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Integrability aspects, Wronskian solution, Grammian solution, lump and lump-multi-kink solutions of an extended (3 + 1)-dimensional Bogoyavlensky-Konopelchenko equation

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Abstract In this article, we examine an extended (3 + 1)-dimensional Bogoyavlensky-Konopelchenko equation, which models the interaction between a Riemann wave and a long wave in a fluid. This equation has significant applications in the study of shallow-water waves, ion-acoustic waves, and water propagation in liquids. We explore the integrability of this model through various approaches. First, we derive the Hirota bilinear form using the Bell polynomial theory. By decoupling the two-field condition, we calculate the bilinear Bäcklund transformation. Subsequently, through the Cole–Hopf transformation and the linearization of the Bäcklund transformation, we obtain the Lax pair. Additionally, we derive infinitely many conservation laws using Bell polynomial theory. We compute one-, two-, and three-soliton solutions directly from the Hirota bilinear form and present their 3-D plot, density plot and 2D plot graphically. We establish the Wronskian condition by employing the Plücker relation, ensuring that the N-soliton solutions of the equation can be represented as Wronskian determinants. Additionally, the use of a suitable transformation and the Wronskian determinant condition in our model establishes the widely known Wronskian solution to the (1 + 1)-dimensional KdV equation. We derive a rational Wronskian solution by selecting a specific coefficient matrix in the resulting Wronskian formulation. Furthermore, we calculate one-, two-, and three-soliton solutions in Wronskian form and visually depict their soliton dynamics using Mathematica with appropriately chosen parameters. Additionally, we present a Grammian determinant solution, utilizing the Jacobi relation. To obtain the lump solution, we employ a quadratic function as a test function within the Hirota bilinear form. Furthermore, we calculate two sets of lump-multi-kink solutions employing two distinct test functions. We provide a visual comparison of the evolutionary dynamics of the lump-multi-kink solutions corresponding to two distinct test functions.

1 Introduction

In recent decades, researchers worldwide have increasingly directed their focus towards nonlinear partial differential equations (PDEs) rather than linear models, driven by major advancements in science and computer technology. A significant motivation for studying nonlinear models lies in their ability to describe virtually every phenomenon encountered in daily life. Nonlinear evolution equations (NLEEs) are particularly vital across numerous scientific and engineering domains, capturing real-life phenomena observed in fields like oceanography [1], nonlinear optics [2], fluid mechanics [3], chemical physics, solid-state physics [4], geochemistry, plasma physics [5] and wave propagation in shallow water [6]. The quest for exact solutions of nonlinear models is arduous yet essential for accurately understanding their characteristics. The integrability of a nonlinear PDE holds immense importance as it ensures the exact solvability of such equations. Over the past decades, various sophisticated methods have been devised for checking the integrability of NLEEs, including the Inverse scattering transform method [7], Painlevè analysis [8], Hirotas bilinear method [9], Lax pair, KP hierarchy reduction method, residual neural network method [10], Bäcklund transformation [11], Wronskian technique [12], Darboux transformation [13], Lie symmetry analysis [14] and bilinear neural network method [15]. Notably, the Hirota bilinear method distinguishes itself as both the most straightforward and the most efficient approach among them. To apply the Hirota bilinear

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method, the problem needs to be transformed into Hirota bilinear form, enabling the derivation of lump solutions, multi-soliton solutions, Wronskian solutions, periodic solutions, breather solutions, Grammian solutions, rogue wave solutions, and various types of rational solutions. Gilson et al. [16], Lambert and Springael [17] pioneered a revolutionary method for deriving the Hirota bilinear form of NLEEs using Bell polynomial theory, allowing for the direct deduction of the bilinear Bäcklund transformation and Lax pair from Hirota bilinear form. This method was extended by E. Fan to calculate infinitely many conservation laws directly from the bilinear Bäcklund transformation [18]. Recently, researchers have checked the integrability of various NLEEs using this method [19–22].The Wronskian technique also emerges as an uncomplicated and highly effective approach for determining exact solutions of integrable systems. In 1979, Satsuma [23] pioneered the derivation of the N-soliton solution through the Wronskian expression, followed by Freeman and Nimmo in 1983, who introduced a systematic methodology for confirming the existence of Wronskian solutions for equations such as the Kadomtsev–Petviashvili (KP) equation and the Korteweg-de Vries (KdV) equation [12]. Hirota et al. subsequently broadened the scope of Wronskian solutions to discrete systems by substituting the Wronskian determinant with the Casorati determinant [24]. Several nonlinear evolution equations, such as, the KdV equation, sine-Gordon equation, Boussinesq equation, KP equation, and mKdV equation have been demonstrated to exhibit solutions through the Wronskian formalism [12, 25]. In recent years utilizing this method, researchers have derived a range of solutions including rational solutions, solitons, complexitons, positons, and negatons [26–30]. Researchers have increasingly explored the fascinating dynamical properties of exact solutions to NLEEs, with a particular emphasis on phenomena such as rogue wave, soliton, breather wave, and lump wave. When waves travel through nonlinear media, their amplitudes and widths typically fluctuate. Nevertheless, under certain conditions, a balance between nonlinearity and dispersion can lead to the formation of stable, localized waves known as solitons. Lump waves are a class of rational function solutions that display localization in all spatial directions. They were initially discovered in 1977 by Manakov et al. [31] and have since been observed in various research areas such as soliton theory, optical media, plasma, and shallow water waves [32–34]. Several methods, including the long wave limit of N-soliton solutions [35], inverse scattering transformation [7], Darboux transformation [36], Lie symmetry method [37], and Bäcklund transformation [38] have been employed to derive exact lump solutions of NLEEs. In 2015, Ma [39] introduced an innovative approach to computing lump waves. This technique has enabled researchers worldwide to derive effective lump solutions for various NLEEs. The exploration of the dynamic interplay among lump solutions and other exact solutions, such as rogue waves and solitons, breathers, stripes, kinks etc., has opened up new frontiers in our understanding of wave phenomena [40–44]. Recently, Mandal et al. [45] introduced a comprehensive criterion for identifying lump and lump-multi-kink solutions of NLEEs manifesting a special type of Hirota bilinear form. Additionally, Mandal et al. [46] presented the interaction phenomena of kink waves with higher-order breather waves for an extended B-type KP equation.

A $(2+1)$-dimensional Bogoyavlensky-Konopelchenko equation [47] can be written as

$$u_{xt} + a_1 u_{4x} + a_2 u_{3xy} + a_3 u_x u_{2x} + a_4 u_x u_{xy} + a_5 u_{2x} u_y = 0, \tag{1}$$

which is the generalized form of $(2+1)$-dimensional KdV equation and have practical applications in elucidating the dynamics of internal waves within deep water. A generalized $(2+1)$-dimensional Bogoyavlensky-Konopelchenko equation can be present as

$$u_{xt} + a_1 u_{4x} + a_2 u_{3xy} + a_3 u_x u_{2x} + a_4 u_x u_{xy} + a_5 u_{2x} u_y + p_1 u_{2x} + p_2 u_{xy} + p_3 u_{2y} = 0. \tag{2}$$

Equations (1) and (2) have been extensively explored in the literature [48–50], with investigations into Lie symmetry analysis, conservation laws, soliton solutions, and lump solutions provided.

Recently, Wazwaz et al. extended Eq. (2) into a novel $(3+1)$-dimensional Bogoyavlensky-Konopelchenko equation [51]

$$u_{xt} + u_{yt} + u_{xxxx} + u_{xxxy} + 6u_x u_{xx} + 3u_x u_{xy} + 3u_{xx} u_y + \alpha u_{xy} + \beta u_{xz} + \beta u_{yz} + \gamma_1 u_{xx} + \gamma_2 u_{yy} = 0, \tag{3}$$

which describes the three-dimensional interaction of a long wave and a Riemann wave in nonlinear media. The equation holds significance across a multitude of scientific domains, encompassing plasma physics, nonlinear optics, fluid dynamics, biological systems, and even differential geometry. In [51], the authors demonstrated that Eq. (3) successfully passes the Painlevè integrability test and obtained lump and multi-soliton solutions.

However, we observe that aspects of integrability such as the Hirota bilinear form, bilinear Bäcklund transformation, Lax pair, infinitely many conservation laws, Wronskian solution and various types of exact solutions like kink solution, lump solution and their interactions have not been explored yet. This observation has motivated our investigation into these aspects in the present article.

The structure of our article is as follows. In Sect. 2, a brief introduction of the multi-dimensional Bell polynomials is given. In Sect. 3, we extensively investigate the Hirota bilinear form, bilinear Bäcklund transformation and the related Lax pair formulation for the model under consideration, utilizing Bell polynomial theory respectively. Section 4 is dedicated to deriving an infinite array of conservation laws. Section 5 delves into the computation and visual representation of one-, two-, and three-soliton solutions. In Sect. 6, we thoroughly investigate the Wronskian condition, Wronskian rational solution, and multi-soliton solutions in Wronskian form. Section 7 is dedicated to deriving the Grammian determinant solution. Section 8 focuses on investigating the lump solution. Section 9 is devoted to computing two sets of lump-multi-kink solutions. In Sect. 10, the obtained solutions are presented, with a thorough discussion of their propagation and their nonlinear interaction with other solutions. Finally, in Sect. 11, conclusions are drawn based on the findings.

2 Multi-dimensional Bell polynomials

In this section, we present a concise overview of the fundamental principles and representations of Bell polynomials [16, 17]. Let's assume that φ is a C^∞ function of x. The one-dimensional Bell polynomial [16] can be represented as

$$Y_{nx}(\varphi) = Y_n(\varphi_1, \varphi_2, \ldots, \varphi_{nx}) = e^{-\varphi}\partial_x{}^n e^{\varphi},\ n = 1, 2, 3, \ldots. \tag{4}$$

The definition above leads to a variety of one-dimensional Bell polynomials

$$Y_x = \varphi_x,\ Y_{2x} = \varphi_{2x} + \varphi_x{}^2,\ Y_{3x} = \varphi_{3x} + 3\varphi_x\varphi_{2x} + \varphi_x{}^3, \ldots. \tag{5}$$

The one-dimensional Bell polynomials described in Eq. (5) can be obtained using the following formula

$$Y_{nx}(\varphi) = \sum \frac{n!}{a_1!\,a_2!\ldots a_n!}\left(\frac{\varphi_x}{1!}\right)^{a_1}\left(\frac{\varphi_{2x}}{2!}\right)^{a_2}\ldots\left(\frac{\varphi_{nx}}{n!}\right)^{a_n}, \tag{6}$$

where the sum run over all partitions of $n = a_1 + 2a_2 + \cdots + na_n$. Assuming $\varphi = \varphi(x_1, x_2, \ldots, x_s)$ as a C^∞ multi-variable function, we have the opportunity to expand the dimension of the Bell polynomial, allowing for a more comprehensive and powerful representation. This extension results in the multi-dimensional Bell polynomial, which can be elegantly expressed as follows

$$Y_{n_1x_1,\ldots,n_sx_s}(\varphi) \equiv Y_{n_1,\ldots,n_s}(\varphi_{m_1x_1,\ldots,m_sx_s}) = e^{-\varphi}\partial_{x_1}^{n_1}\ldots\partial_{x_s}^{n_s} e^{\varphi}, \tag{7}$$

where $\varphi_{m_1x_1,\ldots,m_sx_s} = \partial_{x_1}^{m_1}\ldots\partial_{x_s}^{m_s}\varphi$, $m_i = 0, 1, \ldots, n_i$ $(i = 1, 2, \ldots, s)$. Here $Y_{n_1x_1,\ldots,n_sx_s}(\varphi)$ denotes the multi-variable Bell polynomial with respect to $\varphi_{m_1x_1,\ldots,m_sx_s}$. In the specific case where we choose a function $\varphi(x, y)$, the associated lower-order two-dimensional Bell polynomials can be obtained through the following derivation.

$$Y_{2x(\varphi)} = \varphi_{2x} + \varphi_x^2, \quad Y_{3x(\varphi)} = \varphi_{3x} + 3\varphi_{2x}\varphi_x + \varphi_x^3, \tag{8}$$

$$Y_{x,z} = \varphi_{x,z} + \varphi_x\varphi_z, \quad Y_{2x,z}(\varphi) = \varphi_{2x,z} + \varphi_{2x}\varphi_z + 2\varphi_{x,z}\varphi_x + \varphi_x^2\varphi_z, \ldots \tag{9}$$

From the previously mentioned one-dimensional Bell polynomials in Eq. (7), we can also represent multi-dimensional binary Bell polynomials in the following way.

$$\mathscr{Y}_{n_1x_1,\ldots,n_sx_s}(\mathscr{f}, \mathscr{g}) = Y_{n_1x_1,\ldots,n_sx_s}(\varphi), \tag{10}$$

where

$$\varphi_{m_1x_1,\ldots,m_sx_s} = \begin{cases} \mathscr{f}_{m_1x_1,\ldots,m_sx_s}, & m_1 + \cdots + m_s \text{ is odd}, \\ \mathscr{g}_{m_1x_1,\ldots,m_sx_s}, & m_1 + \cdots + m_s \text{ is even}. \end{cases} \tag{11}$$

Two-dimensional binary Bell polynomials that can be derived from the previously mentioned statement include

$$\begin{aligned} &\mathscr{Y}_x(\mathscr{f}) = \mathscr{f}_x, \quad \mathscr{Y}_{2x}(\mathscr{f}, \mathscr{g}) = \mathscr{g}_{2x} + \mathscr{f}_x^2, \quad \mathscr{Y}_{x,z}(\mathscr{f}, \mathscr{g}) = \mathscr{g}_{x,z} + \mathscr{f}_x\mathscr{f}_z, \\ &\mathscr{Y}_{2x,z}(\mathscr{f}, \mathscr{g}) = \mathscr{f}_{2x,z} + \mathscr{g}_{2x}\mathscr{f}_z + 2\mathscr{g}_{x,z}\mathscr{f}_x + \mathscr{f}_x^2\mathscr{f}_z, \quad \mathscr{Y}_{3x} = \mathscr{f}_{3x} + 3\mathscr{g}_{2x}\mathscr{f}_x + \mathscr{f}_x^3, \ldots \end{aligned} \tag{12}$$

By leveraging the identity

$$(\varphi\psi)^{-1}D_{x_1}^{n_1}\ldots D_{x_s}^{n_s}\varphi.\psi = \mathscr{Y}_{n_1x_1,\ldots,n_sx_s}(\mathscr{f} = \ln\varphi/\psi,\ \mathscr{g} = \ln\varphi\psi), \tag{13}$$

a relationship can be established between the conventional Hirota bilinear expression $D_{x_1}^{n_1}\ldots D_{x_s}^{n_s}\varphi.\psi$ and binary Bell polynomials. Here, the D-operator is elucidated by Hirota [9] as follows

$$D_{x_1}^{n_1}\ldots D_{x_s}^{n_s}\varphi.\psi = (\partial_{x_1} - \partial_{x_1'})^{n_1}\ldots(\partial_{x_s} - \partial_{x_s'})^{n_s}\varphi(x_1, \ldots, x_s).\psi(x_1', \ldots x_s')|_{x_1'=x_1,\ldots,x_s'=x_s}. \tag{14}$$

The identity Eq. (13) becomes in the scenario where $\varphi = \psi$

$$(\varphi)^{-2}D_{x_1}^{n_1}\ldots D_{x_s}^{n_s}\varphi.\varphi = \mathscr{Y}_{n_1x_1,\ldots,n_sx_s}(\mathscr{f} = 0, \mathscr{g} = 2\ln\varphi) = \begin{cases} 0, & n_1 + \cdots + n_s \text{ is odd}, \\ \mathscr{P}_{n_1x_1,\ldots,n_sx_s}(\mathscr{p}), & n_1 + \cdots + n_s \text{ is even}. \end{cases} \tag{15}$$

Here, even-ordered $\mathscr{Y}$-polynomials are represented by $\mathscr{P}$-polynomials. Below are a few of the lower-order $\mathscr{Y}$-polynomials.

$$\mathscr{P}_{2x}(\mathscr{p}) = \mathscr{p}_{2x}, \quad \mathscr{P}_{x,z}(\mathscr{p}) = \mathscr{p}_{x,z}, \quad \mathscr{P}_{3x,z}(\mathscr{p}) = \mathscr{p}_{3x,z} + 3\mathscr{p}_{x,z}\mathscr{p}_{2x}, \quad \mathscr{P}_{4x}(\mathscr{p}) = \mathscr{p}_{4x} + 3\mathscr{p}_{2x}^2. \tag{16}$$

A linear combination of $\mathscr{P}$-polynomials and Bell polynomials $Y_{n_1x_1,\ldots,n_sx_s}(\mathscr{f})$ is employed to express the binary Bell polynomial $\mathscr{Y}_{n_1x_1,\ldots,n_sx_s}(\mathscr{f}, \mathscr{g})$ as

$$\begin{aligned} (\varphi\psi)^{-1}D_{x_1}^{n_1}\ldots D_{x_s}^{n_s}\varphi.\psi &= \mathscr{Y}_{n_1x_1,\ldots,n_sx_s}(\mathscr{f}, \mathscr{g}), \quad \text{where } \mathscr{f} = \ln\varphi/\psi \text{ and } \mathscr{g} = \ln\varphi\psi \\ &= \mathscr{Y}_{n_1x_1,\ldots,n_sx_s}(\mathscr{f}, \mathscr{f} + \mathscr{p}), \quad \text{where } \mathscr{f} = \ln\varphi/\psi \text{ and } \mathscr{p} = 2\ln\psi \end{aligned}$$

$$= \sum_{m_1=0}^{n_1} \cdots \sum_{m_s=0}^{n_s} \prod_{i=1}^{s} \binom{n_i}{m_i} \mathscr{P}_{m_1x_1,\ldots,m_sx_s}(\wp)\, Y_{(n_1-m_1)x_1,\ldots,(n_s-m_s)x_s}(\ell). \tag{17}$$

Utilizing the Hopf-Cole transformation $\ell = \ln\psi$, the binary Bell polynomial can be expressed as follows

$$Y_{n_1x_1,\ldots,n_sx_s}(\ell = \ln\psi) = \frac{\psi_{n_1x_1,\ldots,n_sx_s}}{\psi}, \tag{18}$$

which enables us to rewrite Eq. (17) as

$$(\varphi\psi)^{-1} D_{t_1}^{n_1} \ldots D_{t_s}^{n_s} \varphi.\psi = \psi^{-1} \sum_{m_1=0}^{n_1} \ldots \sum_{m_s=0}^{n_s} \prod_{i=1}^{s} \binom{n_i}{m_i} \mathscr{P}_{m_1x_1,\ldots,m_sx_s}(\wp)\, \psi_{(n_1-m_1)x_1,\ldots,(n_s-m_s)x_s}. \tag{19}$$

The methodology outlined in Eq. (19) furnishes the most straightforward and systematic framework for the explicit construction of the corresponding Lax pair associated with the relevant nonlinear evolution equation.

3 Hirota bilinear form, bilinear Bäcklund transformation and Lax pair

A new potential field $\wp$ is introduced by setting

$$u = \wp_x \tag{20}$$

to achieve the Hirota bilinear form of Eq. (3). After replacing Eq. (20) to Eq. (3) and integrating with regard to x, we get

$$E(\wp) = \wp_{xt} + \wp_{yt} + \wp_{4x} + \wp_{3xy} + 3\wp_{2x}^2 + 3\wp_{2x}\wp_{xy} + \alpha\wp_{xy} + \beta\wp_{yz} + \gamma_1\wp_{2x} + \gamma_2\wp_{2y} = 0, \tag{21}$$

Furthermore, by setting $\wp = 2\log\hbar$, we obtain the Hirota bilinear form of Eq. (3) employing the relationship between the P-polynomial and the Hirota D-operator as described in Eq. (15) as

$$(D_xD_t + D_yD_t + D_x^4 + D_x^3D_y + \alpha D_xD_y + \beta D_xD_z + \beta D_yD_z + \gamma_1 D_x^2 + \gamma_2 D_y^2)\hbar\cdot\hbar = 0. \tag{22}$$

To calculate bilinear Bäcklund transformation, we take $\wp' = 2\log\hbar'$ as another solution of Eq. (3). Additionally, we introduce two new variables $\nu = \frac{\wp'-\wp}{2}$ and $\omega = \frac{\wp'+\wp}{2}$. Consequently, the two-field condition can be written as

$$\begin{aligned} E(\wp') - E(\wp) &= 2\Big[\nu_{xt} + \nu_{yt} + \nu_{4x} + \nu_{3xy} + 6\nu_{2x}\omega_{2x} + 3(\omega_{2x}\nu_{xy} + \nu_{2x}\omega_{xy}) + \alpha\nu_{xy} + \beta(\nu_{xz} + \nu_{yz}) + \gamma_1\nu_{2x} + \gamma_2\nu_{2y}\Big] = 0 \\ &= 2\Big[\frac{\partial}{\partial x}\{\mathscr{Y}_t(\nu) + \alpha\mathscr{Y}_y(\nu) + \beta\mathscr{Y}_z(\nu) + \gamma_1\mathscr{Y}_x(\nu) + \mathscr{Y}_{3x}(\nu,\omega)\} + \frac{\partial}{\partial y}\{\mathscr{Y}_t(\nu) + \mathscr{Y}_{3x}(\nu,\omega)\beta\mathscr{Y}_z(\nu) + \gamma_2\mathscr{Y}_y(\nu)\}\Big] \\ &\quad + 6\ \text{Wronskian}\,[\mathscr{Y}_{2x}, \mathscr{Y}_x] + 6\ \text{Wronskian}\,[\mathscr{Y}_{xy}, \mathscr{Y}_x] = 0. \end{aligned} \tag{23}$$

Finally, the bilinear Bäcklund transformation of Eq. (3) can be derived by decoupling the two-field condition in Eq. (23) as

$$\mathscr{Y}_t(\nu) + \alpha\mathscr{Y}_y(\nu) + \beta\mathscr{Y}_z(\nu) + \gamma_1\mathscr{Y}_x(\nu) + \mathscr{Y}_{3x}(\nu,\omega) - c_1 = 0, \tag{24a}$$

$$\mathscr{Y}_t(\nu) + \beta\mathscr{Y}_z(\nu) + \gamma_2\mathscr{Y}_y(\nu) + \mathscr{Y}_{3x}(\nu,\omega) - c_2 = 0, \tag{24b}$$

$$\mathscr{Y}_{2x}(\nu,\omega) + \delta\mathscr{Y}_x(\nu) = 0, \tag{24c}$$

$$\mathscr{Y}_{xy}(\nu,\omega) + \eta\mathscr{Y}_x(\nu) = 0. \tag{24d}$$

Bilinear Bäcklund transformation Eq. (24) can also be expressed in terms of the Hirota D-operator as follows

$$(\hbar'\cdot\hbar)^{-1}[D_t + \alpha D_y + \beta D_z + \gamma_1 D_x + D_x^3 - c_1](\hbar'\cdot\hbar) = 0, \tag{25a}$$

$$(\hbar'\cdot\hbar)^{-1}[D_t + \beta D_z + \gamma_2 D_y + D_x^3 - c_2](\hbar'\cdot\hbar) = 0, \tag{25b}$$

$$(\hbar'\cdot\hbar)^{-1}[D_x^2 + \delta D_y](\hbar'\cdot\hbar) = 0, \tag{25c}$$

$$(\hbar'\cdot\hbar)^{-1}[D_xD_y + \eta D_x](\hbar'\cdot\hbar) = 0. \tag{25d}$$

By employing the Cole–Hopf transformation $v = \log\lambda$ and linearizing the Bell polynomial system in Eq. (24), we successfully establish the Lax pair for Eq. (3) as outlined below.

$$\lambda_t + \beta\lambda_z + \alpha\lambda + \lambda_x(\gamma_1 + 3\wp_{2x}) + \lambda_{3x} - c_1\lambda = 0, \tag{26a}$$

$$\lambda_t + \beta\lambda + \gamma_2\lambda + 3\lambda\wp_{2x} + \lambda_{3x} - c_2\lambda = 0, \tag{26b}$$

$$\lambda_{2x} + \lambda \wp_{2x} + \delta \lambda_x = 0\,, \tag{26c}$$

$$\lambda_{xy} + \lambda \wp_{xy} + \eta \lambda_x = 0\,. \tag{26d}$$

4 Infinitely many conservation laws

In order to establish infinite conservation laws of Eq. (3), we take $\eta = \delta$ and rewrite the two-field condition Eq. (23) as

$$\omega_{2x} + \nu_x^2 + \omega_{xy} + \nu_x \nu_y + 2\delta \nu_x = 0 \tag{27}$$

$$\frac{\partial}{\partial t}[\nu_x + \nu_y] + \frac{\partial}{\partial x}[\alpha \nu_y + \gamma_1 \nu_x + \nu_{3x} + 3\omega_{2x}\nu_x + \nu_x^3] + \frac{\partial}{\partial y}[\nu_{3x} + 3\omega_{2x}\nu_x + \nu_x^3 + \gamma_2 \nu_y] + \frac{\partial}{\partial z}[\beta \nu_x + \beta \nu_y]\,. \tag{28}$$

We introduce a novel potential function $\sigma = \frac{\wp_x' - \wp_x}{2}$, which yields

$$\nu_x = \sigma,\;\; \nu_y = \partial_x^{-1}(\sigma_y),\;\; \nu_z = \partial_x^{-1}(\sigma_z),\;\; \omega_x = \wp_x + \sigma. \tag{29}$$

Substituting Eq. (29) in both Eq. (27) and Eq. (28) results

$$\wp_{xx} + \sigma_x + \sigma^2 + \wp_{xy} + \sigma_y + \sigma\,\partial_x^{-1}(\sigma_y) + 2\delta\sigma = 0\,, \tag{30}$$

$$\begin{aligned}&\partial_t[\sigma + \partial_x^{-1}(\sigma_y)] + \partial_x[\alpha\,\partial_x^{-1}(\sigma_y) + \gamma_1\sigma + \sigma_{2x} + 3(\wp_{2x} + \sigma_x)\sigma + \sigma^3] + \partial_y[\sigma_{2x} + 3(\wp_{2x} + \sigma_x)\sigma + \sigma^3 + \gamma_2\,\partial_x^{-1}(\sigma_y)]\\ &+ \partial_z[\beta\sigma + \partial_x^{-1}(\sigma_y)] = 0\,.\end{aligned} \tag{31}$$

We adopt the infinite series form of σ as

$$\sigma = \sum_{n=1}^{\infty} \mathscr{L}_n(\wp, \wp_x, \wp_{2x}, \ldots)\delta^{-n}. \tag{32}$$

By substituting Eq. (32) into Eq. (30) and equating all similar powers of δ, we derive the recursion relations that govern the conserved densities as

$$\mathscr{L}_1 = -\frac{1}{2}(\wp_{2x} + \wp_{xy})\,, \tag{33a}$$

$$\mathscr{L}_2 = -\frac{1}{2}(\mathscr{L}_{1,x} + \mathscr{L}_{1,y}) = \frac{1}{4}(\wp_{3x} + 2\wp_{2xy} + \wp_{x2y})\,, \tag{33b}$$

$$\mathscr{L}_3 = -\frac{1}{2}\{\mathscr{L}_{2,x} + \mathscr{L}_1^2 + \mathscr{L}_{2,y} + \mathscr{L}_1\,\partial_x^{-1}(\mathscr{L}_{1,y})\}\,, \tag{33c}$$

$$\cdots\cdots$$

$$\mathscr{L}_{n+1} = -\frac{1}{2}\left\{\mathscr{L}_{n,x} + \sum_{k=1}^{n}\mathscr{L}_k\,\mathscr{L}_{n-k} + \mathscr{L}_{n,y} + \sum_{k=1}^{n}\mathscr{L}_k\,\partial_x^{-1}(\mathscr{L}_{n-k})\right\}. \tag{33d}$$

Again, replacing Eq. (32) into Eq. (31) yields

$$\begin{aligned}&\partial_t\left[\sum_{n=1}^{\infty}\mathscr{L}_n\delta^{-n} + \partial_x^{-1}\left(\sum_{n=1}^{\infty}\mathscr{L}_{n,y}\delta^{-n}\right)\right] + \partial_x\left[\alpha\,\partial_x^{-1}\left(\sum_{n=1}^{\infty}\mathscr{L}_{n,y}\delta^{-n}\right) + \gamma_1\left(\sum_{n=1}^{\infty}\mathscr{L}_n\delta^{-n}\right) + \left(\sum_{n=1}^{\infty}\mathscr{L}_{n,2x}\delta^{-n}\right)\right.\\ &\left.+3\left(\wp_{2x} + \sum_{n=1}^{\infty}\mathscr{L}_{n,x}\delta^{-n}\right)\left(\sum_{n=1}^{\infty}\mathscr{L}_n\delta^{-n}\right) + \left(\sum_{n=1}^{\infty}\mathscr{L}_n\delta^{-n}\right)^3\right] + \partial_y\left[\sum_{n=1}^{\infty}\mathscr{L}_{n,2x}\delta^{-n} + 3\left(\wp_{2x} + \sum_{n=1}^{\infty}\mathscr{L}_{n,x}\delta^{-n}\right)\right.\\ &\left.\left(\sum_{n=1}^{\infty}\mathscr{L}_n\delta^{-n}\right) + \left(\sum_{n=1}^{\infty}\mathscr{L}_n\delta^{-n}\right)^3 + \gamma_2\,\partial_x^{-1}\left(\sum_{n=1}^{\infty}\mathscr{L}_n\delta^{-n}\right)\right] + \partial_z\left[\beta\left\{\left(\sum_{n=1}^{\infty}\mathscr{L}_n\delta^{-n}\right) + \partial_x^{-1}\left(\sum_{n=1}^{\infty}\mathscr{L}_{n,y}\delta^{-n}\right)\right\}\right] = 0\,.\end{aligned} \tag{34}$$

Finally, we compare the coefficient of all similar powers of δ from Eq. (3) to obtain the conservation rules for Eq. (34).

$$\mathscr{X}_{n,t} + \mathscr{U}_{n,x} + \mathscr{Z}_{n,y} + \mathscr{W}_{n,z} = 0, \tag{35}$$

where,

$$\mathscr{X}_1 = \mathscr{L}_1 + \partial_x^{-1}(\mathcal{L}\vee\infty\Leftrightarrow\dagger) = -\frac{1}{2}(\wp_{2x} + 2\wp_{xy} + \wp_{2y})\,, \tag{36a}$$

$$\mathscr{X}_2 = \mathscr{L}_2 + \partial_x^{-1}(\mathcal{L}\vee\in\Leftrightarrow\dagger) = \frac{1}{4}(\wp_{3x} + 3\wp_{2xy} + 3\wp_{x2y} + \wp_{3y}), \tag{36b}$$

$$\cdots\cdots$$

$$\mathscr{X}_n = \mathscr{L}_n + \partial x^{-1}\mathscr{L}_{n,y}, \tag{36c}$$

$$\mathscr{U}_1 = \alpha\partial_x^{-1}(\mathscr{L}_{1,y}) + \gamma_1\mathscr{L}_1 + \mathscr{L}_{1,2x} + 3\mathscr{L}_1\wp_{2x} \tag{37a}$$

$$= -\frac{1}{2}\left\{\alpha(\wp_{xy} + \wp_{2y}) + \gamma_1(\wp_{2x} + \wp_{xy}) + \wp_{4x} + \wp_{3xy} - 3\wp_{2x}^2 + 3\wp_{2x} + 3\wp_{2x}\wp_{xy}\right\}, \tag{37b}$$

$$\mathscr{U}_2 = \alpha\partial_x^{-1}(\mathscr{L}_{2,y}) + \gamma_1\mathscr{L}_2 + \mathscr{L}_{2,2x} + 3\mathscr{L}_1\mathscr{L}_{1,x} + 3\mathscr{L}_2\wp_{2x}, \tag{37c}$$

$$\cdots\cdots$$

$$\mathscr{U}_n = \alpha\,\partial_x^{-1}(\mathscr{L}_{n,y}) + \gamma_1\mathscr{L}_n + \mathscr{L}_{n,2x} + 3\sum_{i+j=n}\mathscr{L}_i\mathscr{L}_{j,x} + 3\mathscr{L}_n\wp_{2x} + \sum_{i+j+k=n}\mathscr{L}_i\mathscr{L}_j\mathscr{L}_k, \tag{37d}$$

$$\mathscr{Z}_1 = \mathscr{L}_{1,2x} + 3\mathscr{L}_1\wp_{2x} + \gamma_2\partial_x^{-1}\mathscr{L}_{1,y} \tag{38a}$$

$$= -\frac{1}{2}\left\{\mathscr{L}_{4x} + \mathscr{L}_{3xy} + 3(\wp_{2x}^2 + \wp_{xy}\wp_{2x}) + \gamma_2(\wp_{xy} + \wp_{2y})\right\}, \tag{38b}$$

$$\mathscr{Z}_2 = \mathscr{L}_{2,2x} + 3\mathscr{L}_1\mathscr{L}_{1,2x} + 3\mathscr{L}_2\wp_{2x} + \gamma_2\partial_x^{-1}(\mathscr{L}_{2,y}), \tag{38c}$$

$$\cdots\cdots$$

$$\mathscr{Z}_n = \mathscr{L}_{n,2x} + 3\sum_{i+j=n}\mathscr{L}_i\mathscr{L}_{j,x} + 3\mathscr{L}_n\wp_{2x} + \sum_{i+j+k=n}\mathscr{L}_i\mathscr{L}_j\mathscr{L}_k + \gamma_2\partial_x^{-1}\mathscr{L}_{n,y} \tag{38d}$$

and

$$\mathscr{W}_1 = \beta\mathscr{L}_1 + \beta\partial_x^{-1}(\mathscr{L}_{1,y}) = -\frac{\beta}{2}(\wp_{2x} + 2\wp_{xy} + \wp_{2y}) \tag{39a}$$

$$\mathscr{W}_2 = \beta\mathscr{L}_2 + \beta\partial_x^{-1}(\mathscr{L}_{2,y}), \tag{39b}$$

$$\cdots\cdots$$

$$\mathscr{W}_n = \beta\mathscr{L}_n + \beta\partial_x^{-1}(\mathscr{L}_{n,y}). \tag{39c}$$

The values of $\mathscr{L}_n$ are given via recursion relations Eq. (33).

5 Soliton solutions

5.1 One-soliton solution

To calculate one-soliton solution of Eq. (3), we adopt the following expression for $\hbar$

$$\hbar = 1 + e^{\chi_1}, \tag{40}$$

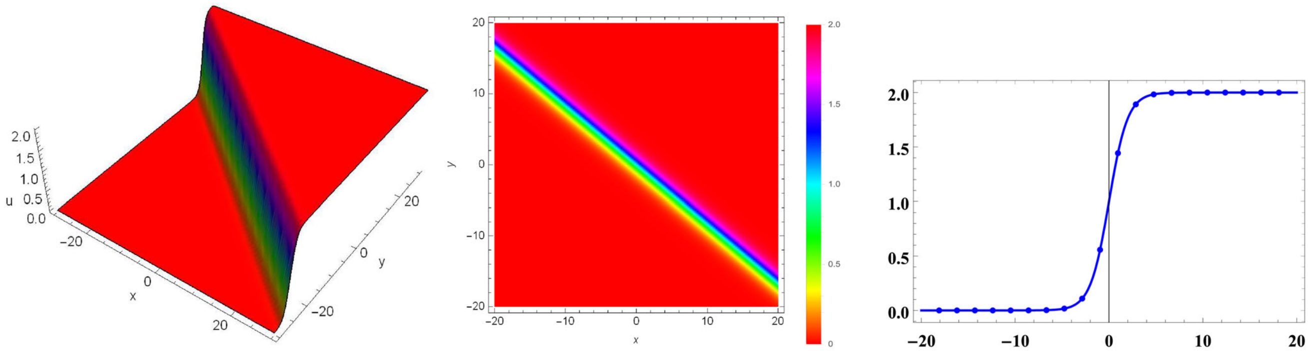
where $\chi_1 = a_1(x + b_1 y + c_1 z + w_1 t) + \chi_1^0$ and a_1, b_1, c_1, χ_1^0 are arbitrary constants. By substituting Eq. (40) into Eq. (22) and setting each exponential function to zero, we derive the dispersion relation as

$$w_1 = -\frac{a_1^2 + a_1^2 b_1 + \alpha b_1 + \beta(b_1 + c_1) + \gamma_1 + \gamma_2 b_1^2}{1 + b_1}. \tag{41}$$

Finally, the one-soliton solution for Eq. (3) is obtained by substituting Eq. (40) along with Eq. (41) into Eq. (22) as

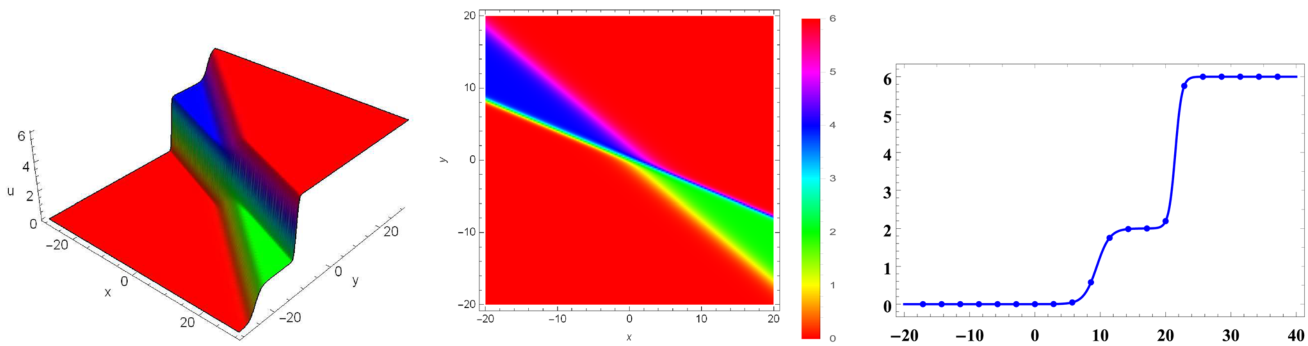
$$u = 2\left[\log(1 + e^{\chi_1})\right]_x. \tag{42}$$

In Fig. 1, we present 3-dimensional plot, density plot and 2-dimensional plot of one-soliton solution Eq. (42), corresponding to parametric values as $\alpha = \beta = \gamma_1 = \gamma_2 = 1,\ a_1 = 1,\ b_1 = 1.2,\ c_1 = 1.3,\ \chi_1^0 = 0$.



(a) (b) (c)

Fig. 1 The one-soliton solution of Eq. (3), represented by Eq. (42) **a** 3D surface plot, **b** density map, **c** 2D projection



(a) (b) (c)

Fig. 2 The two-soliton solution of Eq. (3), represented by Eq. (45) **a** 3D surface plot, **b** density map, **c** 2D projection

5.2 Two-soliton solution

To derive the two-soliton solution for Eq. (3), we consider $\hbar$ in the following form

$$\hbar = 1 + e^{\chi_1} + e^{\chi_2} + C_{12} e^{\chi_1+\chi_2}, \tag{43}$$

where $\chi_i = a_i(x + b_i y + c_i z + w_i t) + \chi_i^0,\ \ (i = 1, 2)$ and $a_i,\ b_i,\ c_i,\ \chi_i^0,\ \ (i = 1, 2)$ are arbitrary constants. Substitution of Eq. (43) into Eq. (22) yields dispersion relation and C_{12} as

$$w_i = -\frac{a_i^2 + a_i^2 b_i + \alpha b_i + \beta(b_i + c_i) + \gamma_1 + \gamma_i b_i^2}{1 + b_i},\ (i = 1, 2) \tag{44a}$$

$$C_{12} = \left[\frac{\begin{array}{c}(a_1w_1 - a_2w_2)(a_1 - a_2 + a_1b_1 - a_2b_2) + (a_1 - a_2)^4 + (a_1 - a_2)^3(a_1b_1 - a_2b_2) + \alpha(a_1 - a_2)(a_1b_1 - a_2b_2)\\ +\beta\{(a_1 - a_2)(a_1c_1 - a_2c_2) + (a_1b_1 - a_2b_2)(a_1c_1 - a_2c_2)\} + \gamma_1(a_1 - a_2)^2 + \gamma_2(a_1b_1 - a_2b_2)^2\end{array}}{\begin{array}{c}(a_1w_1 + a_2w_2)(a_1 + a_2 + a_1b_1 + a_2b_2) + (a_1 + a_2)^4 + (a_1 + a_2)^3(a_1b_1 + a_2b_2) + \alpha(a_1 + a_2)(a_1b_1 + a_2b_2)\\ +\beta\{(a_1 + a_2)(a_1c_1 + a_2c_2) + (a_1b_1 + a_2b_2)(a_1c_1 + a_2c_2)\} + \gamma_1(a_1 + a_2)^2 + \gamma_2(a_1b_1 + a_2b_2)^2\end{array}}\right]. \tag{44b}$$

Again substituting Eq. (43) with Eq. (44) into Eq. (22), we have obtained two-soliton solution of Eq. (3) as

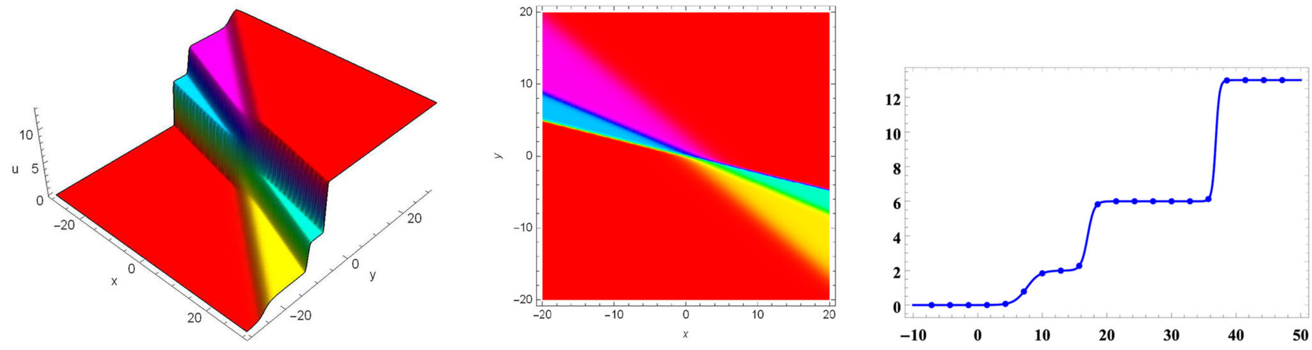
$$\mathcal{U} = 2\left[\log\left(1 + e^{\chi_1} + e^{\chi_2} + B_{12} e^{\chi_1+\chi_2}\right)\right]_x . \tag{45}$$

Figure 2 illustrates a 3-dimensional plot, a density plot, and a 2-dimensional plot of the two-soliton solution given by Eq. (45), corresponding to parametric values as $\alpha = \beta = \gamma_1 = \gamma_2 = 1,\ a_1 = 1,\ a_2 = 2,\ b_1 = 1.2,\ b_2 = 2.4\, c_1 = 1.3,\ c_2 = 2.5,$ $\chi_i^0 = 0\,(i = 1, 2)$.

5.3 Three-soliton solution

Following a similar approach, we can derive the three-soliton solution of Eq. (3) by selecting $\hbar$ in the following form

$$\hbar = 1 + e^{\chi_1} + e^{\chi_2} + e^{\chi_3} + C_{12}e^{\chi_1+\chi_2} + C_{13}e^{\chi_1+\chi_3} + C_{23}e^{\chi_2+\chi_3} + C_{123}e^{\chi_1+\chi_2+\chi_3}, \tag{46}$$



(a) (b) (c)

Fig. 3 The three-soliton solution of Eq. (3), represented by Eq. (92) **a** 3D surface plot, **b** density map, **c** 2D projection

where $\chi_i = a_i(x + b_i y + c_i z + w_i t) + \chi_i^0,\ (i = 1, 2, 3)$ and $a_i,\ b_i,\ c_i,\ \chi_i^0,\ (i = 1, 2, 3)$ are arbitrary constants. Substituting Eq. (46) into Eq. (22), we derive dispersion relation and C_{ij} as

$$w_i = -\frac{a_2^2 + a_2^2 b_2 + \alpha b_2 + \beta(b_2 + c_2) + \gamma_1 + \gamma_2 b_2^2}{1 + b_2},\ (i = 1, 2, 3), \tag{47a}$$

$$C_{12} = \left[\frac{\begin{array}{c}(a_i w_i - a_j w_j)(a_i - a_j + a_i b_i - a_j b_j) + (a_i - a_j)^4 + (a_i - a_j)^3(a_i b_i - a_j b_j) + \alpha(a_i - a_j)(a_i b_i - a_j b_j)\\ +\beta\{(a_i - a_j)(a_i c_i - a_j c_j) + (a_i b_i - a_j b_j)(a_i c_i - a_j c_j)\} + \gamma_1(a_i - a_j)^2 + \gamma_2(a_i b_i - a_j b_j)^2\end{array}}{\begin{array}{c}(a_i w_i + a_j w_j)(a_i + a_j + a_i b_i + a_j b_j) + (a_i + a_j)^4 + (a_i + a_j)^3(a_i b_i + a_j b_j) + \alpha(a_i + a_j)(a_i b_i + a_j b_j)\\ +\beta\{(a_i + a_j)(a_i c_i + a_j c_j) + (a_i b_i + a_j b_j)(a_i c_i + a_j c_j)\} + \gamma_1(a_i + a_j)^2 + \gamma_2(a_i b_i + a_j b_j)^2\end{array}}\right], \tag{47b}$$

$$C_{123} = C_{12}C_{13}C_{23}. \tag{47c}$$

Finally substituting Eq. (46) with Eq. (47) into Eq. (22), we derive three soliton solution of Eq. (3) as

$$u = 2\big[\log\left(1 + e^{\chi_1} + e^{\chi_2} + e^{\chi_3} + B_{12}e^{\chi_1+\chi_2} + B_{13}e^{\chi_1+\chi_3} + B_{23}e^{\chi_2+\chi_3} + B_{123}e^{\chi_1+\chi_2+\chi_3}\right)\big]_x. \tag{48}$$

In Fig. 3, we illustrate three soliton solution Eq. (48) graphically, corresponding to the parametric values as $\alpha = \beta = \gamma_1 = \gamma_2 = 1$, $a_1 = 1,\ a_2 = 2,\ a_3 = 3.5,\ b_1 = 1.2,\ b_2 = 2.4,\ b_3 = 4,\ c_1 = 1.3,\ c_2 = 2,\ c_3 = 3.3,\ \chi_i^0 = 0\,(i = 1, 2, 3)$.

6 Wronskian solution

To achieve $\mathcal{N}^{th}$ order Wronskian solution for Eq. (3), firstly, we adopt the $\mathcal{N}^{th}$ order Wronskian determinant notation proposed by Freeman and Nimmo [12, 25] as follows

$$\mathscr{W} = \mathscr{W}(\phi_1, \phi_2, \ldots, \phi_{\mathcal{N}}) = \begin{vmatrix} \phi_1^{(0)} & \phi_1^{(1)} & \cdots & \phi_1^{(\mathcal{N}-1)} \\ \phi_2^{(0)} & \phi_2^{(1)} & \cdots & \phi_2^{(\mathcal{N}-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{\mathcal{N}}^{(0)} & \phi_{\mathcal{N}}^{(1)} & \cdots & \phi_{\mathcal{N}}^{(\mathcal{N}-1)} \end{vmatrix} = |\widehat{\mathcal{N}-1}| \tag{49}$$

where $\mathcal{N} \geq 1$, an arbitrary integer and $\phi = (\phi_1, \phi_2, \ldots, \phi_{\mathcal{N}})^T$, a sequence of smooth polynomial functions defined as $\phi_i^{(0)} = \phi_i,\ \phi_i^{(j)} = \frac{\partial^j \phi_i}{\partial x^j},\ \ 1 \leq i \leq \mathcal{N},\ \ 0 \leq j \leq (\mathcal{N} - 1)$.

Employing the Hirota operator mentioned in Eq. (14) into the bilinear form Eq. (22), the Hirota bilinear form can be expressed as

$$\begin{aligned}&\hbar(\hbar_{xxxx} + \hbar_{xxxy} + \gamma_1\hbar_{xx} + \gamma_2\hbar_{yy} + \alpha\hbar_{xy} + \beta\hbar_{xz} + \beta\hbar_{yz} + \hbar_{xt} + \hbar_{yt}) - 4\hbar_x\hbar_{xxx} + 3\hbar_{xx}^2 - \hbar_{xxx}\hbar_y - 3\hbar_x\hbar_{xxy}\\ &\quad + 3\hbar_{xx}\hbar_{xy} - \gamma_1\hbar_x^2 - \gamma_2\hbar_y^2 - \alpha\hbar_x\hbar_y - \beta\hbar_x\hbar_z - \beta\hbar_y\hbar_z - \hbar_x\hbar_t - \hbar_y\hbar_t = 0\end{aligned} \tag{50}$$

A sufficient condition that the bilinear equation Eq. (50) of the (3 + 1)-dimensional extended BK equation Eq. (3) has Wronskian determinant solutions is mentioned below.

Theorem A: Consider a set of functions $\phi_i = \phi_i(x, y, z, t)$, $1 \leq i \leq \mathscr{N}$ that meet the subsequent linear partial differential conditions

$$\phi_{i,xx} = \sum_{j=1}^{\mathscr{N}} \lambda_{ij}\phi_j\,, \tag{51a}$$

$$\phi_{i,y} = \mathscr{A}\phi_{i,x}\,, \tag{51b}$$

$$\phi_{i,z} = \mathscr{B}\phi_{i,xxx}\,, \tag{51c}$$

$$\phi_{i,t} = \mathscr{C}\phi_{i,xxx}\,, \tag{51d}$$

where

$$\mathscr{A} = \left(\frac{-\alpha \pm \sqrt{\alpha^2 - 4\gamma_1\gamma_2}}{2\gamma_2}\right),\ \mathscr{C} = -4 - \beta\mathscr{B},\ 2\gamma_2 - \alpha \pm \sqrt{\alpha^2 - 4\gamma_1\gamma_2} \neq 0 \tag{52}$$

and $\mathscr{B}$ is a free parameter. Then, Wronskian determinant $\hbar = \hbar_{\mathscr{N}} = \mathscr{W}(\phi_1, \phi_2, \ldots, \phi_{\mathscr{N}})$ yields a solution $u = 2[\log \hbar]_x$ to the extended BK equation Eq. (3).

Before starting the proof, we introduce some lemmas to achieve the result.

Lemma A.1: For Wronskian determinant, the Plücker relation [9] is defined as follows

$$|\, \mathscr{P}, a, b \,||\, \mathscr{P}, c, d \,| - |\, \mathscr{P}, a, c \,||\, \mathscr{P}, b, d \,| + |\, \mathscr{P}, a, d \,||\, \mathscr{P}, b, c \,| = 0\,, \tag{53}$$

where $\mathscr{P}$ is an $\mathscr{N} \times (\mathscr{N} - 2)$ matrix and a, b, c, d represents the column vectors of $\mathscr{N}$-dimension.

Lemma A.2: Let ϕ_i in the Wronskian determinant fulfill the condition specified in Eq. (51a), then the following equalities [52] hold

$$\sum_{j=1}^{\mathscr{N}} \lambda_{jj}(t)\,|\,\widehat{\mathscr{N}-1}\,| = |\,\widehat{\mathscr{N}-2}, \mathscr{N}+1\,| - |\,\widehat{\mathscr{N}-3}, \mathscr{N}-1, \mathscr{N}\,|\,, \tag{54}$$

$$\begin{aligned}\left(\sum_{j=1}^{\mathscr{N}} \lambda_{jj}(t)\right)^2 |\,\widehat{\mathscr{N}-1}\,| &= |\,\widehat{\mathscr{N}-5}, \mathscr{N}-3, \mathscr{N}-2, \mathscr{N}-1, \mathscr{N}\,| - |\,\widehat{\mathscr{N}-4}, \mathscr{N}-2, \mathscr{N}-1, \mathscr{N}+1\,| \\ &- |\,\widehat{\mathscr{N}-3}, \mathscr{N}-1, \mathscr{N}+2\,| + 2\,|\,\widehat{\mathscr{N}-3}, \mathscr{N}, \mathscr{N}+1\,| + |\,\widehat{\mathscr{N}-2}, \mathscr{N}+3\,|\,.\end{aligned} \tag{55}$$

By virtue of these identities the following equality holds

$$|\,\widehat{\mathscr{N}-1}\,|\sum_{j=1}^{\mathscr{N}} \lambda_{jj}(t)\left(\sum_{j=1}^{\mathscr{N}} \lambda_{jj}(t)\,|\,\widehat{\mathscr{N}-1}\,|\right) = \left(\sum_{j=1}^{\mathscr{N}} \lambda_{jj}(t)\,|\,\widehat{\mathscr{N}-1}\,|\right)^2. \tag{56}$$

Proof: Utilizing the differential conditions Eq. (51) and the properties of the determinant, we are able to derive the following results

$$\hbar_{\mathscr{N}} = |\,\widehat{\mathscr{N}-1}\,|\,, \tag{57a}$$

$$\hbar_{\mathscr{N},x} = |\,\widehat{\mathscr{N}-2}, \mathscr{N}\,|\,, \tag{57b}$$

$$\hbar_{\mathscr{N},y} = \mathscr{A}\hbar_{\mathscr{N},x}\,, \tag{57c}$$

$$\hbar_{\mathscr{N},z} = \mathscr{B}\{|\,\widehat{\mathscr{N}-4}, \mathscr{N}-2, \mathscr{N}-1, \mathscr{N}\,| - |\,\widehat{\mathscr{N}-3}, \mathscr{N}-1, \mathscr{N}+1\,| + |\,\widehat{\mathscr{N}-2}, \mathscr{N}+2\,|\}\,, \tag{57d}$$

$$\hbar_{\mathscr{N},t} = \mathscr{C}\{|\,\widehat{\mathscr{N}-4}, \mathscr{N}-2, \mathscr{N}-1, \mathscr{N}\,| - |\,\widehat{\mathscr{N}-3}, \mathscr{N}-1, \mathscr{N}+1\,| + |\,\widehat{\mathscr{N}-2}, \mathscr{N}+2\,|\}\,, \tag{57e}$$

$$\hbar_{\mathscr{N},xx} = |\,\widehat{\mathscr{N}-3}, \mathscr{N}-1, \mathscr{N}\,| + |\,\widehat{\mathscr{N}-2}, \mathscr{N}+1\,|\,, \tag{57f}$$

$$\hbar_{\mathscr{N},xy} = \mathscr{A}\hbar_{\mathscr{N},xx}\,, \tag{57g}$$

$$\hbar_{\mathscr{N},yy} = \mathscr{A}^2\hbar_{\mathscr{N},xx}\,, \tag{57h}$$

$$\hbar_{\mathscr{N},xz} = \mathscr{B}\{|\,\widehat{\mathscr{N}-5}, \mathscr{N}-3, \mathscr{N}-2, \mathscr{N}-1, \mathscr{N}\,| - |\,\widehat{\mathscr{N}-3}, \mathscr{N}, \mathscr{N}+1\,| + |\,\widehat{\mathscr{N}-2}, \mathscr{N}+3\,|\}\,, \tag{57i}$$

$$\hbar_{\mathscr{N},yz} = \mathscr{A}\hbar_{\mathscr{N},xz}\,, \tag{57j}$$

$$\hbar_{\mathscr{N},xt} = \mathscr{C}\{|\,\widehat{\mathscr{N}-5}, \mathscr{N}-3, \mathscr{N}-2, \mathscr{N}-1, \mathscr{N}\,| - |\,\widehat{\mathscr{N}-3}, \mathscr{N}, \mathscr{N}+1\,| + |\,\widehat{\mathscr{N}-2}, \mathscr{N}+3\,|\}\,, \tag{57k}$$

$$\hbar_{\mathscr{N},yt} = \mathscr{A}\hbar_{\mathscr{N},xt}\,, \tag{57l}$$

$$\hbar_{\mathcal{N},xxx} = |\widehat{\mathcal{N}-4}, \mathcal{N}-2, \mathcal{N}-1, \mathcal{N}| + 2|\widehat{\mathcal{N}-3}, \mathcal{N}-1, \mathcal{N}+1| + |\widehat{\mathcal{N}-2}, \mathcal{N}+2|, \tag{57m}$$

$$\hbar_{\mathcal{N},xxy} = \mathscr{A}\hbar_{\mathcal{N},xxx}, \tag{57n}$$

$$\hbar_{\mathcal{N},xxxx} = |\widehat{\mathcal{N}-5}, \mathcal{N}-3, \mathcal{N}-2, \mathcal{N}-1, \mathcal{N}| + 3|\widehat{\mathcal{N}-4}, \mathcal{N}-2, \mathcal{N}-1, \mathcal{N}| + 3|\widehat{\mathcal{N}-3}, \mathcal{N}-1, \mathcal{N}+2| \tag{57o}$$

$$+2|\widehat{\mathcal{N}-3}, \mathcal{N}, \mathcal{N}+1| + |\widehat{\mathcal{N}-2}, \mathcal{N}+3|, \tag{57p}$$

$$\hbar_{\mathcal{N},xxxy} = \mathscr{A}\hbar_{\mathcal{N},xxxx}, \tag{57q}$$

Combining the bilinear form Eq. (50) with Eq. (57) and using the **Lemma A.1** and **Lemma A.2** we obtain

$$\begin{aligned}
&\hbar_{\mathcal{N}}(\hbar_{\mathcal{N},xxxx} + \hbar_{\mathcal{N},xxxy} + \gamma_1\hbar_{\mathcal{N},xx} + \gamma_2\hbar_{\mathcal{N},yy} + \alpha\hbar_{\mathcal{N},xy} + \beta\hbar_{\mathcal{N},xz} + \beta\hbar_{\mathcal{N},yz} + \hbar_{\mathcal{N},xt} + \hbar_{\mathcal{N},yt}) - 4\hbar_{\mathcal{N},x}\hbar_{\mathcal{N},xxx} \\
&\quad + 3\hbar^2_{\mathcal{N},xx} - \hbar_{\mathcal{N},xxx}\hbar_{\mathcal{N},y} - 3\hbar_{\mathcal{N},x}\hbar_{\mathcal{N},xxy} + 3\hbar_{\mathcal{N},xx}\hbar_{\mathcal{N},xy} - \gamma_1\hbar^2_{\mathcal{N},x} - \gamma_2\hbar^2_{\mathcal{N},y} - \alpha\hbar_{\mathcal{N},x}\hbar_{\mathcal{N},y} - \beta\hbar_{\mathcal{N},x}\hbar_{\mathcal{N},z} \\
&\quad - \beta\hbar_{\mathcal{N},y}\hbar_{\mathcal{N},z} - \hbar_{\mathcal{N},x}\hbar_{\mathcal{N},t} - \hbar_{\mathcal{N},y}\hbar_{\mathcal{N},t} \\
&= 12(1+\mathscr{A})[|\widehat{\mathcal{N}-3}, \mathcal{N}-2, \mathcal{N}-1||\widehat{\mathcal{N}-3}, \mathcal{N}, \mathcal{N}+1| \\
&\quad - |\widehat{\mathcal{N}-3}, \mathcal{N}-2, \mathcal{N}||\widehat{\mathcal{N}-3}, \mathcal{N}-1, \mathcal{N}+1| + |\widehat{\mathcal{N}-3}, \mathcal{N}-1, \mathcal{N}||\widehat{\mathcal{N}-3}, \mathcal{N}-2, \mathcal{N}+1|] \\
&= 0.
\end{aligned}$$

This demonstrates that $\hbar_{\mathcal{N}} = |\widehat{\mathcal{N}-1}|$ resolves the bilinear extended BK Eq. (20). In the consequence, $u = 2[\log\hbar_{\mathcal{N}}]_x$ is a solution of Eq. (3).

Reflecting the transformation

$$t' = \frac{t}{2} + \frac{z}{2\beta} \tag{58}$$

into Eq. (3) delivers

$$u_{xt'} + u_{yt'} + u_{xxxx} + u_{xxxy} + 6u_x u_{xx} + 3u_x u_{xy} + 3u_{xx}u_y + \alpha u_{xy} + \gamma_1 u_{xx} + \gamma_2 u_{yy} = 0 \tag{59}$$

and transforms Hirota bilinear form Eq. (22) into

$$(D_x D_{t'} + D_y D_{t'} + D_x^4 + D_x^3 D_y + \alpha D_x D_y + \gamma_1 D_x^2 + \gamma_2 D_y^2)\hbar\cdot\hbar = 0. \tag{60}$$

Equation (60) is the Hirota bilinear form of the (2 + 1)-dimensional combined equation (3.10) presented in [53] corresponding to the specific values $a_1 = 1, a_2 = 1, a_3 = \gamma_1, a_4 = \alpha, a_5 = \gamma_2$. By considering the polynomial $P(x, y, t') = xt' + yt' + x^4 + x^3y + \alpha xy + \gamma_1 x^2 + \gamma_2 y^2$ associated with the bilinear equation Eq. (60), it can be demonstrated that Eq. (60) satisfies Hirota's N-soliton condition [53].

Further, by employing the condition Eq. (51b) into the Wronskian determinant solution $\hbar = \hbar_{\mathcal{N}} = \mathscr{W}(\phi_1, \phi_2, \ldots, \phi_{\mathcal{N}})$, we perceive that

$$\begin{aligned}
\hbar_y &= \frac{\partial}{\partial y}\mathscr{W}(\phi_1, \phi_2, \ldots, \phi_{\mathcal{N}}) \\
&= \sum_{i=1}^{\mathcal{N}} |\phi_1, \phi_2, \ldots, \phi_{i,y} \ldots, \phi_{\mathcal{N}}| \\
&= \sum_{i=1}^{\mathcal{N}} |\phi_1, \phi_2, \ldots, \mathscr{A}\phi_{i,x} \ldots, \phi_{\mathcal{N}}| \\
&= \mathscr{A}\sum_{i=1}^{\mathcal{N}} |\phi_1, \phi_2, \ldots, \phi_{i,x} \ldots, \phi_{\mathcal{N}}| \\
&= \mathscr{A}\hbar_x.
\end{aligned} \tag{61}$$

Consequently, Eq. (60) with the help of Eq. (61) and Eq. (52) reduces to

$$(D_x D_{t'} + D_x^4)\hbar\cdot\hbar = 0, \tag{62}$$

the Hirota bilinear form of the (1 + 1)-dimensional KdV equation and the solution presented in Theorem A reduces to a well-known solution of the KdV equation [54]. A broad set of sufficient conditions for the Wronskian determinant solution of the Korteweg-de Vries equation and various type of exact solutions including rational solutions, solitons, positons, negatons, breathers, complexiton are briefly discussed in [27]. By choosing the polynomial $P(x, t') = xt' + x^4$ associated with the bilinear equation Eq. (62), it can be shown that Eq. (62) satisfies Hirota's N-soliton condition [55]. This demonstrates the validity of our obtained result.

6.1 Wronskian rational solution

In the subsequent discussion, we are going to explore lower-order rational Wronskian solutions to the extended (3 + 1)-dimensional BK equation Eq. (3) through the revealed Wronskian formulation. We are familiar if a similar transformation is used on the coefficient matrix $\mathscr{J}$,

$$\mathscr{J} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}_{\mathscr{N}\times\mathscr{N}}, \tag{63}$$

the soliton equations have identical Wronskian solutions. So for erecting Wronskian rational solution to the equation Eq. (3), we only emphasize different states of the coefficient matrix $\mathscr{J}$. We denote the Wronskian solution $u = 2[\log \mathscr{W}(\phi_1, \phi_2, \ldots, \phi_{\mathscr{N}})]_x$ corresponding to each $\hbar \geq 1$, in relation to the Jordan block $\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}_{\|\times\|}$ as a rational Wronskian solution of order $\hbar - 1$. To attain Wronskian rational solutions for Eq. (3), we presume $\lambda_i = 0$ in Eq. (51) which yields the condition

$$\phi_{1,xx} = 0, \quad \phi_{i+1,xx} = \phi_i, \quad \phi_{i,y} = \mathscr{A}\phi_{i,x}, \quad \phi_{i,z} = \mathscr{B}\phi_{i,xxx}, \quad \phi_{i,t} = \mathscr{C}\phi_{i,xxx}, \quad i \geq 1. \tag{64}$$

In the subsequent discussion, we will reveal several rational Wronskian solutions of lower order for the generalized (3+1)-dimensional BK equation Eq. (3).

6.1.1 Zero order

Taking $\phi_1 = c_1(x+\mathscr{A}y)+c_2$, we obtain associated Wronskian determinant as $\hbar = \mathscr{W}(\phi_1) = c_1(x+\mathscr{A}y)+c_2$ and the corresponding rational solution of Eq. (3) as

$$u = 2[\log \mathscr{W}(\phi_1)]_x = \frac{2c_1}{c_1(x+\mathscr{A}y)+c_2}. \tag{65}$$

More specifically, considering $\phi_1 = x + \mathscr{A}y$ produces Wronskian solution of Eq. (3) as $u = \frac{2}{x+\mathscr{A}y}$.

6.1.2 First order

Setting $\phi_1 = 1$, a direct computation leads to $\phi_2 = \frac{x^2}{2} + \mathscr{A}xy + \frac{\mathscr{A}^2y^2}{2}$. Then the associated first order Wronskian determinant is $\hbar = \mathscr{W}(\phi_1, \phi_2) = x + \mathscr{A}y$ and the corresponding first order Wronskian rational solution of Eq. (3) is given by

$$u = 2[\log(\mathscr{W}(\phi_1, \phi_2)]_x = \frac{2}{x+\mathscr{A}y}. \tag{66}$$

Next taking $\phi_1 = x + \mathscr{A}y$, we arrive at $\phi_2 = \frac{x^3}{6} + \frac{\mathscr{A}x^2y}{2} + \frac{\mathscr{A}^2xy^2}{2} + \frac{\mathscr{A}^3y^3}{6} + \mathscr{B}z + \mathscr{C}t$. Therefore, the corresponding first order Wronskian determinant and it′s associated Wronskian rational solution of Eq. (3) is presented respectively by $\hbar = \mathscr{W}(\phi_1, \phi_2) = \frac{x^3}{3} + \mathscr{A}x^2y + \mathscr{A}^2xy^2 + \frac{\mathscr{A}^3y^3}{2} - \mathscr{B}z - \mathscr{C}t$ and

$$u = 2[\log \mathscr{W}(\phi_1, \phi_2)]_x = \frac{2(x^2 + 2\mathscr{A}xy + \mathscr{A}^2y^2)}{\frac{x^3}{3} + \mathscr{A}x^2y + \mathscr{A}^2xy^2 + \frac{\mathscr{A}^3y^3}{2} - \mathscr{B}z - \mathscr{C}t}. \tag{67}$$

6.1.3 Second order

As the choice of $\phi_1 = 1$ yields $\phi_2 = \frac{x^2}{2} + \mathscr{A}xy + \frac{\mathscr{A}^2y^2}{2}$ and by a direct computation gives $\phi_3 = \frac{x^4}{24} + \frac{\mathscr{A}x^3y}{6} + \frac{\mathscr{A}^2x^2y^2}{4} + \mathscr{B}xz + \mathscr{C}xt$, the second order Wronskian determinant $\hbar = \mathscr{W}(\phi_1, \phi_2, \phi_3) = \frac{x^3}{3} + \mathscr{A}x^2y + \mathscr{A}^2xy^2 + \frac{\mathscr{A}^3y^3}{2} - \mathscr{B}z - \mathscr{C}t$ and associated second order Wronskian rational solution is expressed as

$$u = 2[\log \mathscr{W}(\phi_1, \phi_2, \phi_3)]_x = \frac{2(x^2 + 2\mathscr{A}xy + \mathscr{A}^2y^2)}{\frac{x^3}{3} + \mathscr{A}x^2y + \mathscr{A}^2xy^2 + \frac{\mathscr{A}^3y^3}{2} - \mathscr{B}z - \mathscr{C}t}. \tag{68}$$

Moreover, the choice of $\phi_1 = x + \mathscr{A} y$ delivers $\phi_2 = \frac{x^3}{6} + \frac{\mathscr{A} x^2 y}{2} + \frac{\mathscr{A}^2 x y^2}{2} + \frac{\mathscr{A}^3 y^3}{6} + \mathscr{B} z + \mathscr{C} t$ which offers $\phi_3 = \frac{x^5}{120} + \frac{\mathscr{A}^5 y^5}{120} + \frac{\mathscr{A} x^4 y}{24} + \frac{\mathscr{A}^4 x y^4}{24} + \frac{\mathscr{A}^2 x^3 y^2}{12} + \frac{\mathscr{A}^3 x^2 y^3}{12} + \frac{\mathscr{A}^2 \mathscr{B} y^2 z}{2} + \frac{\mathscr{A}^2 \mathscr{C} y^2 t}{2} + \frac{\mathscr{B} x^2 z}{2} + \frac{\mathscr{C} x^2 t}{2} + \mathscr{A}\mathscr{B} xyz + \mathscr{A}\mathscr{C} xyt$. This offers the third order Wronskian determinant $\hbar = \mathscr{W}(\phi_1, \phi_2, \phi_3) = \frac{x^6}{45} + \frac{\mathscr{A}^6 y^6}{45} + \frac{2\mathscr{A} x^5 y}{15} + \frac{2\mathscr{A}^5 x y^5}{15} + \frac{\mathscr{A}^2 x^4 y^2}{3} + \frac{\mathscr{A}^4 x^2 y4}{3} + \frac{4\mathscr{A}^3 x^3 y^3}{9} - \frac{\mathscr{B} x^3 z}{3} - \frac{\mathscr{C} x^3 t}{3} - \frac{\mathscr{A}^3 \mathscr{B} y^3 z}{3} - \frac{\mathscr{A}^3 \mathscr{C} y^3 t}{3} - \mathscr{B}^2 z^2 - \mathscr{C}^2 t^2 - \mathscr{A}\mathscr{B} x^2 yz - \mathscr{A}\mathscr{C} x^2 yt - \mathscr{A}^2 \mathscr{B} x y^2 z - \mathscr{A}^2 \mathscr{C} x y^2 t - 2\mathscr{B}\mathscr{C} zt$ and corresponding third order Wronskian rational solution

$$\mathscr{u} = 2[\log \mathscr{W}(\phi_1, \phi_2, \phi_3)]_x = \frac{2}{\hbar_3}\Bigg[\frac{2x^5}{15} + \frac{2\mathscr{A} x^4 y}{3} + \frac{2\mathscr{A}^5 y^5}{15} + \frac{4\mathscr{A}^2 x^3 y^2}{3} + \frac{2\mathscr{A}^4 x y^4}{3} + \frac{4\mathscr{A}^3 x^2 y^3}{3} \tag{69}$$

$$-\mathscr{B} x^2 z - \mathscr{C} x^2 t - 2\mathscr{A}\mathscr{B} xyz - 2\mathscr{A}\mathscr{C} xyt - \mathscr{A}^2 \mathscr{B} y^2 z - \mathscr{A}^2 \mathscr{C} y^2 t\Bigg]. \tag{70}$$

6.2 Soliton solutions

To construct soliton solutions, it is imperative to select precise forms of $\phi_i' s$ within $\mathscr{N}^{th}$-order analytic solutions. Considering the special form of $\lambda_{ij} = \Im_i$ in Eq. (51), where $\Im_i$'s are real constants,

$$\phi_{i,xx} = \Im_i^2 \phi_i, \quad \phi_{i,y} = \mathscr{A}\phi_{i,x}, \quad \phi_{i,z} = \mathscr{B}\phi_{i,xxx}, \quad \phi_{i,t} = \mathscr{C}\phi_{i,xxx}, \quad 1 \le i \le \mathscr{N}. \tag{71}$$

These partial differential conditions concede the solutions such as

$$\phi_i = \exp^{\frac{\vartheta_i}{2}} + (-1)^{i+1}\exp^{-\frac{\vartheta_i}{2}}, \quad \vartheta_i = \Im_i x + \mathscr{A}\Im_i y + \mathscr{B}\Im_i^3 z + \mathscr{C}\Im_i^3 t, \quad 1 \le i \le \mathscr{N} \tag{72}$$

and simply written as

$$\phi_i = \cosh(\Im_i x + \mathscr{A}\Im_i y + \mathscr{B}\Im_i^3 z + \mathscr{C}\Im_i^3 t), \quad \text{i} \quad \text{odd}; \tag{73a}$$

$$\phi_i = \sinh(\Im_i x + \mathscr{A}\Im_i y + \mathscr{B}\Im_i^3 z + \mathscr{C}\Im_i^3 t), \quad \text{i} \quad \text{even}. \tag{73b}$$

6.2.1 One-soliton solution

For $i = 1$, we obtain the Wronskian determinant as $\hbar = \mathscr{W}(\phi_1) = cosh(\Im_1 x + \mathscr{A}\Im_1 y + \mathscr{B}\Im_1^3 z + \mathscr{C}\Im_1^3 t)$ and related one soliton solution of Eq. (3)

$$\mathscr{u} = 2[\log \mathscr{W}(\phi_1)]_x = 2\Im_1 \tanh(\Im_1 x + \mathscr{A}\Im_1 y + \mathscr{B}\Im_1^3 z + \mathscr{C}\Im_1^3 t). \tag{74}$$

6.2.2 Two-soliton solution

$i = 2$ provides the Wronskian determinant $\hbar = \mathscr{W}(\phi_1, \phi_2)$, in presence of $\phi_1 = \cosh(\Im_1 x + \mathscr{A}\Im_1 y + \mathscr{B}\Im_1^3 z + \mathscr{C}\Im_1^3 t)$, $\phi_2 = \sinh(\Im_2 x + \mathscr{A}\Im_2 y + \mathscr{B}\Im_2^3 z + \mathscr{C}\Im_2^3 t)$ and corresponding two soliton solution of Eq. (3)

$$\mathscr{u} = 2[\log(\mathscr{W}(\phi_1, \phi_2))]_x. \tag{75}$$

6.2.3 Three-soliton solution

In case of $i = 3$, $\phi_1 = \cosh(\Im_1 x + \mathscr{A}\Im_1 y + \mathscr{B}\Im_1^3 z + \mathscr{C}\Im_1^3 t)$, $\phi_2 = \sinh(\Im_2 x + \mathscr{A}\Im_2 y + \mathscr{B}\Im_2^3 z + \mathscr{C}\Im_2^3 t)$ and $\phi_3 = \cosh(\Im_3 x + \mathscr{A}\Im_3 y + \mathscr{B}\Im_3^3 z + \mathscr{C}\Im_3^3 t)$, delivers the three soliton solution of Eq. (3)

$$\mathscr{u} = 2[\log(\mathscr{W}(\phi_1, \phi_2, \phi_3)]_x \tag{76}$$

accompanied by the Wronskian determinant $\hbar = \mathscr{W}(\phi_1, \phi_2, \phi_3)$.

7 Grammian solution

In this section, we present a Grammian condition for Eq. (3) and derive its $\mathscr{N}$-soliton solution in terms of a Grammian determinant. For that, at first we define Grammian determinant[9] as follows

$$\hbar_{\mathscr{N}} = \det(\mathscr{a}_{ij}), \quad 1 \le i, j \le \mathscr{N}, \quad \mathscr{a}_{ij} = \mathscr{c}_{ij} + \int^x \mathscr{r}_i \mathscr{s}_j \, dx,$$

$$\mathscr{r}_i = \mathscr{r}(x, y, z, t), \quad \mathscr{s}_i = \mathscr{s}(x, y, z, t) \quad \text{and} \quad \mathscr{c}_{ij} = \text{constant}. \tag{77}$$

A sufficient condition for the existence of the Grammian solution is given below

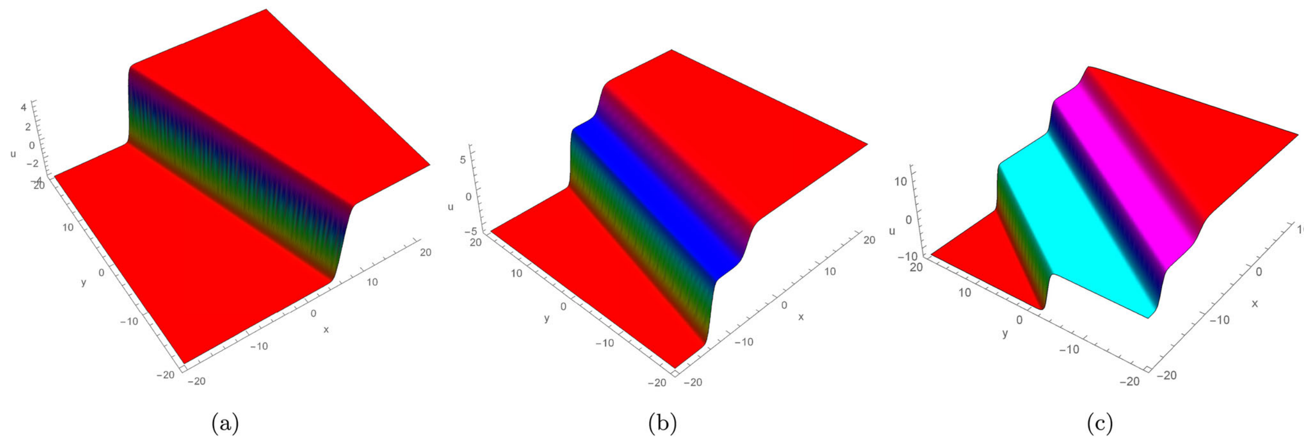


Fig. 4 **a** Wronskian one- soliton Eq. (74), **b** Wronskian two- soliton Eq. (75) and **c** Wronskian three- soliton Eq. (76)

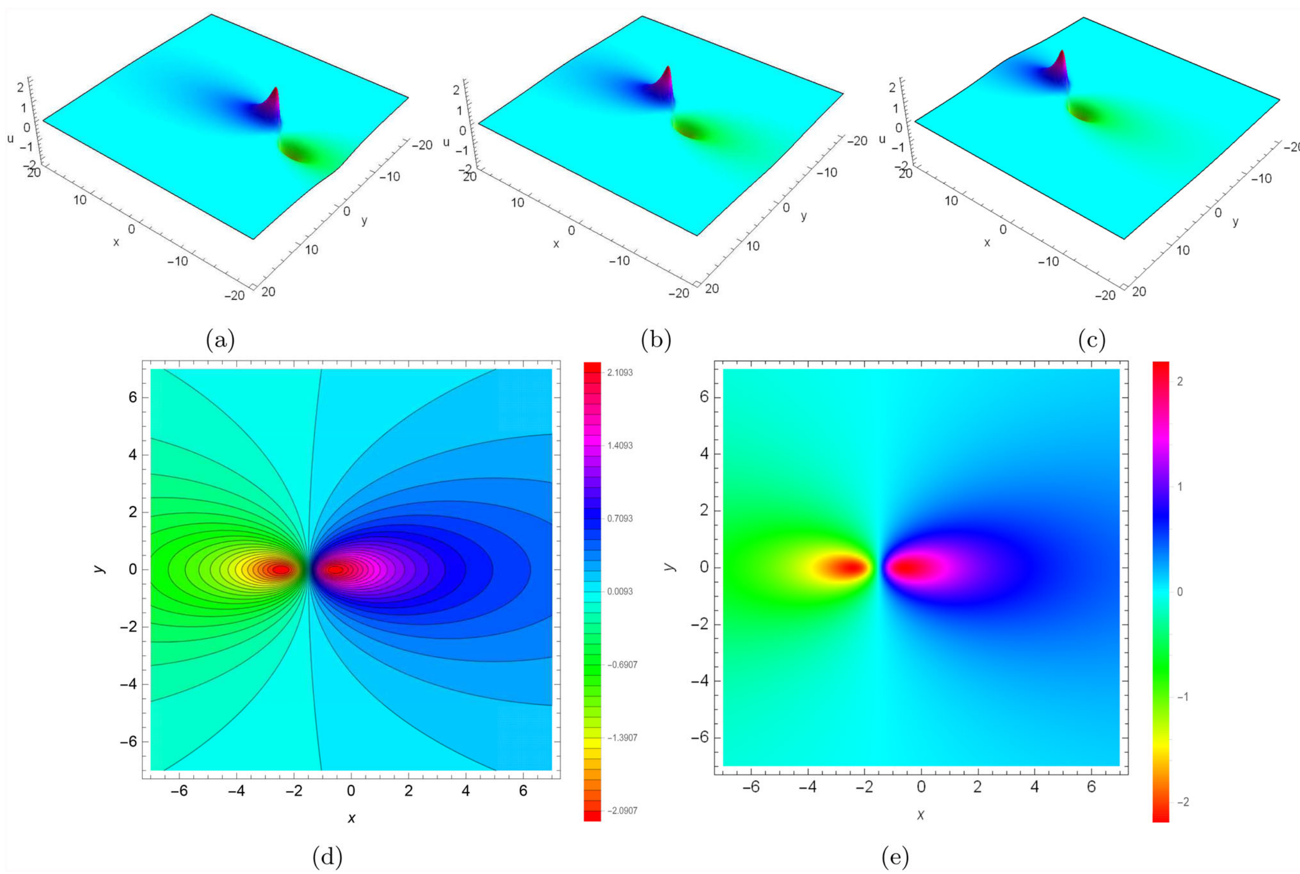


Fig. 5 Lump solution of Eq. (3) at different time frames **a** $t = -1.5$, **b** $t = 0$, **c** $t = 1.5$, **d** contour plot, **e** density plot

Theorem B: If both $\mathscr{r}(x, y, z, t)$ and $\mathscr{s}(x, y, z, t)$ fulfil the conditions mentioned below

$$\mathscr{r}_{i,y} = \mathscr{A}\mathscr{r}_{i,x}, \quad \mathscr{r}_{i,z} = \mathscr{B}\mathscr{r}_{i,xxx}, \quad \mathscr{r}_{i,t} = \mathscr{C}\mathscr{r}_{i,xxx}, \; 1 \leq i \leq \mathscr{N}, \tag{78a}$$

$$\mathscr{s}_{j,y} = \mathscr{A}\mathscr{s}_{j,x}, \quad \mathscr{s}_{j,z} = \mathscr{B}\mathscr{s}_{j,xxx}, \quad \mathscr{s}_{j,t} = \mathscr{C}\mathscr{s}_{j,xxx}, \; 1 \leq j \leq \mathscr{N}, \tag{78b}$$

where

$$\mathscr{A} = \left(\frac{-\alpha \pm \sqrt{\alpha^2 - 4\gamma_1\gamma_2}}{2\gamma_2}\right), \; \mathscr{C} = -4 - \beta\mathscr{B}, \; 2\gamma_2 - \alpha \pm \sqrt{\alpha^2 - 4\gamma_1\gamma_2} \neq 0 \tag{79}$$

and $\mathscr{B}$ is free parameter, then the Grammian determinant Eq. (77) provides a solution for the bilinear form Eq. (50) and solution for Eq. (3) is obtainable as $u = 2(\log h_{\mathscr{N}})_x$.

Proof For simplicity, at first we recast $h_{\mathscr{N}}$ as a Pfaffian in which i and j are integers and to distinguish j and j^*, we employ the superscript $*$

$$h_{\mathscr{N}} = (1, 2, \ldots, \mathscr{N}, \mathscr{N}^*, \ldots, 2^*, 1^*) = (\bullet), \tag{80}$$

$$(i, j^*) = a_{ij} = c_{ij} + \int^x r_i s_j \, dx, \quad (i, j) = (i^*, j^*) = 0, \tag{81}$$

in conjunction with the Pfaffian entries defined by

$$(d_n, j^*) = \frac{\partial^n s}{\partial x^n}, \ (d_m, d_n^*) = 0, \ (d_n^*, i) = \frac{\partial^n r}{\partial x^n}, \ (d_n, i) = (d_m^*, j^*) = 0, \tag{82}$$

where d_n and d_m are Pfaffian characters and the derivatives of a_{ij} are articulated as

$$\frac{\partial}{\partial x} a_{ij} = r_i s_j = (d_0, d_0^*, i, j^*), \tag{83}$$

$$\frac{\partial}{\partial y} a_{ij} = \mathscr{A} \int^x (r_{i,x} s_j + r_i s_{j,x}) dx = \mathscr{A} r_i s_j = \mathscr{A}(d_0, d_0^*, i, j^*), \tag{84}$$

$$\begin{aligned} \frac{\partial}{\partial z} a_{ij} &= \mathscr{B} \int^x (r_{i,xxx} s_j + r_i s_{j,xxx}) dx = \mathscr{B}(r_{i,xx} s_j - r_{i,x} s_{j,x} + r_i s_{j,xx}) \\ &= \mathscr{B}[(d_2, d_0^*, i, j^*) - (d_1, d_1^*, i, j^*) + (d_0, d_2^*, i, j^*)], \end{aligned} \tag{85}$$

$$\begin{aligned} \frac{\partial}{\partial t} a_{ij} &= \mathscr{C} \int^x (r_{i,xxx} s_j + r_i s_{j,xxx}) dx = \mathscr{C}(r_{i,xx} s_j - r_{i,x} s_{j,x} + r_i s_{j,xx}) \\ &= \mathscr{C}[(d_2, d_0^*, i, j^*) - (d_1, d_1^*, i, j^*) + (d_0, d_2^*, i, j^*)]. \end{aligned} \tag{86}$$

To begin the proof, we will consider some lemmas as follows. □

Lemma B.1: For Grammian determinant, the Jacobi relation [9] defined as follows

$$(d_1, d_1^*, d_0, d_0^*, \bullet)(\bullet) - (d_1, d_1^*, \bullet)(d_0, d_0^*, \bullet) + (d_1, d_0^*, \bullet)(d_0, d_1^*, \bullet) = 0. \tag{87}$$

Lemma B.2: By leveraging the properties of determinants, we can easily derive the following identity

$$2(d_1, d_1^*, d_0, d_0^*, \bullet)(\bullet) + (d_3, d_0^*, \bullet)(\bullet) + (d_0, d_3^*, \bullet)(\bullet) - (d_1, d_2^*, \bullet)(\bullet) - (d_2, d_1^*, \bullet)(\bullet) = [(d_1, d_0^*, \bullet) - (d_0, d_1^*, \bullet)]^2. \tag{88}$$

Computing various states derivatives of the Grammian determinant $h_{\mathscr{N}} = a_{ij}$ regarding the variables x, y, z, t as listed below

$$h_{\mathscr{N}} = (\bullet), \tag{89a}$$

$$h_{\mathscr{N},x} = (d_0, d_0^*, \bullet), \tag{89b}$$

$$h_{\mathscr{N},y} = \mathscr{A} h_{\mathscr{N},x}, \tag{89c}$$

$$h_{\mathscr{N},z} = \mathscr{B}[(d_0, d_2^*, \bullet) - (d_1, d_1^*, \bullet) + (d_2, d_0^*, \bullet)], \tag{89d}$$

$$h_{\mathscr{N},t} = \mathscr{C}[(d_0, d_2^*, \bullet) - (d_1, d_1^*, \bullet) + (d_2, d_0^*, \bullet)], \tag{89e}$$

$$h_{\mathscr{N},xx} = (d_0, d_1^*, \bullet) + (d_1, d_0^*, \bullet), \tag{89f}$$

$$h_{\mathscr{N},xy} = \mathscr{A} h_{\mathscr{N},xx}, \tag{89g}$$

$$h_{\mathscr{N},yy} = \mathscr{A}^2 h_{\mathscr{N},xx}, \tag{89h}$$

$$h_{\mathscr{N},xz} = \mathscr{B}[(d_0, d_3^*, \bullet) - (d_1, d_1^*, d_0, d_0^*, \bullet) + (d_3, d_0^*, \bullet)], \tag{89i}$$

$$h_{\mathscr{N},yz} = \mathscr{A} h_{\mathscr{N},xz}, \tag{89j}$$

$$h_{\mathscr{N},xt} = \mathscr{C}[(d_0, d_3^*, \bullet) - (d_1, d_1^*, d_0, d_0^*, \bullet) + (d_3, d_0^*, \bullet)], \tag{89k}$$

$$h_{\mathscr{N},yt} = \mathscr{A} h_{\mathscr{N},xt}, \tag{89l}$$

$$h_{\mathscr{N},xxx} = (d_0, d_2^*, \bullet) + 2(d_1, d_1^*, \bullet) + (d_2, d_0^*, \bullet), \tag{89m}$$

$$\hbar_{\mathcal{N},xxy} = \mathcal{A}\hbar_{\mathcal{N},xxx}, \tag{89n}$$

$$\hbar_{\mathcal{N},xxxx} = (d_0, d_3^*, \bullet) + 2(d_1, d_1^*, d_0, d_0^*, \bullet) + 3(d_1, d_2^*, \bullet) + 3(d_2, d_1^*, \bullet) + (d_3, d_0^*, \bullet), \tag{89o}$$

$$\hbar_{\mathcal{N},xxxy} = \mathcal{A}\hbar_{\mathcal{N},xxxx}. \tag{89p}$$

By combining Eqs. (89) and (50), along with the support of the **Lemma B.1** and **Lemma B.2**,

$$\begin{aligned}
&\hbar_{\mathcal{N}}(\hbar_{\mathcal{N},xxxx} + \hbar_{\mathcal{N},xxxy} + \gamma_1\hbar_{\mathcal{N},xx} + \gamma_2\hbar_{\mathcal{N},yy} + \alpha\hbar_{\mathcal{N},xy} + \beta\hbar_{\mathcal{N},xz} + \beta\hbar_{\mathcal{N},yz} + \hbar_{\mathcal{N},xt} + \hbar_{\mathcal{N},yt}) - 4\hbar_{\mathcal{N},x}\hbar_{\mathcal{N},xxx}\\
&\quad + 3\hbar^2_{\mathcal{N},xx} - \hbar_{\mathcal{N},xxx}\hbar_{\mathcal{N},y} - 3\hbar_{\mathcal{N},x}\hbar_{\mathcal{N},xxy} + 3\hbar_{\mathcal{N},xx}\hbar_{\mathcal{N},xy} - \gamma_1\hbar^2_{\mathcal{N},x} - \gamma_2\hbar^2_{\mathcal{N},y} - \alpha\hbar_{\mathcal{N},x}\hbar_{\mathcal{N},y} - \beta\hbar_{\mathcal{N},x}\hbar_{\mathcal{N},z}\\
&\quad - \beta\hbar_{\mathcal{N},y}\hbar_{\mathcal{N},z} - \hbar_{\mathcal{N},x}\hbar_{\mathcal{N},t} - \hbar_{\mathcal{N},y}\hbar_{\mathcal{N},t}\\
&= 12(1+\mathcal{A})[(d_1, d_1^*, d_0, d_0^*, \bullet)(\bullet) - (d_1, d_1^*, \bullet)(d_0, d_0^*, \bullet) + (d_1, d_0^*, \bullet)(d_0, d_1^*, \bullet)]\\
&= 0.
\end{aligned}$$

This indicates the function $h_{\mathcal{N}}$ given by Eq. (77) is a Grammian determinant solution for the bilinear equation Eq. (50) and consequently, $u = 2[\log(1, 2, \ldots, \mathcal{N}, \mathcal{N}^*, \ldots, 2^*, 1^*)]_x$ is a solution of Eq. (3).

Upon incorporating the condition Eq. (84) into the solution for the Grammian determinant $\hbar = \hbar_{\mathcal{N}} = \det(a_{ij})$, $1 \le i, j \le \mathcal{N}$, we can discern that

$$\begin{aligned}
\hbar_y &= \frac{\partial}{\partial y}\det(a_{ij}) = \sum_{1\le i,j\le\mathcal{N}} \frac{\partial}{\partial y}a_{ij}\Delta_{ij}, \ \ \Delta_{ij} \text{ is the cofactor of } a_{ij}\\
&= \sum_{1\le i,j\le\mathcal{N}} \mathcal{A}r_i s_j\Delta_{ij} = \mathcal{A}\sum_{1\le i,j\le\mathcal{N}} r_i s_j\Delta_{ij} = \mathcal{A}\sum_{1\le i,j\le\mathcal{N}} \frac{\partial}{\partial x}a_{ij}\Delta_{ij} = \mathcal{A}\frac{\partial}{\partial x}\det(a_{ij})\\
&= \mathcal{A}\hbar_x
\end{aligned} \tag{90}$$

Utilizing the relation Eq. (90), along with Eq. (79) in previously obtained Hirota bilinear form of (2+1)-dimensional combined Eq. (60), we obtain

$$(D_x D_{t'} + D_x^4)\hbar\cdot\hbar = 0, \tag{91}$$

the Hirota bilinear form of the (1 + 1)-dimensional KdV equation.

8 Lump solution

To determine the lump solution [39, 51] of Eq. (3), we consider the test function in the following quadratic form

$$\hbar = h_1^2 + h_2^2 + m_1, \tag{92}$$

where

$$h_1 = p_1x + p_2y + p_3z + p_4t + p_5, \tag{93a}$$

$$h_2 = p_6x + p_7y + p_8z + p_9t + p_{10}, \tag{93b}$$

where m_1, p_i, $(i = 1, 2, \ldots, 10)$ are arbitrary constants that to be calculated later and the constants $p_i (i = 1, 2, 3, 6, 7, 8)$ satisfy the condition $(p_1, p_2, p_3) \nparallel (p_6, p_7, p_8)$. By setting $\gamma_1 = \gamma_2$ and substituting Eq. (92) into Eq. (22), then equating coefficients of different powers of $x^a y^b z^c t^d$ to zero, we form a system of equations. Solving this system yields the following constraint between the parameters.

$$\begin{aligned}
&p_2 = -\frac{p_6p_7}{p_1},\ p_3 = \frac{p_1\{\beta p_8(p_1^2+p_7^2) + p_7(\alpha-\gamma_1)(p_1^2+p_6^2)\}}{\beta p_6(p_1^2+p_7^2)},\ p_4 = \frac{p_1\{\beta p_8(p_1^2+p_7^2) + (\alpha-\gamma_1)p_1^2p_7 + p_6(\alpha p_7^2+\gamma_1p_1^2)\}}{p_6(p_1^2+p_7^2)},\\
&p_5 = \frac{p_1p_{10}}{p_6},\ p_9 = \frac{p_4p_6}{p_1},\ p_{11} = \frac{(p_1^2+p_6^2)(p_1^2+p_7^2)}{(\alpha-\gamma_1)p_7^2}, \alpha \ne \gamma_1, p_7 \ne 0.
\end{aligned} \tag{94}$$

Finally, we utilize Eq. (92) and Eq. (94) to replace Eq. (20) in order to obtain the lump solution of Eq. (3).

Figure 5 depicts the propagation of the lump wave through various visual representations, including 3D plot, contour plot and density plot, considering the parameter values $\alpha = 3$, $\beta = 1$, $\gamma_1 = \gamma_2 = 1.5$, $p_1 = 2$, $p_6 = 1$, $p_7 = 4$, $p_8 = 1.5$, $p_{10} = 1.5$.

9 Lump-multi-kink solutions

We have utilized the following two sets of test functions to compute lump-multi-kink solutions for Eq. (3).

$$(i)\ \hbar = h_1^2 + h_2^2 + d_1 + \sum_{i=1}^{n} e^{\chi_i}\,, \tag{95}$$

$$(ii)\ \hbar = h_1^2 + h_2^2 + d_1 + \sum_{i=1}^{n} \cosh \chi_i\,. \tag{96}$$

The following relations between the parameters can be obtained by substituting Eq. (95) into Eq. (22).

$$p_2 = -\frac{p_6 p_7}{p_1},\ p_3 = \frac{p_1\{\beta p_8(p_1^2+p_7^2) + p_7(\alpha-\gamma_1)(p_1^2+p_6^2)\}}{\beta p_6(p_1^2+p_7^2)},\ p_4 = \frac{p_1\{\beta p_8(p_1^2+p_7^2) + (\alpha-\gamma_1)p_1^2 p_7 + p_6(\alpha p_7^2+\gamma_1 p_1^2)\}}{p_6(p_1^2+p_7^2)},$$
$$p_5 = \frac{p_1 p_{10}}{p_6},\ p_9 = \frac{p_4 p_6}{p_1},\ p_{11} = \frac{(p_1^2+p_6^2)(p_1^2+p_7^2)}{(\alpha-\gamma_1)p_7^2},\ \omega_i = -\frac{a_2^2 + a_2^2 b_2 + \alpha b_2 + \beta(b_2+c_2) + \gamma_1 + \gamma_2 b_2^2}{1+b_2},\ (i=1,2,\ldots,n),$$
$$\alpha \neq \gamma_1,\ p_7 \neq 0. \tag{97}$$

Substituting Eq. (95) and Eq. (97) into Eq. (20) produces the lump-multi-kink solution of Eq. (3) as

$$u = 2\left[\frac{2p_1h_1 + 2p_6h_2 + \sum_{i=1}^{n} a_i e^{\chi_i}}{h_1^2 + h_2^2 + d_1 + \sum_{i=1}^{n} e^{\chi_i}}\right], \tag{98}$$

with

$$h_1 = p_1 x - \left(\frac{p_6 p_7}{p_1}\right)y + \left[\frac{p_1\{\beta p_8(p_1^2+p_7^2) + p_7(\alpha-\gamma_1)(p_1^2+p_6^2)\}}{\beta p_6(p_1^2+p_7^2)}\right]z$$
$$+ \left[\frac{p_1\{\beta p_8(p_1^2+p_7^2) + (\alpha-\gamma_1)p_1^2 p_7 + p_6(\alpha p_7^2+\gamma_1 p_1^2)\}}{p_6(p_1^2+p_7^2)}\right]t + \frac{p_1 p_{10}}{p_6}, \tag{99a}$$

$$h_2 = p_6 x + p_7 y + p_8 z + \left(\frac{p_4 p_6}{p_1}\right)t + p_{10}, \tag{99b}$$

$$\chi_i = a_i\left\{x + b_i y + c_i z - \frac{a_i^2 + a_i^2 b_i + \alpha b_i + \beta(b_i+c_i) + \gamma_1 + \gamma_2 b_i^2}{(1+b_i)}t\right\} + \chi_i^0,\ (i=1,2,\ldots,n). \tag{99c}$$

In similar way, we substitute Eq. (96) into Eq. (22) and obtain the same relation between the parameters as Eq. (97) and the corresponding lump-multi-kink solution of Eq. (3) can be obtained as

$$u = 2\left[\frac{2p_1h_1 + 2p_6h_2 + \sum_{i=1}^{n} a_i \sinh \chi_i}{h_1^2 + h_2^2 + d_1 + \sum_{i=1}^{n} \cosh \chi_i}\right], \tag{100}$$

with

$$h_1 = p_1 x - \left(\frac{p_6 p_7}{p_1}\right)y + \left[\frac{p_1\{\beta p_8(p_1^2+p_7^2) + p_7(\alpha-\gamma_1)(p_1^2+p_6^2)\}}{\beta p_6(p_1^2+p_7^2)}\right]z$$
$$+ \left[\frac{p_1\{\beta p_8(p_1^2+p_7^2) + (\alpha-\gamma_1)p_1^2 p_7 + p_6(\alpha p_7^2+\gamma_1 p_1^2)\}}{p_6(p_1^2+p_7^2)}\right]t + \frac{p_1 p_{10}}{p_6}\,, \tag{101a}$$

$$h_2 = p_6 x + p_7 y + p_8 z + \left(\frac{p_4 p_6}{p_1}\right)t + p_{10}\,, \tag{101b}$$

$$\chi_i = a_i\left\{x + b_i y + c_i z - \frac{a_i^2 + a_i^2 b_i + \alpha b_i + \beta(b_i+c_i) + \gamma_1 + \gamma_2 b_i^2}{(1+b_i)}t\right\} + \chi_i^0,\ (i=1,2,\ldots,n)\,. \tag{101c}$$

9.1 Lump-multi-kink solutions using test function I

9.1.1 Lump-one-kink solution

Lump-one-kink solution can be obtained by choosing $n = 1$ in Eq. (98) as

$$u = 2\left[\frac{2p_1h_1 + 2p_6h_2 + a_1 e^{\chi_1}}{h_1^2 + h_2^2 + d_1 + e^{\chi_1}}\right], \tag{102}$$

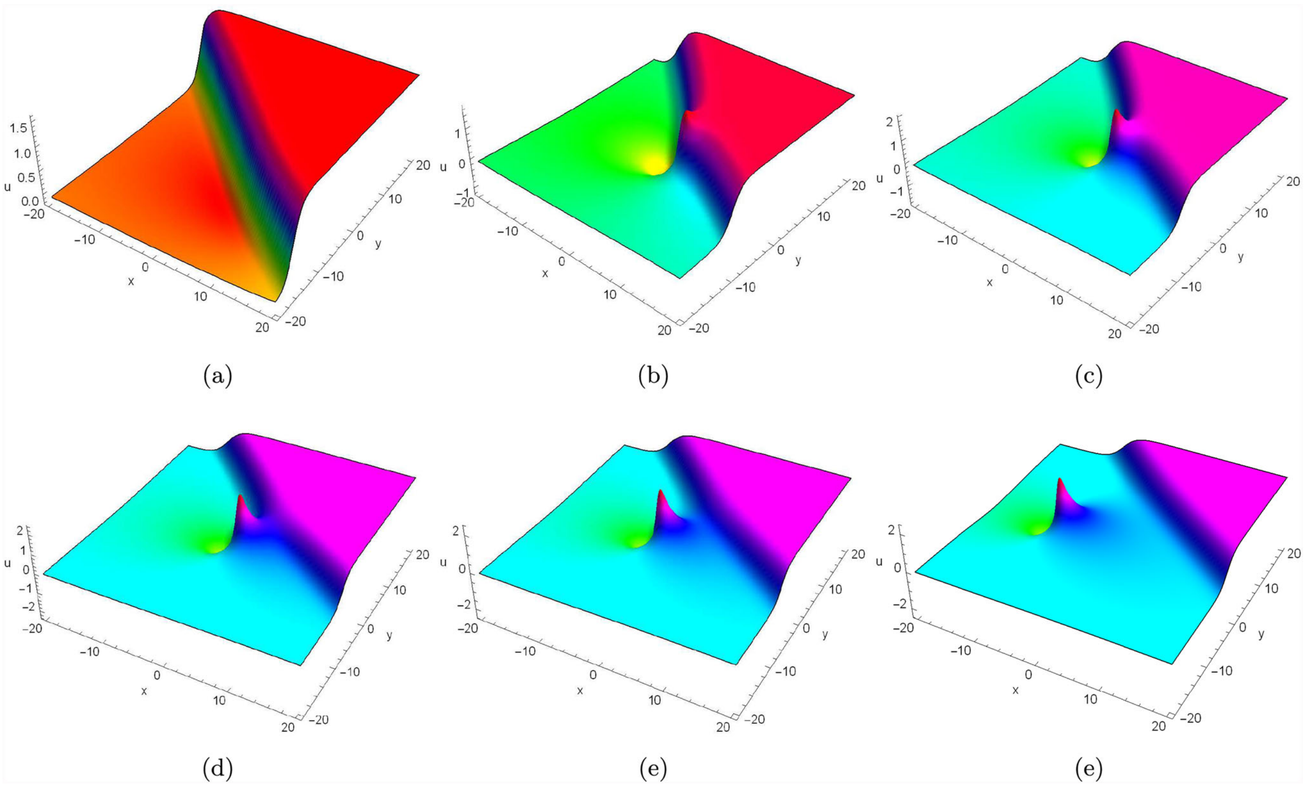


Fig. 6 Progression of the lump-one-kink solution for Eq. (3) at various time points: **a** $t = -2.5$, **b** $t = -0.5$, **c** $t = -0.1$, **d** $t = 0.4$, **e** $t = 1$, **f** 2.5

with

$$h_1 = p_1 x - \left(\frac{p_6 p_7}{p_1}\right) y + \left[\frac{p_1\{\beta p_8(p_1^2 + p_7^2) + p_7(\alpha - \gamma_1)(p_1^2 + p_6^2)\}}{\beta p_6(p_1^2 + p_7^2)}\right] z + \left[\frac{p_1\{\beta p_8(p_1^2 + p_7^2) + (\alpha - \gamma_1)p_1^2 p_7 + p_6(\alpha p_7^2 + \gamma_1 p_1^2)\}}{p_6(p_1^2 + p_7^2)}\right] t + \frac{p_1 p_{10}}{p_6}, \tag{103a}$$

$$h_2 = p_6 x + p_7 y + p_8 z + \left(\frac{p_4 p_6}{p_1}\right) t + p_{10}, \tag{103b}$$

$$\chi_1 = a_1 \left\{ x + b_1 y + c_1 z - \frac{a_1^2 + a_1^2 b_1 + \alpha b_1 + \beta(b_1 + c_1) + \gamma_1 + \gamma_2 b_1^2}{(1 + b_1)} t \right\} + \chi_1^0. \tag{103c}$$

Figure 6 showcases the interaction dynamics between a lump wave and a kink wave, describing their evolutionary behavior across various time frames for the parametric values as $\alpha = \beta = 1$, $\gamma_1 = \gamma_2 = -1.5$, $a_1 = 0.8$, $b_1 = 1.2$, $c_1 = 1.3$, $p_1 = -2.5$, $p_6 = -1$, $p_7 = 3$, $p_8 = 1$, $p_1 0 = 1.5$, $\chi_1^0 = 0$. At $t = -2.5$, the solution features a kink wave. By $t = 0$, a lump wave begins to emerge from the kink. As time progresses to $t = 1$, the lump wave fully detaches from the kink and propagates independently.

9.1.2 Lump-two-kink solution

Lump-two-kink solution can be obtained by choosing $n = 2$ in Eq. (98) as

$$u = 2\left[\frac{2p_1 h_1 + 2p_6 h_2 + a_1 e^{\chi_1} + a_2 e^{\chi_2}}{h_1^2 + h_2^2 + d_1 + e^{\chi_1} + e^{\chi_1}}\right], \tag{104}$$

with

$$h_1 = p_1 x - \left(\frac{p_6 p_7}{p_1}\right) y + \left[\frac{p_1\{\beta p_8(p_1^2 + p_7^2) + p_7(\alpha - \gamma_1)(p_1^2 + p_6^2)\}}{\beta p_6(p_1^2 + p_7^2)}\right] z + \left[\frac{p_1\{\beta p_8(p_1^2 + p_7^2) + (\alpha - \gamma_1)p_1^2 p_7 + p_6(\alpha p_7^2 + \gamma_1 p_1^2)\}}{p_6(p_1^2 + p_7^2)}\right] t + \frac{p_1 p_{10}}{p_6}, \tag{105a}$$

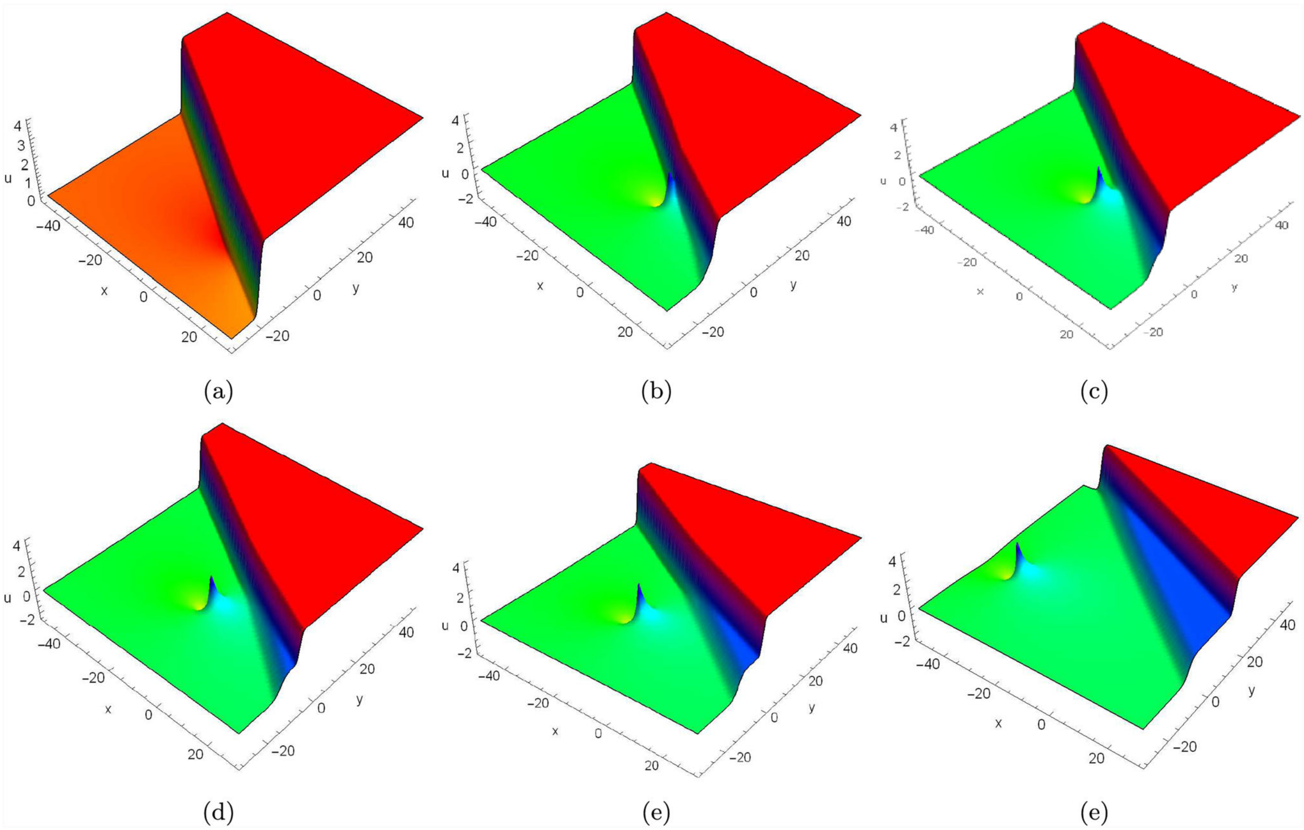


Fig. 7 Progression of the lump-two-kink solution for Eq. (3) at various time points: **a** $t = -2$, **b** $t = 0.8$, **c** $t = 1.9$, **d** $t = 3$, **e** $t = 4.5$, **f** $t = 10$

$$h_2 = p_6 x + p_7 y + p_8 z + \left(\frac{p_4 p_6}{p_1}\right) t + p_{10} \tag{105b}$$

$$\chi_i = a_i \left\{ x + b_i y + c_i z - \frac{a_i^2 + a_i^2 b_i + \alpha b_i + \beta(b_i + c_i) + \gamma_1 + \gamma_2 b_i^2}{(1 + b_i)} t \right\} + \chi_i^0, \; (i = 1, 2). \tag{105c}$$

Figure 7 presents the interaction phenomena between a lump wave and a two-kink wave, narrating their evolutionary behavior over different time frames for the parametric values as $\alpha = \beta = 1$, $\gamma_1 = \gamma_2 = -1.5$, $a_1 = 0.8$, $b_1 = 1.2$, $c_1 = 1.3$, $a_2 = 2$, $b_2 = 1.7$, $c_2 = 1.8$, $p_1 = -2.5$, $p_6 = -1$, $p_7 = 3$, $p_8 = 1$, $p_1 0 = 3.5$, $\chi_i^0 = 0$, $(i = 1, 2)$. At $t = -2$, there is only one kink wave. At $t = 0.8$, lump wave start originating from kink wave. At $t = 1.9$, lump wave completely separates from kink wave. Consequently, at $t = 0$, kink wave start splitting into two-kink wave and at $t = 10$, kink wave completely splits into two distinct kink waves.

9.1.3 Lump-three-kink solution

Lump-three-kink solution can be obtained by choosing $n = 3$ in Eq. (98) as

$$u = 2\left[\frac{2p_1 h_1 + 2p_6 h_2 + a_1 e^{\chi_1} + a_2 e^{\chi_2} + a_3 e^{\chi_3}}{h_1^2 + h_2^2 + d_1 + e^{\chi_1} + e^{\chi_2} + e^{\chi_3}}\right], \tag{106}$$

with

$$h_1 = p_1 x - \left(\frac{p_6 p_7}{p_1}\right) y + \left[\frac{p_1\{\beta p_8(p_1^2 + p_7^2) + p_7(\alpha - \gamma_1)(p_1^2 + p_6^2)\}}{\beta p_6(p_1^2 + p_7^2)}\right] z$$
$$+ \left[\frac{p_1\{\beta p_8(p_1^2 + p_7^2) + (\alpha - \gamma_1) p_1^2 p_7 + p_6(\alpha p_7^2 + \gamma_1 p_1^2)\}}{p_6(p_1^2 + p_7^2)}\right] t + \frac{p_1 p_{10}}{p_6}, \tag{107a}$$

$$h_2 = p_6 x + p_7 y + p_8 z + \left(\frac{p_4 p_6}{p_1}\right) t + p_{10} \tag{107b}$$

$$\chi_i = a_i \left\{ x + b_i y + c_i z - \frac{a_i^2 + a_i^2 b_i + \alpha b_i + \beta(b_i + c_i) + \gamma_1 + \gamma_2 b_i^2}{(1 + b_i)} t \right\} + \chi_i^0, \; (i = 1, 2, 3). \tag{107c}$$

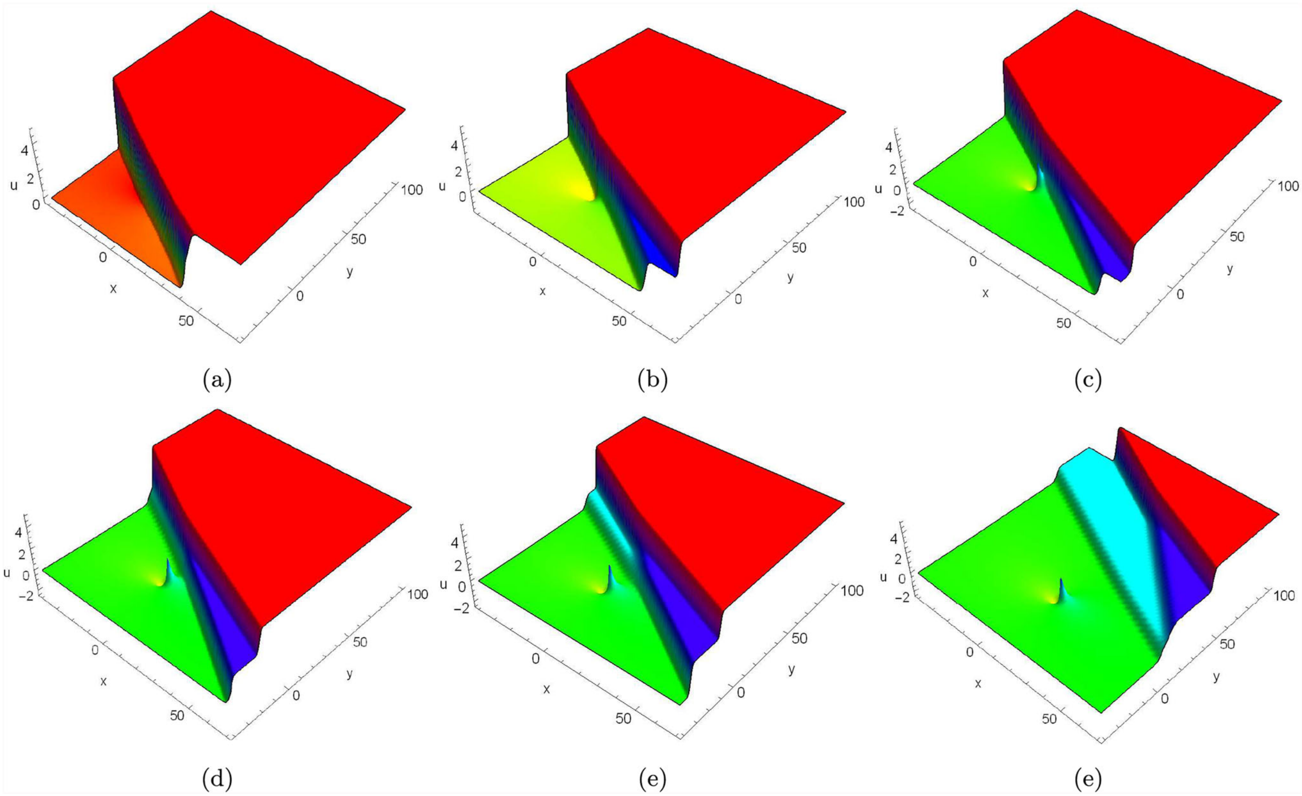


Fig. 8 Progression of the lump-three-kink solution for Eq. (3) at various time points: **a** $t = -3$, **b** $t = 1$, **c** $t = 3$, **d** $t = 5$, **e** $t = 8$, **f** $t = 15$

Figure 8 presents the intricate interaction dynamics between a lump wave and a three-kink wave, capturing their evolution across various time frames. This progression is mapped out according to the specific parametric values $\alpha = \beta = 1$, $\gamma_1 = \gamma_2 = -1.5$, $a_1 = 0.8$, $b_1 = 1.2$, $c_1 = 1.3$, $a_2 = 2$, $b_2 = 1.7$, $c_2 = 1.8$, $a_3 = 2.5$, $b_3 = 1.5$, $c_3 = -1.5$, $p_1 = -2.5$, $p_6 = -1$, $p_7 = -3$, $p_8 = 1$, $p_10 = 1.5$, $\chi_i^0 = 0$, $(i = 1, 2, 3)$. At $t = -3$, there is only one kink wave. At $t = 3$, kink wave start splitting into two kink wave and in the same time lump wave start originating from kink wave. At $t = 5$, lump wave completely separates from kink wave. At $t = 8$, another kink wave originates and consequently separates from other kink waves.

9.2 Lump-multi-kink solutions using test function II

9.2.1 Lump-one-kink solution

We derive lump-one-kink solution by choosing $n = 1$ in Eq. (100) as

$$u = 2\left[\frac{2p_1h_1 + 2p_6h_2 + a_1\cosh\chi_1}{h_1^2 + h_2^2 + d_1 + \cosh\chi_1}\right], \tag{108}$$

with

$$h_1 = p_1x - \left(\frac{p_6p_7}{p_1}\right)y + \left[\frac{p_1\{\beta p_8(p_1^2 + p_7^2) + p_7(\alpha - \gamma_1)(p_1^2 + p_6^2)\}}{\beta p_6(p_1^2 + p_7^2)}\right]z$$
$$+ \left[\frac{p_1\{\beta p_8(p_1^2 + p_7^2) + (\alpha - \gamma_1)p_1^2p_7 + p_6(\alpha p_7^2 + \gamma_1 p_1^2)\}}{p_6(p_1^2 + p_7^2)}\right]t + \frac{p_1p_{10}}{p_6}, \tag{109a}$$

$$h_2 = p_6x + p_7y + p_8z + \left(\frac{p_4p_6}{p_1}\right)t + p_{10} \tag{109b}$$

$$\chi_1 = a_1\left\{x + b_1y + c_1z - \frac{a_1^2 + a_1^2b_1 + \alpha b_1 + \beta(b_1 + c_1) + \gamma_1 + \gamma_2b_1^2}{(1 + b_1)}t\right\} + \chi_1^0. \tag{109c}$$

In Fig. 9, we showcase the evolutionary dynamics of the lump-one-kink solution of Eq. (3), as derived from Eq. (108), across distinct time frames. This phenomena is observed under the corresponding parametric values outlined as $\alpha = 1$, $\beta = 2$, $\gamma_1 = \gamma_2 = -1.5$,

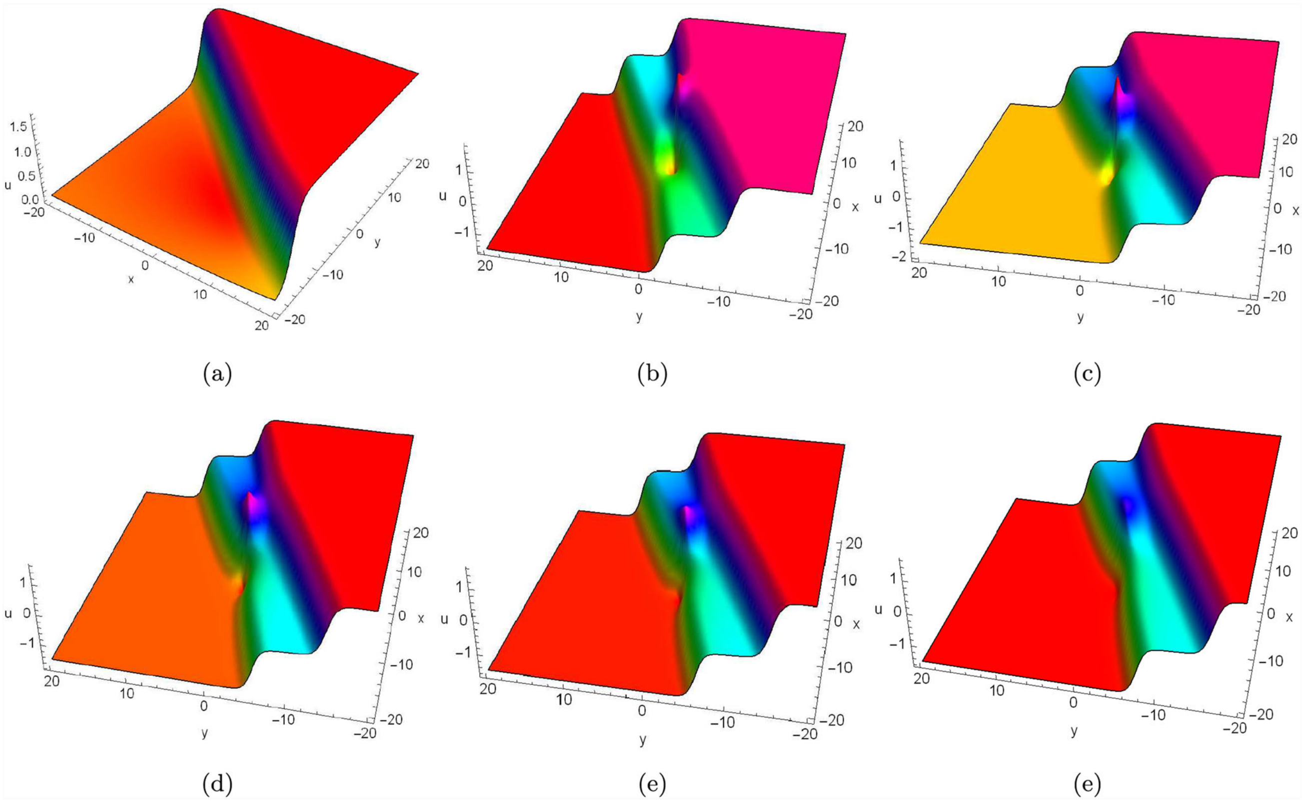


Fig. 9 Progression of the lump-one-kink solution for Eq. (3) at various time points: **a** $t = -1.5$, **b** $t = -0.5$, **c** $t = 0$, **d** $t = 0.5$, **e** $t = 1.5$

$a_1 = 0.8,\ b_1 = -2.5,\ c_1 = -2.1,\ p_1 = 1,\ p_6 = 1.7,\ p_7 = 1.9,\ p_8 = -1.6,\ p_{10} = -1,\ \chi_1^0 = 0$. Initially, there is only one kink solution, then lump solution start originating and reach its peak. Consequently, lump solution disappear in kink solution with time.

9.2.2 Lump-two-kink solution

We derive lump-two-kink solution by choosing $n = 2$ in Eq. (100) as

$$u = 2\left[\frac{2p_1h_1 + 2p_6h_2 + a_1\cosh\chi_1 + a_2\cosh\chi_2}{h_1^2 + h_2^2 + d_1 + \cosh\chi_1 + \cosh\chi_2}\right], \tag{110}$$

with

$$h_1 = p_1x - \left(\frac{p_6p_7}{p_1}\right)y + \left[\frac{p_1\{\beta p_8(p_1^2 + p_7^2) + p_7(\alpha - \gamma_1)(p_1^2 + p_6^2)\}}{\beta p_6(p_1^2 + p_7^2)}\right]z$$
$$+ \left[\frac{p_1\{\beta p_8(p_1^2 + p_7^2) + (\alpha - \gamma_1)p_1^2p_7 + p_6(\alpha p_7^2 + \gamma_1p_1^2)\}}{p_6(p_1^2 + p_7^2)}\right]t + \frac{p_1p_{10}}{p_6}, \tag{111a}$$

$$h_2 = p_6x + p_7y + p_8z + \left(\frac{p_4p_6}{p_1}\right)t + p_{10} \tag{111b}$$

$$\chi_i = a_i\left\{x + b_iy + c_iz - \frac{a_i^2 + a_i^2b_i + \alpha b_i + \beta(b_i + c_i) + \gamma_1 + \gamma_2b_i^2}{(1 + b_i)}t\right\} + \chi_i^0,\ (i = 1, 2). \tag{111c}$$

In Fig. 10, we showcase the evolutionary dynamics of the lump-two-kink solution of Eq. (3), as derived from Eq. (110), across distinct time frames. This phenomena is observed under the corresponding parametric values outlined as $\alpha = \beta = 1,\ \gamma_1 = \gamma_2 = -1.5,\ a_1 = 0.5,\ b_1 = 1.8,\ c_1 = -1.9,\ a_2 = -.8,\ b_2 = 1.1,\ c_2 = -1.5,\ p_1 = 2,\ p_6 = .5,\ p_7 = -3,\ p_8 = 1.4,\ p_10 = 1.2,$ $\chi_i^0 = 0,\ (i = 1, 2)$. At $t = -2$, there is only one kink wave. At $t = 0.8$, lump wave start originating from kink wave. At $t = 1.9$, lump wave completely separates from kink wave. Consequently, at $t = 0$, kink wave start splitting into two-kink wave and at $t = 10$, kink wave completely splits into two distinct kink waves.

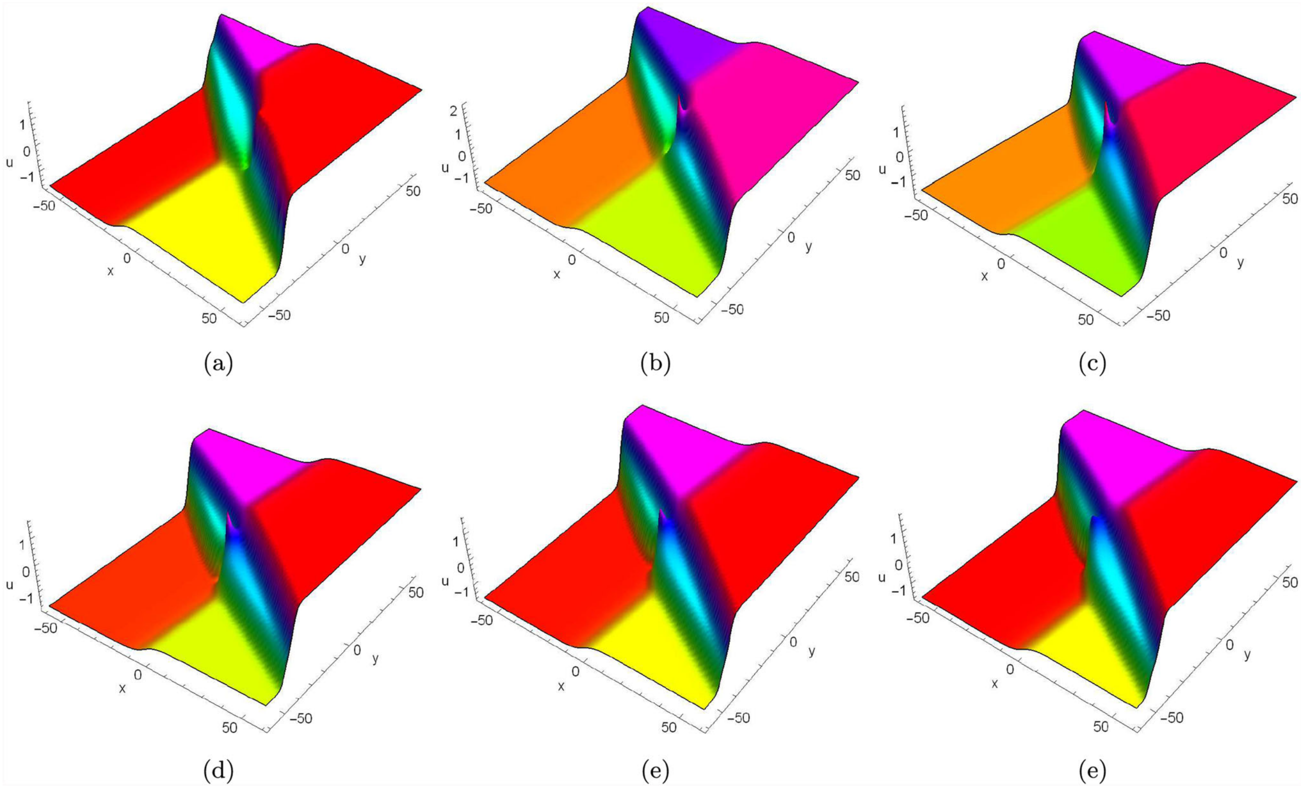


Fig. 10 Progression of the lump-two-kink solution for Eq. (3) at various time points: **a** $t = -1.5$, **b** $t = -0.5$, **c** $t = 0$, **d** $t = 0.5$, **e** $t = 1.5$

10 Graphical illustrations

In Fig. 1, we depict a 3D figure, a density plot, and a 2D plot corresponding to the one-kink solution of Eq. (3). In Fig. 2, we showcase the same visualizations for the two-kink solution and in Fig. 3, we illustrate them for the three-kink solution. In Fig. 4, we present Wronskian-multi-soliton solutions. Figure 5 showcases the 3D representation of the lump solution described by Eq. (3) across various times, illustrating its characteristic localization in all directions. Additionally, contour and density plots of the lump solution are provided at $t = 0$, corresponding to specific parameter values, with $\alpha = 3,\ \beta = 1,\ \gamma_1 = \gamma_2 = 1.5,\ p_1 = 2,\ p_6 = 1,\ p_7 = 4,\ p_8 = 1.5,\ p_{10} = 1.5$. Figure 6, depicts the evolution of a lump solution emerging from a one-kink wave, with parameter values as $\alpha = \beta = 1,\ \gamma_1 = \gamma_2 = -1.5,\ a_1 = 0.8,\ b_1 = 1.2,\ c_1 = 1.3,\ p_1 = -2.5,\ p_6 = -1,\ p_7 = 3,\ p_8 = 1,\ p_1 0 = 1.5,\ \chi_1^0 = 0$. Initially, only a single kink wave is present. Over time, a lump solution gradually emerges from the kink wave, eventually separating entirely from it as the kink wave continues to propagate. In Fig. 7, we illustrate the entire evolutionary process of lump-two-kink solution from a single kink wave, corresponding to the parametric values as $\alpha = \beta = 1,\ \gamma_1 = \gamma_2 = -1.5,\ a_1 = 0.8,\ b_1 = 1.2,\ c_1 = 1.3,\ a_2 = 2,\ b_2 = 1.7,\ c_2 = 1.8,\ p_1 = -2.5,\ p_6 = -1,\ p_7 = 3,\ p_8 = 1,\ p_1 0 = 3.5,\ \chi_i^0 = 0,\ (i = 1, 2)$. Initially, only one kink wave is present. Subsequently, a lump wave emerges from the kink wave and completely separate from it. Following this, the kink wave bifurcates into two distinct kink waves. Figure 8, showcases the complete evolutionary journey of the lump-three-kink solution, originating from a singular kink wave with parameter values as $\alpha = \beta = 1,\ \gamma_1 = \gamma_2 = -1.5,\ a_1 = 0.8,\ b_1 = 1.2,\ c_1 = 1.3,\ a_2 = 2,\ b_2 = 1.7,\ c_2 = 1.8,\ a_3 = 2.5,\ b_3 = 1.5,\ c_3 = -1.5,\ p_1 = -2.5,\ p_6 = -1,\ p_7 = -3,\ p_8 = 1,\ p_1 0 = 1.5,\ \chi_i^0 = 0,\ (i = 1, 2, 3)$. Initially, a solitary kink wave appears. Over time, a lump wave emerges from this kink wave, leading to its division into two separate kink waves. Subsequently, the lump wave detaches entirely from the kink wave, which then further divides into three distinct kink waves, eventually propagating forward. In Fig. 9 we depict the evolution of lump-kink wave, illustrating how the lump wave originates from the kink wave and eventually dissipates back into it over time. In Fig. 10, we portray the progression of a lump-two-kink wave, demonstrating how the lump wave emerges from the two-kink wave and eventually diminishes back into it as time progresses.

11 Conclusions

This article thoroughly investigates the integrability properties of a (3+1)-dimensional extended BK equation. Utilizing relations between Hirota D-operator and binary Bell polynomials, we directly derive the Hirota bilinear form and achieve the bilinear

Bäcklund transformation by decoupling the two-field condition. By incorporating the Cole–Hopf transformation within the bilinear Bäcklund transformation and subsequently linearizing it, we establish a Lax pair formulation. Additionally, Our analysis also explores the integrability of the model equation, leading to the discovery of an infinite sequence of conservation laws. The Hirota bilinear form's analytical power enables us to derive one-, two-, and three-soliton solutions, which we illustrate with detailed by presenting 3-dimensional plot, 2-dimensional plot and density plot. We establish the Wronskian condition for the $(3+1)$-dimensional extended BK equation, employing the Plücker relation to ensure that the N-soliton solutions of the equation can be represented as Wronskian determinants. Additionally, by applying a transformation to our considered model, we obtain a specific type of $(2+1)$-dimensional combined Hirota bilinear equation (3.10) as described in [53]. Moreover, through the use of the obtained Wronskian determinant condition, we successfully derive the $(1+1)$-dimensional KdV equation. As a result, Theorem A leads to a solution of the KdV equation, which notably holds significant recognition. Also, rational Wronskian solutions are obtained by selecting a specific coefficient matrix in the resulting Wronskian formulation. Furthermore, explicit one-, two-, and three-soliton solutions in Wronskian form are derived and their soliton dynamics are visually depicted using Mathematica by selecting appropriate parameters. We also present a Grammian determinant solution, utilizing the Jacobi relation for the $(3+1)$-dimensional extended BK equation. These aspects collectively ensure the complete integrability of our model. Additionally, we acquire the lump solution by considering the test function in quadratic form and present it's localized characteristic in all directions at various times. Moreover, we present a new method that utilizes a combination of an exponential function and a quadratic function as a test function, resulting lump-multi-kink solutions. This depicts that the lump solution originates from a single kink wave and over time separates from the kink wave as the kink wave propagates.

Additionally, a new set of lump-multi-kink solutions is obtained by using a quadratic function and hyperbolic cosine function as test functions. This demonstrates that the lump wave emerges from the kink wave, reaches its peak, and then gradually diminishes back into the kink wave. Notably, these revelations advance our insight into nonlinear wave phenomena spanning a multitude of disciplines, such as nonlinear optics, fluid mechanics, shallow water dynamics, plasma physics and oceanography.

In future, our focus remain towards exploring Lie symmetry analysis to derive diverse types of invariant solutions. Another avenue for our future research entails computing higher-order breather solutions and investigating their interaction phenomena with other exact waves.

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Data Availability Statement All data generated or analyzed during this study are included in this article.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

1. A. Kubokawa, Growing solitary disturbance in a baroclinic boundary current. J. Phys. Oceanogr. **19**, 182–192 (1989)
2. G.P. Agrawal, *Nonlinear Fiber Optics* (Academic Press, San Diego, 2006)
3. B.R. Munson, T.H. Okiishi, W.W. Huebsch, A.P. Rothmayer, *Fundamentals of Fluid Mechanics* (Wiley, New York, 2013)
4. J.M. Luttinger, W. Kohn, Motion of electrons and holes in perturbed periodic fields. Phys. Rev. **97**, 869 (1955)
5. A. Hasegawa, *Plasma Instabilities and Nonlinear Effects* (Springer, Berlin, 2012)
6. C.S. Gardner, J.M. Greene, M.D. Kruskal, R.M. Miura, Method for solving the Korteweg-de Vries equation. Phys. Rev. Lett. **19**, 1095 (1967)
7. M.J. Ablowitz, M.A. Ablowitz, P.A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering Ttransform* (Cambridge University Press, Cambridge, 1991)
8. J. Weiss, M. Tabor, G. Carnevale, The Painlevè property for partial differential equations. J. Math. Phys. **24**, 522–526 (1983)
9. R. Hirota, *The Direct Method in Soliton Theory* (Cambridge University Press, Cambridge, 2004)
10. R.F. Zhang, M.C. Li, Bilinear residual network method for solving the exactly explicit solutions of nonlinear evolution equations. Nonlinear Dyn. **108**, 521–531 (2022)
11. Y.H. Yin, X. Lü, W.X. Ma, Bäcklund transformation, exact solutions and diverse interaction phenomena to a (3 + 1)-dimensional nonlinear evolution equation. Nonlinear Dyn. **108**, 4181–4194 (2022)
12. N.C. Freeman, J.C. Nimmo, Soliton solutions of the Korteweg-de Vries and Kadomtsev-Petviashvili equations: the Wronskian technique. Phys. Lett. A **95**, 1–3 (1983)
13. V.B. Matveed, M.A. Salle, *Darboux Transformation and Solitons* (Springer, Berlin, 1991)
14. P.J. Olver, *Applications of Lie Groups to Differential Equations* (Springer, Berlin, 2000)
15. R.F. Zhang, S. Bilige, Bilinear neural network method to obtain the exact analytical solutions of nonlinear partial differential equations and its application to p-gBKP equation. Nonlinear Dyn. **95**, 3041–3048 (2019)
16. C. Gilson, F. Lambert, J. Nimmo, R. Willox, On the combinatorics of the Hirota D-Operators. Proc. R. Soc. Lond. A **452**, 223–234 (1996)
17. F. Lambert, J. Springael, On a direct procedure for the disclosure of Lax pairs and Bäcklund transformations. Chaos Solitons Fractals **12**, 2821–2832 (2001)
18. E. Fan, The integrability of nonisospectral and variable-coefficient KdV equation with binary Bell polynomials. Phys. Lett. A **52**, 493 (2011)

19. S. Singh, S. Saha Ray, Integrability and new periodic, kink-antikink and complex optical soliton solutions of (3 + 1)-dimensional variable coefficient DJKM equation for the propagation of nonlinear dispersive waves in inhomogeneous media. Chaos Solitons Fractals **168**, 113184 (2023)
20. A. Das, U.K. Mandal, Integrability, bilinearization, solitons and exact three wave solutions for a forced Korteweg-de Vries equation. Commun. Nonlinear Sci. Numer. Simul. **102**, 105936 (2021)
21. X. Lü, S.J. Chen, N-soliton solutions and associated integrability for a novel (2+1)-dimensional generalized KdV equation. Chaos Solitons Fractals **169**, 113291 (2023)
22. U.K. Mandal, S. Malik, S. Kumar, A. Das, A generalized (2 + 1)-dimensional Hirota bilinear equation: integrability, solitons and invariant solutions. Nonlinear Dyn. **111**, 4593–4611 (2023)
23. J. Satsuma, A Wronskian representation of N-Soliton solutions of nonlinear evolution equations. J. Phys. Soc. Jpn. **46**, 359–360 (1979)
24. R. Hirota, M. Ito, F. Kako, Two-dimensional Toda lattice equations. Progr. Theor. Phys. Suppl. **94**, 42–58 (1988)
25. J.C. Nimmo, N.C. Freeman, A method of obtaining the N-soliton solution of the Boussinesq equation in terms of a Wronskian. Phys. Lett. A **95**, 4 (1983)
26. Y. Chen, X. Lü, Wronskian solutions and linear superposition of rational solutions to B-type Kadomtsev-Petviashvili equation. Phys. Fluids **35**, 106613 (2023)
27. W.X. Ma, Y. You, Solving the Korteweg-de Vries equation by its bilinear form: Wronskian solutions. Trans. Am. Math. Soc. **357**, 1753–1778 (2004)
28. S. Shah, W. Masood, M. Siddiq, H. Rizvi, Nonlinear ion acoustic waves in dense magnetoplasmas: analyzing interaction solutions of the KdV equation using Wronskian formalism for electron trapping with Landau diamagnetism and thermal excitations. Chaos Solitons Fractals **181**, 114638 (2024)
29. L. Cheng, Y. Zhang, W.X. Ma, Y.W. Hu, Wronskian rational solutions to the generalized (2 + 1)-dimensional Date-Jimbo-Kashiwara-Miwa equation in fluid dynamics. Phys. Fluids **36**, 017116 (2024)
30. A. Silem, J. Lin, Exact solutions for a variable-coefficients nonisospectral nonlinear Schrödinger equation via Wronskian technique. Appl. Math. Lett. **135**, 108397 (2023)
31. S.V. Manakov, V.E. Zakharov, L.A. Bordag, A.R. Its, V.B. Matveev, Two-dimensional solitons of the Kadomtsev-Petviashvili equation and their interaction. Phys. Lett. A **63**, 205–206 (1977)
32. D.R. Solli, C. Ropers, P. Koonath, B. Jalali, Optical rogue waves. Nature **450**, 1054–1057 (2007)
33. D.R. Solli, C. Ropers, B. Jalali, Active control of rogue waves for stimulated supercontinuum generation. Phys. Rev. Lett. **101**, 233902 (2008)
34. D.S. Wang, J. Liu, L.Z. Wang, Non-autonomous matter-wave solitons in hybrid atomic-molecular Bose-Einstein condensates with tunable interactions and harmonic potential. Phys. Lett. A **382**, 799–805 (2018)
35. M.J. Ablowitz, J. Satsuma, Solitons and rational solutions of nonlinear evolution equations. J. Math. Phys. **19**, 2180–2186 (1978)
36. H.H. Sheng, G.F. Yu, Solitons, breathers and rational solutions for a (2 + 1)-dimensional dispersive long wave system. Physica D **432**, 133671 (2022)
37. G.W. Bluman, S. Kumei, *Symmetries and Differential Equations* (Springer, New York, 1989)
38. D.J. Kaup, The lump solutions and the Bäcklund transformation for the three dimensional three-wave resonant interaction. J. Math. Phys. **22**, 1176–1181 (1981)
39. W.X. Ma, Lump solutions to the Kadomtsev-Petviashvili equation. Phys. Lett. A **379**, 1975–1978 (2015)
40. G. Han, X. Li, Q. Zhao, C. Li, Interaction structures of multi localized waves within the Kadomtsev-Petviashvili I equation. Physica D **446**, 133671 (2023)
41. U.K. Mandal, A. Das, W.X. Ma, Integrability, breather, rogue wave, lump, lump-multi-stripe, and lump-multi-soliton solutions of a (3+1)-dimensional nonlinear evolution equation. Phys. Fluids **36**, 037151 (2024)
42. H.F. Ismael, T.A. Sulaiman, On the dynamics of the nonautonomous multi-soliton, multi-lump waves and their collision phenomena to a (3 + 1)-dimensional nonlinear model. Chaos Solitons Fractals **169**, 113213 (2023)
43. W.X. Ma, X. Yong, H.Q. Zhang, Diversity of interaction solutions to the (2+1)-dimensional Ito equation. Comput. Math. Appl. **75**, 289–295 (2018)
44. Y. Liu, Z. Zhao, Periodic line wave, rogue waves and the interaction solutions of the (2 + 1)-dimensional integrable Kadomtsev-Petviashvili-based system. Chaos Solitons Fractals **183**, 114883 (2024)
45. U.K. Mandal, B. Karmakar, A. Das, W.X. Ma, Integrability, bilinearization, exact traveling wave solutions, lump and lump-multi-kink solutions of a (3 + 1)-dimensional negative order KdV-Calogero-Bogoyavlenskii-Schiff equation. Nonlinear Dyn. **112**, 4727–4748 (2024)
46. U.K. Mandal, S. Malik, S. Kumar, Y. Zhang, A. Das, Integrability aspects, rational type solutions and invariant solutions of an extended (3+1)-dimensional B-type Kadomtsev-Petviashvili equation. Chaos Solitons Fractals **181**, 114689 (2024)
47. G. Falqui, F. Magri, M. Pedroni, Bihamiltonian geometry, Darboux coverings, and linearization of the KP hierarchy. Commun. Math. Phys. **197**, 303–324 (1998)
48. A. Yokus, S. Duran, D. Kaya, An expansion method for generating travelling wave solutions for the (2 + 1)-dimensional Bogoyavlensky-Konopelchenko equation with variable coefficients. Chaos Solitons Fractals **178**, 114316 (2024)
49. Q. Li, T. Chaolu, Y.H. Wang, Lump-type solutions and lump solutions for the (2 + 1)-dimensional generalized Bogoyavlensky-Konopelchenko equation. Comput. Math. Appl. **77**, 2077–2085 (2019)
50. S. Saha Ray, On conservation laws by Lie symmetry analysis for (2 + 1)-dimensional Bogoyavlensky-Konopelchenko equation in wave propagation. Comput. Math. Appl. **74**, 1158–1165 (2017)
51. A.M. Wazwaz, R.A. Alharbey, S.A. El-Tantawy, Painlevè integrability for an extended (3 + 1)-dimensional Bogoyavlensky-Konopelchenko equation: lumps and ltiple soliton solutions. Rom. Rep. Phys. **75**, 119 (2023)
52. S. Sirianunpiboon, S.D. Howard, S.K. Roy, A note on the Wronskian form of solutions of the KdV equation. Phys. Lett. A **134**, 31 (1988)
53. W.X. Ma, N-soliton solution and the Hirota condition of a (2+ 1)-dimensional combined equation. Math. Comput. Simul. **190**, 270–279 (2021)
54. W.X. Ma, Wronskians, generalized Wronskians and solutions to the Korteweg-de Vries equation. Chaos Solitons Fractals **19**, 163–170 (2004)
55. W.X. Ma, N-soliton solutions and the Hirota conditions in (1 + 1)-dimensions. Int. J. Nonlinear Sci. Numer. Simul. **33**, 123–133 (2022)