right\}^2 + \{b_6 x + b_7 y + (-b_6 - b_7 - b_9)z + b_9 t + b_{10}\}^2 + c_1 + \cosh \varsigma_1. \tag{119c}$$

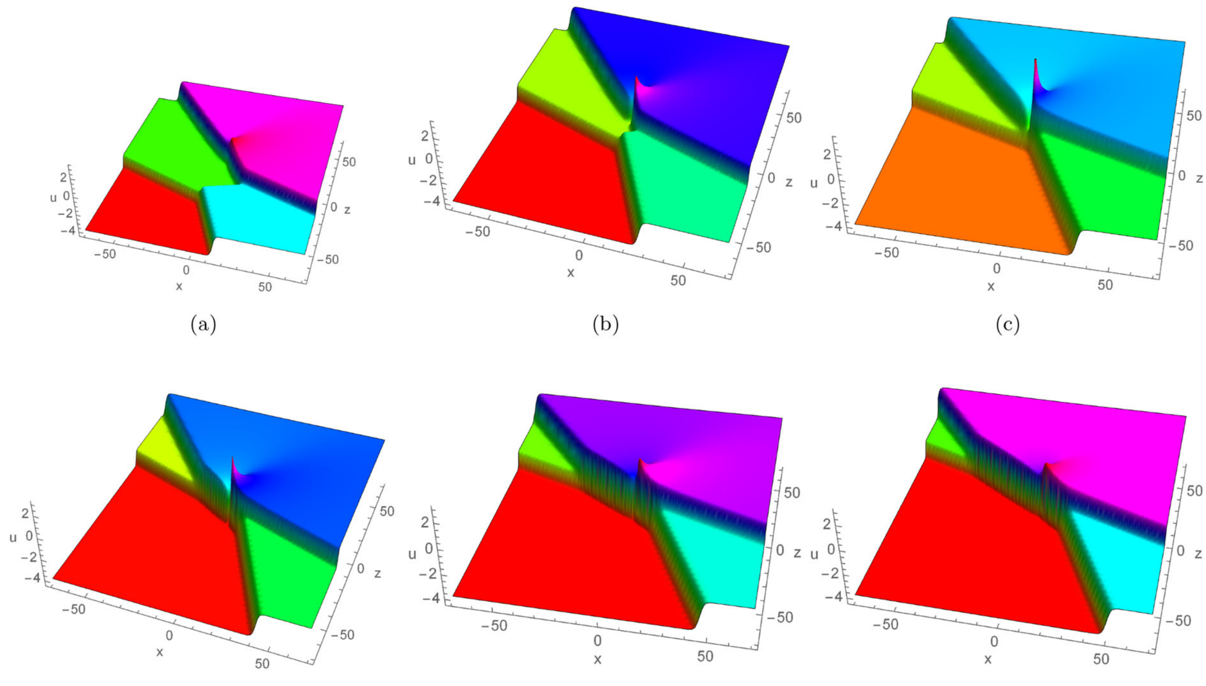


Fig. 9 The propagation of lump-three-kink solution of the Eq. (11) given by Eq. (116) at **a** $t=-2.7$, **b** $t=-1$, **c** $t=0$, **d** $t=1$, **e** $t=1.5$, **f** $t=2$, when $b_2=1.5, b_4=1, b_5=.5, b_6=2, b_7=2, b_9=-1, b_{10}=2, c_1=1, k_1=-2, p_1=1, q_1=1.5, k_2=1.3, p_2=-2, q_2=0.8, k_3=-0.8, p_3=2, q_3=3$

7.5.2 Lump-two-kink solution

We choose $n=2$ in Eq. (110) and obtain lump-two-kink solution as

$$u=-\frac{2}{f}\left\{-\frac{2b_6b_7}{b_2}f_1+2b_6f_2+k_1\sinh\varsigma_1+k_2\sinh\varsigma_2\right\}, \tag{120}$$

where

$$f_1=-\frac{b_6b_7}{b_2}x+b_2y+\left(\frac{b_6b_7}{b_2}-b_2-b_4\right)z+b_4t+b_5, \tag{121a}$$

$$f_2=b_6x+b_7y+(-b_6-b_7-b_9)z+b_9t+b_{10}, \tag{121b}$$

$$\begin{aligned}f=&\left\{-\frac{b_6b_7}{b_2}x+b_2y+\left(\frac{b_6b_7}{b_2}-b_2-b_4\right)z+b_4t+b_5\right\}^2\\&+\{b_6x+b_7y+(-b_6-b_7-b_9)z+b_9t+b_{10}\}^2\\&+c_1+\cosh\varsigma_1+\cosh\varsigma_2.\end{aligned} \tag{121c}$$

7.5.3 Lump-three-kink solution

We choose $n=3$ in Eq. (110) and we have obtain lump-three-kink solution as

$$\begin{aligned}u=&-\frac{2}{f}\left\{-\frac{2b_6b_7}{b_2}f_1+2b_6f_2+k_1\sinh\varsigma_1\right.\\&\left.+k_2\sinh\varsigma_2+k_3\sinh\varsigma_3\right\},\end{aligned} \tag{122}$$

where

$$f_1=-\frac{b_6b_7}{b_2}x+b_2y+\left(\frac{b_6b_7}{b_2}-b_2-b_4\right)z+b_4t+b_5, \tag{123a}$$

$$f_2=b_6x+b_7y+(-b_6-b_7-b_9)z+b_9t+b_{10}, \tag{123b}$$

$$\begin{aligned}f=&\left\{-\frac{b_6b_7}{b_2}x+b_2y+\left(\frac{b_6b_7}{b_2}-b_2-b_4\right)z+b_4t+b_5\right\}^2\\&+\{b_6x+b_7y+(-b_6-b_7-b_9)z+b_9t+b_{10}\}^2\\&+c_1+\cosh\varsigma_1\\&+\cosh\varsigma_2+\cosh\varsigma_3.\end{aligned} \tag{123c}$$

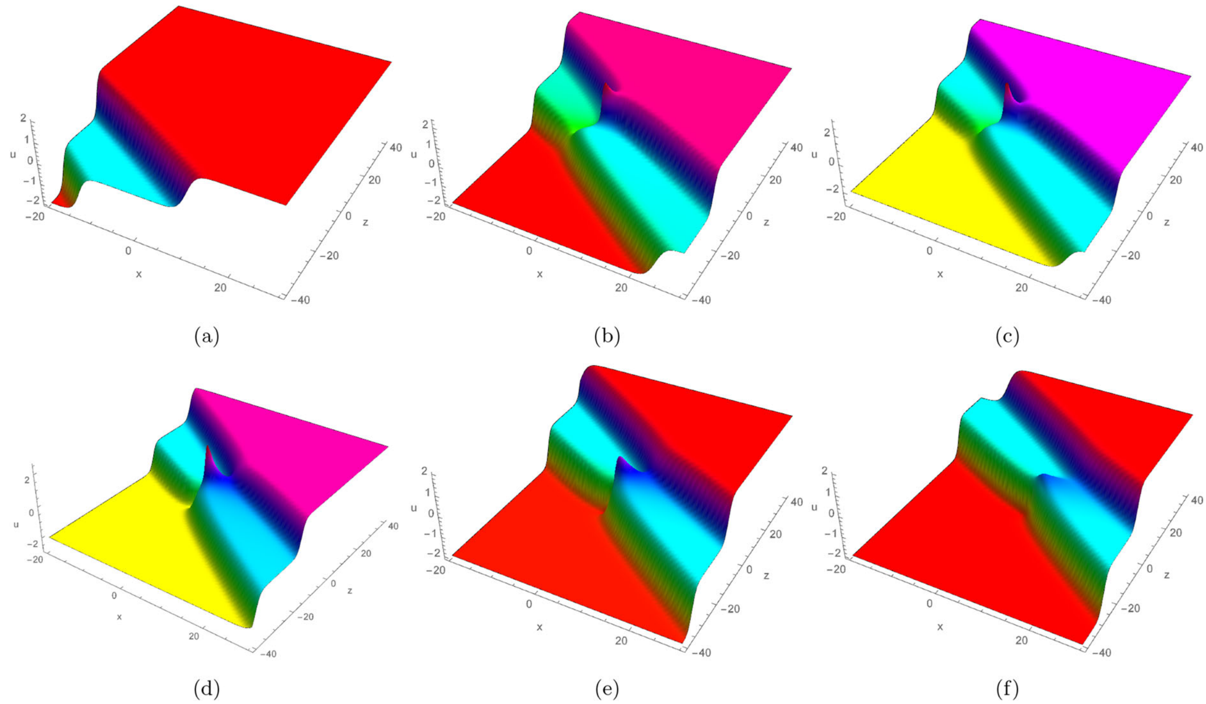


Fig. 10 The propagation of lump-one-kink solution of Eq. (11) given by Eq. (118) at **a** $t = -10$, **b** $t = -.5$, **c** $t = 0$, **d** $t = 1.3$, **e** $t = 2.2$, **f** $t = 4$, when $b_2 = 3, b_4 = 1, b_5 = 3, b_6 = -2, b_7 = 4, b_9 = 1, b_{10} = -1, c_1 = 2, k_1 = -1, p_1 = 1, q_1 = 1$

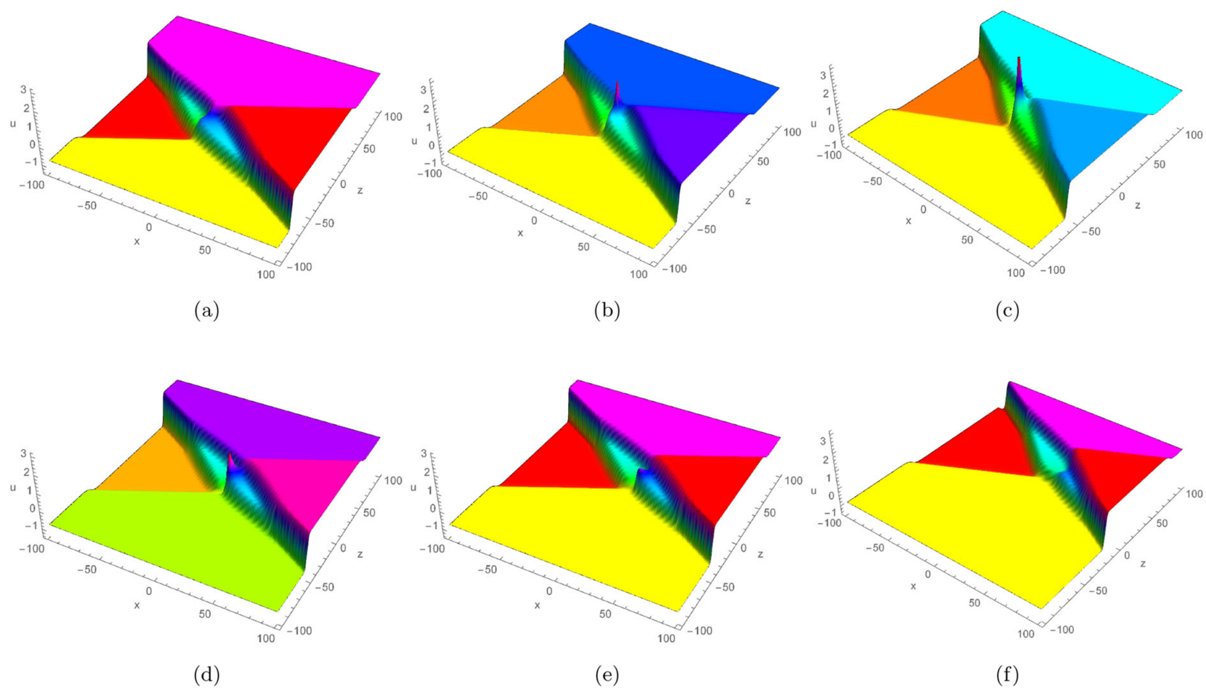


Fig. 11 The propagation of lump-two-kink solution of Eq. (11) given by Eq. (120) at **a** $t = -5$, **b** $t = -1$, **c** $t = 1.5$, **d** $t = 4$, **e** $t = 7$, and **f** $t = 15$, when $b_2 = 1, b_4 = 1, b_5 = 2, b_6 = 3, b_7 = 2, b_9 = 1, b_{10} = .1, c_1 = 4, k_1 = 0.6, p_1 = 1, q_1 = 3, k_2 = 0.4, p_2 = -3, and q_2 = 2.5$.

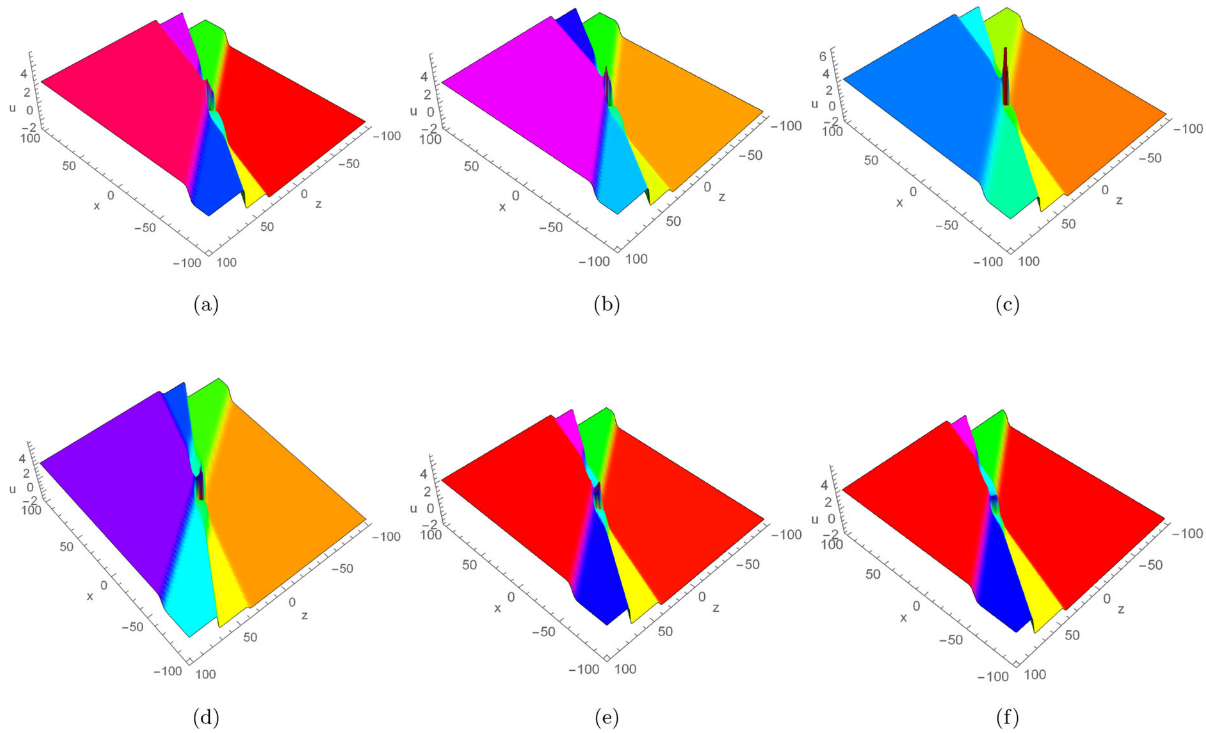


Fig. 12 The propagation of lump-three-kink solution of Eq. (11) given by Eq. (122) at **a** $t = -1$, **b** $t = -0.3$, **c** $t = 1$, **d** $t = 2$, **e** $t = 3$, and **f** $t = 4.5$, when $b_2 = -2.9$, $b_4 = 3$, $b_5 = -2$, $b_6 = -9$, $b_7 = 7$, $b_9 = -1$, $b_{10} = 2$, $c_1 = 2$, $k_1 = 1.2$, $p_1 = 1$, $q_1 = 1.5$, $k_2 = 0.8$, $p_2 = 2$, $q_2 = 0.8$, $k_3 = 0.4$, $p_3 = 2$, $and q_3 = 3$

8 Conclusion

In this current exposition, we have successfully delved into the integrability features of the (3+1)-dimensional negative-order KDV–CBS equation. Our investigation included a pivotal step, where we identified the Hirota bilinear form through the utilization of binary Bell polynomials. We also introduced some fundamental concepts from the theory of binary Bell polynomials, laying the foundation for deriving the Hirota bilinear form by employing the notion of P-polynomials. Furthermore, we achieved a significant breakthrough by decoupling two-field conditions to obtain a bilinear Bäcklund transformation. To illuminate our findings, we incorporated the Cole–Hopf transformation into the Bäcklund transformation process, thereby linearizing it and establishing the Lax pair for the system. This led us to rephrase the two-field conditions, from which we deduced a divergence-type equation and a Riccati-type equation using a novel potential function. Notably, when we expanded this potential function into a series, we unveiled an infinite sequence of conservation laws. The presence and verification of the Lax pair, Bäcklund transformation, Hirota bilinear form and the existence of conservation laws provide undeniable evidence of the complete integrability of the model under consideration. This significance is further underscored by our discovery of one-, two- and three-kink solutions, as visually represented in Fig. 1. The attainment of closed-form analytic solutions, in the form of traveling waves, assumes paramount importance as it offers valuable insights into the dynamic behavior of the model. In the subsequent sections, we delve into the existence of traveling wave solutions within the (3+1)-dimensional negative-order KDV–CBS model. We elucidate the conditions governing the existence of these solutions and present a range of solitary wave solutions, including homoclinic breather wave solutions (Figs. 2, 3), and kink wave solutions (Figs. 4, 5) expressed in terms of Jacobi elliptic functions and incomplete elliptic integrals of the second kind. These solutions are not only significant but also exhibit a rich and rigorous ability to describe various physical phenomena. Within the realm of integrable models, we encounter lump waves characterized by rational functions that are localized in all spatial directions. In exploring the long wave limits of soliton solutions, we unearth the potential for generating lump wave solutions. Our approach builds upon the work of Chen and Lü [52], who established a necessary and sufficient condition for the existence of lump solutions in selected NLLEs with specific Hirota bilinear forms. We expand upon this concept by introducing additional terms into the previously mentioned Hirota bilinear form, thus demonstrating that lump solutions can also manifest in these types of bilinear forms. Our examination includes the incorporation of a quadratic form of a test function into the Hirota bilinear form, leading to the discovery of lump wave solutions (Figure 6). In an exciting development, we introduce two novel forms of test functions, enabling us to discern lump-multi-kink solutions. By adeptly employing these test functions, we uncover and illustrate specific instances of lump-one-kink, lump-two-kink and lump-three-kink solutions (Figs. 7, 8, 9) for the considered model. This exploration extends to a second set of test functions, yielding further insights into lump-one-kink, lump-two-kink and lump-three-kink solutions (Figs. 10, 11, 12). Our future research endeavors will focus on investigating higher-order rogue waves, breather waves and hybrid solutions, exploring their interactions within the framework of the concerned theories. Additionally, we plan to apply the linear superposition principle to the Hirota bilinear form in the context of complex fields, with the aim of identifying complextions, resonant solitons and other intriguing phenomena. In summary, our comprehensive study sheds light on the intricate integrability aspects of the (3+1)-dimensional negative-order KDV–CBS equation, unveiling a wealth of novel solutions and paving the way for further exploration in the field of nonlinear wave dynamics.

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Data availability All data generated or analyzed during this study are included in this article.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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