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Symbolic Computation of Lump Solutions to a Combined Equation Involving Three Types of Nonlinear Terms

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Abstract. This paper aims to compute lump solutions to a combined fourth-order equation involving three types of nonlinear terms in (2+1)-dimensions via symbolic computations. The combined nonlinear equation contains all second-order linear terms and it possesses a Hirota bilinear form under two logarithmic transformations. Two classes of explicit lump solutions are determined, which are associated with two cases of the coefficients in the model equation. Two illustrative examples of the combined nonlinear equation are presented, along with lump solutions and their representative three-dimensional plots, contour plots and density plots.

AMS subject classifications: 35Q51, 35Q53, 37K40

Key words: Soliton equation, lump solution, Hirota derivative, symbolic computation, dispersion relation.

1. Introduction

The Hirota bilinear method [3, 14] is effective in constructing soliton solutions to integrable equations generated from zero curvature equations [1, 46]. Soliton solutions are analytic, and usually exponentially localised in space and time. Assume that a polynomial P defines a Hirota bilinear differential equation

$$P(D_x, D_y, D_t)f \cdot f = 0$$

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in (2+1)-dimensions. Here D_x, D_y and D_t are Hirota's bilinear derivatives [14]. An associated partial differential equation (PDE) with a dependent variable u is often determined by some logarithmic transformation of $u = 2(\ln f)_x$ and $u = 2(\ln f)_{xx}$. Within the Hirota bilinear formulation, the N-soliton solution — cf. [13], can be presented through

$$f = \sum_{\mu=0,1} \exp\left(\sum_{i=1}^{N} \mu_i \xi_i + \sum_{i<j} \mu_i \mu_j a_{ij}\right),$$

where $\sum_{\mu=0,1}$ is the sum over all possibilities for $\mu_1, \mu_2, \ldots, \mu_N$ taking either 0 or 1, and the wave variables and the phase shifts are defined by

$$\xi_i = k_i x + l_i y - \omega_i t + \xi_{i,0}, \quad 1 \leq i \leq N,$$

and

$$\mathrm{e}^{a_{ij}} = -\frac{P(k_i - k_j, l_i - l_j, \omega_j - \omega_i)}{P(k_i + k_j, l_i + l_j, \omega_j + \omega_i)}, \quad 1 \leq i < j \leq N,$$

in which k_i, l_i and ω_i, $1 \leq i \leq N$ satisfy the associated dispersion relation and the phases shifts $\xi_{i,0}$, $1 \leq i \leq N$ are arbitrary.

Recent studies show the remarkable richness of lump solutions to integrable equations, which describe various dispersive wave phenomena. Lumps are rational solutions, which are analytic and localised in all directions in space [42, 43, 49] and they can also be derived from taking long wave limits of soliton equations [47]. The KPI equation possesses diverse lump solutions [24] and its special lump solutions are generated from soliton solutions [44]. Other integrable equations which have lump solutions include the three-dimensional three-wave resonant interaction [17], the Davey-Stewartson II equation [47], the BKP equation [10, 64], the Ishimori-I equation [16], the KPI and mKPI equation with a self-consistent source [71, 72]. Moreover, non integrable equations can have lump solutions, and such equations contain the generalised KP, BKP, KP-Boussinesq, Sawada-Kotera, Calogero-Bogoyavlenskii-Schiff and Bogoyavlensky-Konopelchenko equations in (2+1)-dimensions [4, 21, 31, 37, 39, 74]. It is worth noting that the second KPI equation exhibits a new kind of lump solutions with higher-order rational dispersion relations [41]. The starting point in constructing lump solutions is to determine positive quadratic function solutions to Hirota bilinear equations [42]. Then from positive quadratic function solutions, lump solutions to nonlinear PDEs are constructed by using the logarithmic transformations.

In this paper, we would like to discuss a combined fourth-order equation in (2+1)-dimensional dispersive waves and determine its diverse lump solutions. The Hirota bilinear form plays a crucial role in our analysis [23, 42, 43, 82]. The combined nonlinear equation includes three types of fourth-order nonlinear terms and all second-order linear terms. To conduct symbolic computation of lump solutions with Maple, we will analyze two cases of the coefficients in the model equation. Illustrative examples of the considered model equation will be made, together with specific lump solutions and their three-dimensional plots, contour plots and density plots. A few concluding remarks will be given in the final section.

2. An Equation Involving Three Types of Nonlinear Terms

We would like to consider a combined fourth-order nonlinear equation:

$$\begin{aligned} P(u) =&\alpha_1(6u_xu_{xx}+u_{xxxx})+\alpha_2[3(u_xu_y)_x+u_{xxxy}] \\ &+\alpha_3(4u_yu_{xy}+u_xu_{yy}+u_{xx}v+u_{xxyy}) \\ &+\delta_1u_{yt}+\delta_2u_{xx}+\delta_3u_{xt}+\delta_4u_{xy}+\delta_5u_{yy}+\delta_6u_{tt}=0, \end{aligned} \tag{2.1}$$

where $v_x=u_{yy}$, and the constants $\alpha_i, 1\le i\le 3$ and $\delta_i, 1\le i\le 6$ are generally arbitrary. The coefficients $\alpha_i, 1\le i\le 3$ correspond to three types of nonlinear terms.

It is direct to show that the above combined nonlinear equation (2.1) possesses a Hirota bilinear form

$$\begin{aligned} B(f)=&\big(\alpha_1D_x^4+\alpha_2D_x^3D_y+\alpha_3D_x^2D_y^2+\delta_1D_yD_t+\delta_2D_x^2 \\ &+\delta_3D_xD_t+\delta_4D_xD_y+\delta_5D_y^2+\delta_6D_t^2\big)f\cdot f=0 \end{aligned} \tag{2.2}$$

under the logarithmic transformations

$$u=2(\ln f)_x=\frac{2f_x}{f},\quad v=2(\ln f)_{yy}=\frac{2\left(f_{yy}f-f_y^2\right)}{f^2}. \tag{2.3}$$

Precisely, we can have the connection between the combined nonlinear and bilinear equations: $P(u)=(B(f)/f^2)_x$, when u,v and f satisfy the link (2.3). The combined bilinear equation (2.2) contains three types of fourth-order derivative terms and all second-order derivative terms, and it reduces to the standard bilinear KP equation, when $\alpha_1=\delta_3=1$, $\delta_5=-1$ and all other coefficients are zero.

Moreover, on one hand, upon taking $\alpha_2=1$, $\delta_3=\delta_5=1$ and all other coefficients as zero, the combined nonlinear equation (2.1) gives a generalised Calogero-Bogoyavlenskii-Schiff equation [4]:

$$3(u_xu_y)_x+u_{xxxy}+u_{xt}+u_{yy}=0,$$

which also possesses a Hirota bilinear form

$$\left(D_x^3D_y+D_xD_t+D_y^2\right)f\cdot f=0,$$

under $u=2(\ln f)_x$, and whose lump solutions have been computed in [4].

On the other hand, upon taking $\alpha_1=\alpha_2=1$, $\delta_2=\delta_3=\delta_5=1$ and all other coefficients as zero, the combined nonlinear equation (2.1) gives a generalised Bogoyavlensky-Konopelchenko equation [5]:

$$6u_xu_{xx}+u_{xxxx}+3(u_xu_y)_x+u_{xxxy}+u_{xt}+u_{xx}+u_{yy}=0$$

whose Hirota bilinear form is given by

$$\left(D_x^4+D_x^3D_y+D_xD_t+D_x^2+D_y^2\right)f\cdot f=0$$

under $u=2(\ln f)_x$. This equation has lump solutions, too [5].

When $\alpha_3\ne 0$, the combined nonlinear equation (2.1) presents a new model, due to the fourth-order term $D_x^2D_y^2f\cdot f$ in the corresponding bilinear form.

3. Computing Lump Solutions

In this section, we would like to compute lump solutions to the combined fourth-order nonlinear equation (2.1), through symbolic computations with Maple.

A general ansatz on lump solutions in (2+1)-dimensions [24] is to start to determine positive quadratic solutions

$$f=(a_1x+a_2y+a_3t+a_4)^2+(a_5x+a_6y+a_7t+a_8)^2+a_9 \tag{3.1}$$

for the combined Hirota bilinear equation (2.2). The job is to compute the involved constant parameters $a_i, 1\le i\le 9$ by trial and error. In the following, we present two sets of such constant parameters, which correspond to two cases of the coefficients.

3.1. The case of $\delta_6=0$

Let us first consider the case of $\delta_6=0$ for the combined nonlinear equation (2.1). A straightforward symbolic computation tells a set of solutions for the parameters, where

$$\begin{aligned} a_3&=-\frac{b_1}{(a_2\delta_1+a_1\delta_3)^2+(a_6\delta_1+a_5\delta_3)^2},\\ a_7&=-\frac{b_2}{(a_2\delta_1+a_1\delta_3)^2+(a_6\delta_1+a_5\delta_3)^2},\\ a_9&=-\frac{3(a_1^2+a_5^2)(\alpha_1b_3+\alpha_2b_4)+\alpha_3b_5}{(a_1a_6-a_2a_5)^2(\delta_1^2\delta_2-\delta_1\delta_3\delta_4+\delta_3^2\delta_5)}, \end{aligned} \tag{3.2}$$

and all other a_i are arbitrary. The involved five constants $b_i, 1\le i\le 5$ are given by

$$\begin{aligned} b_1=&\left[\left(a_1^2a_2+2a_1a_5a_6-a_2a_5^2\right)\delta_2+a_1\left(a_2^2+a_6^2\right)\delta_4+a_2\left(a_2^2+a_6^2\right)\delta_5\right]\delta_1\\ &+\left[a_1\left(a_1^2+a_5^2\right)\delta_2+a_2\left(a_1^2+a_5^2\right)\delta_4+\left(a_1a_2^2+2a_2a_5a_6-a_1a_6^2\right)\delta_5\right]\delta_3,\\ b_2=&\left[\left(-a_1^2a_6+2a_1a_2a_5+a_5^2a_6\right)\delta_2+a_5\left(a_2^2+a_6^2\right)\delta_4+a_6\left(a_2^2+a_6^2\right)\delta_5\right]\delta_1\\ &+\left[a_5\left(a_1^2+a_5^2\right)\delta_2+a_6\left(a_1^2+a_5^2\right)\delta_4+\left(-a_2^2a_5+2a_1a_2a_6+a_5a_6^2\right)\delta_5\right]\delta_3,\\ b_3=&\left(a_1^2+a_5^2\right)\left[(a_2\delta_1+a_1\delta_3)^2+(a_6\delta_1+a_5\delta_3)^2\right],\\ b_4=&(a_1a_2+a_5a_6)\left[(a_2\delta_1+a_1\delta_3)^2+(a_6\delta_1+a_5\delta_3)^2\right],\\ b_5=&\left(3a_1^2a_2^2+a_1^2a_6^2+4a_1a_2a_5a_6+a_2^2a_5^2+3a_5^2a_6^2\right)\left[(a_2\delta_1+a_1\delta_3)^2+(a_6\delta_1+a_5\delta_3)^2\right]. \end{aligned} \tag{3.3}$$

The above expressions of a_3 and a_7 generate abundant dispersion relations in (2+1)-dimensional dispersive waves.

3.2. The case of $\delta_5=0$

Let us second consider the case of $\delta_5=0$ for the combined nonlinear equation (2.1). A similar straightforward symbolic computation determines a set of solutions for the parameters, where

$$
\begin{aligned}
a_2 &= -\frac{c_1}{(a_3\delta_1+a_1\delta_4)^2+(a_7\delta_1+a_5\delta_4)^2},\\
a_6 &= -\frac{c_2}{(a_3\delta_1+a_1\delta_4)^2+(a_7\delta_1+a_5\delta_4)^2},\\
a_9 &= -\frac{3(a_1^2+a_5^2)(\alpha_1c_3-\alpha_2c_4)}{(a_1a_7-a_3a_5)^2(\delta_1^2\delta_2-\delta_1\delta_3\delta_4+\delta_4^2\delta_6)}\\
&\quad -\frac{\alpha_3c_5}{(a_1a_7-a_3a_5)^2(\delta_1^2\delta_2-\delta_1\delta_3\delta_4+\delta_4^2\delta_6)[(a_3\delta_1+a_1\delta_4)^2+(a_7\delta_1+a_5\delta_4)^2]},
\end{aligned}
\tag{3.4}
$$

and all other a_i's are arbitrary. The involved five constants $c_i, 1\le i\le 5$, are given by

$$
\begin{aligned}
c_1 =&\left[\left(a_1^2a_3+2a_1a_5a_7-a_3a_5^2\right)\delta_2+a_1\left(a_3^2+a_7^2\right)\delta_3+a_3\left(a_3^2+a_7^2\right)\delta_6\right]\delta_1\\
&+\left[a_1\left(a_1^2+a_5^2\right)\delta_2+a_3\left(a_1^2+a_5^2\right)\delta_3+\left(a_1a_3^2+2a_3a_5a_7-a_1a_7^2\right)\delta_6\right]\delta_4,\\
c_2 =&\left[\left(-a_1^2a_7+2a_1a_3a_5+a_5^2a_7\right)\delta_2+a_5\left(a_3^2+a_7^2\right)\delta_3+a_7\left(a_3^2+a_7^2\right)\delta_6\right]\delta_1\\
&+\left[a_5\left(a_1^2+a_5^2\right)\delta_2+a_7\left(a_1^2+a_5^2\right)\delta_3+\left(-a_3^2a_5+2a_1a_3a_7+a_5a_7^2\right)\delta_6\right]\delta_4,\\
c_3 =&\left(a_1^2+a_5^2\right)\left[(a_1\delta_4+a_3\delta_1)^2+(a_5\delta_4+a_7\delta_1)^2\right],\\
c_4 =&\left(a_1^2+a_5^2\right)\left(a_1a_3+a_5a_7\right)\left(\delta_1\delta_2+\delta_3\delta_4\right)+\left(a_1^2+a_5^2\right)\left(a_3^2+a_7^2\right)\delta_1\delta_3\\
&+\left(a_1^2+a_5^2\right)^2\delta_2\delta_4+\left(a_3^2+a_7^2\right)\left(a_1a_3+a_5a_7\right)\delta_1\delta_6\\
&+\left[(a_1a_3+a_5a_7)^2-(a_1a_7-a_3a_5)^2\right]\delta_4\delta_6,
\end{aligned}
\tag{3.5}
$$

and

$$
\begin{aligned}
c_5 =&\left[\left(a_1^2+a_5^2\right)^2p_1\delta_3^2+3\left(a_1^2+a_5^2\right)^4\delta_2^2\right]\delta_4^2\\
&+6\left(a_1^2+a_5^2\right)^3\left(a_1a_3+a_5a_7\right)\delta_2\left(\delta_1\delta_2\delta_4+\delta_3\delta_4^2\right)\\
&+6\left(a_3^2+a_7^2\right)\left(a_1^2+a_5^2\right)^2\left(a_1a_3+a_5a_7\right)\delta_1\left(\delta_1\delta_2\delta_3+\delta_3^2\delta_4\right)\\
&+3\left(a_3^2+a_7^2\right)^2\left(a_1^2+a_5^2\right)^2\delta_1^2\delta_3^2+\left(a_1^2+a_5^2\right)^2p_1\delta_1^2\delta_2^2\\
&+4\left(a_1^2+a_5^2\right)^2p_1\delta_1\delta_2\delta_3\delta_4\\
&+\Big\{\left[2\left(a_1^2+a_5^2\right)\left(a_1a_3+a_5a_7\right)p_2\delta_3+6\left(a_1^2+a_5^2\right)^2p_3\delta_2\right]\delta_4^2\\
&+\left[4\left(a_3^2+a_7^2\right)\left(a_1^2+a_5^2\right)p_2\delta_1\delta_3+4\left(a_1^2+a_5^2\right)\left(a_1a_3+a_5a_7\right)p_2\delta_1\delta_2\right]\delta_4\\
&+6\left(a_3^2+a_7^2\right)^2\left(a_1^2+a_5^2\right)\left(a_1a_3+a_5a_7\right)\delta_1^2\delta_3+2\left(a_3^2+a_7^2\right)\left(a_1^2+a_5^2\right)p_2\delta_1^2\delta_2\Big\}\delta_6\\
&+\left[\left(a_3^2+a_7^2\right)^2p_1\delta_1^2+2\left(a_3^2+a_7^2\right)\left(a_1a_3+a_5a_7\right)p_2\delta_1\delta_4+p_4\delta_4^2\right]\delta_6^2,
\end{aligned}
\tag{3.6}
$$

where

$$
\begin{aligned}
p_1 =&3a_1^2a_3^2+a_1^2a_7^2+4a_1a_3a_5a_7+a_3^2a_5^2+3a_5^2a_7^2,\\
p_2 =&3a_1^2a_3^2-a_1^2a_7^2+8a_1a_3a_5a_7-a_3^2a_5^2+3a_5^2a_7^2,\\
p_3 =&(a_1a_3+a_1a_7-a_3a_5+a_5a_7)(a_1a_3-a_1a_7+a_3a_5+a_5a_7),
\end{aligned}
\tag{3.7}
$$

$$
\begin{aligned}
p_4 =& 3a_1^4a_3^4-2a_1^4a_3^2a_7^2+3a_1^4a_7^4+16a_1^3a_3^3a_5a_7-16a_1^3a_3a_5a_7^3 \\
&-2a_1^2a_3^4a_5^2+44a_1^2a_3^2a_5^2a_7^2-2a_1^2a_5^2a_7^4-16a_1a_3^3a_5^3a_7 \\
&+16a_1a_3a_5^3a_7^3+3a_3^4a_5^4-2a_3^2a_5^4a_7^2+3a_5^4a_7^4.
\end{aligned}
$$

We point out that all the above formulas for wave frequencies and wave numbers in (3.2)-(3.7) have been presented through direct simplifications with Maple. Based on those solution formulas, we require the following two basic conditions:

$$
\left(\delta_1^2+\delta_3^2\right)\left(\delta_1^2\delta_2-\delta_1\delta_3\delta_4+\delta_3^2\delta_5\right)\neq 0
$$

in the case of $\delta_6=0$, and

$$
\left(\delta_1^2+\delta_4^2\right)\left(\delta_1^2\delta_2-\delta_1\delta_3\delta_4+\delta_4^2\delta_6\right)\neq 0 \tag{3.8}
$$

in the case of $\delta_5=0$, to generate lump solutions to the combined nonlinear equation (2.1).

In the case of $\delta_6=0$, we can work out

$$
\begin{aligned}
&a_1a_7-a_3a_5 \\
=&\frac{(a_1a_6-a_2a_5)[(a_1^2+a_5^2)(\delta_1\delta_2-\delta_3\delta_4)-(a_2^2+a_6^2)\delta_1\delta_5-2(a_1a_2+a_5a_6)\delta_3\delta_5]}{(a_2\delta_1+a_1\delta_3)^2+(a_6\delta_1+a_5\delta_3)^2},
\end{aligned}
$$

and in the case of $\delta_5=0$ we can get

$$
\begin{aligned}
&a_1a_6-a_2a_5 \\
=&\frac{(a_1a_7-a_3a_5)[(a_1^2+a_5^2)(\delta_1\delta_2-\delta_3\delta_4)-(a_3^2+a_7^2)\delta_1\delta_6-2(a_1a_3+a_5a_7)\delta_4\delta_6]}{(a_3\delta_1+a_1\delta_4)^2+(a_7\delta_1+a_5\delta_4)^2}.
\end{aligned}
$$

Therefore, we can see that in the case of $\delta_5=0$, the condition $a_1a_6-a_2a_5\neq 0$, ensuing the existence of lumps, is equivalent to the following two conditions:

$$
\begin{aligned}
&a_1a_7-a_3a_5\neq 0, \\
&\left(a_1^2+a_5^2\right)\left(\delta_1\delta_2-\delta_3\delta_4\right)-\left(a_3^2+a_7^2\right)\delta_1\delta_6-2\left(a_1a_3+a_5a_7\right)\delta_4\delta_6\neq 0
\end{aligned}
$$

besides (3.8). Along with $a_9>0$, those three conditions guarantee that the set of the associated parameters yields lump solutions.

4. Equivalence Between Two Classes of Lumps

When $\delta_5=\delta_6=0$, we can have two sets of the parameters, which yield lump solutions, determined via symbolic computation in the last section. Below, we show an equivalence between those two classes of associated lump solutions.

While $\delta_5=\delta_6=0$ is taken, the combined Hirota bilinear equation (2.2) becomes

$$
\left(\alpha_1D_x^4+\alpha_2D_x^3D_y+\alpha_3D_x^2D_y^2+\delta_1D_yD_t+\delta_2D_x^2+\delta_3D_xD_t+\delta_4D_xD_y\right)f\cdot f=0.
$$

Two classes of lump solutions defined by (3.2) with (3.3) and (3.4) with (3.5)-(3.7) are equivalent to each other. In another word, the one can be obtained from the other.

The first set of the parameters by (3.2) with (3.3) reads

$$
\begin{aligned}
a_3 &= -\frac{(a_1^2 a_2 + 2a_1a_5a_6 - a_2a_5^2)\delta_1\delta_2 + a_1(a_2^2+a_6^2)\delta_1\delta_4 + (a_1^2+a_5^2)(a_1\delta_2+a_2\delta_4)\delta_3}{(a_1\delta_3+a_2\delta_1)^2+(a_5\delta_3+a_6\delta_1)^2},\\
a_7 &= -\frac{(-a_1^2a_6+2a_1a_2a_5+a_5^2a_6)\delta_1\delta_2+a_5(a_2^2+a_6^2)\delta_1\delta_4+(a_1^2+a_5^2)(a_5\delta_2+a_6\delta_4)\delta_3}{(a_1\delta_3+a_2\delta_1)^2+(a_5\delta_3+a_6\delta_1)^2},\\
a_9 &= -\frac{3(a_1^2+a_5^2)(\alpha b_3+\alpha_2 b_4)+\alpha_3 b_5}{(a_1a_6-a_2a_5)^2\delta_1(\delta_1\delta_2-\delta_3\delta_4)},
\end{aligned}
$$

where b_3, b_4 and b_5 are defined as in (3.3).

The second set of the parameters by (3.4) with (3.5) and (3.6) reads

$$
\begin{aligned}
a_2 &= -\frac{(a_1^2a_3+2a_1a_5a_7-a_3a_5^2)\delta_1\delta_2+a_1(a_3^2+a_7^2)\delta_1\delta_3+(a_1^2+a_5^2)(a_1\delta_2+a_3\delta_3)\delta_4}{(a_3\delta_1+a_1\delta_4)^2+(a_7\delta_1+a_5\delta_4)^2},\\
a_6 &= -\frac{(-a_1^2a_7+2a_1a_3a_5+a_5^2a_7)\delta_1\delta_2+a_5(a_3^2+a_7^2)\delta_1\delta_3+(a_1^2+a_5^2)(a_5\delta_2+a_7\delta_3)\delta_4}{(a_3\delta_1+a_1\delta_4)^2+(a_7\delta_1+a_5\delta_4)^2},\\
a_9 &= -\frac{3(a_1^2+a_5^2)(\alpha_1 d_3-\alpha_2 d_4)}{(a_1a_7-a_3a_5)^2\delta_1(\delta_1\delta_2-\delta_3\delta_4)}\\
&\quad -\frac{\alpha_3 d_5}{(a_1a_7-a_3a_5)^2\delta_1(\delta_1\delta_2-\delta_3\delta_4)[(a_3\delta_1+a_1\delta_4)^2+(a_7\delta_1+a_5\delta_4)^2]},
\end{aligned}
$$

where

$$
\begin{aligned}
d_3 &= \left(a_1^2+a_5^2\right)\left[\left(a_1\delta_4+a_3\delta_1\right)^2+\left(a_5\delta_4+a_7\delta_1\right)^2\right],\\
d_4 &= \left(a_1^2+a_5^2\right)\left(a_1a_3+a_5a_7\right)\left(\delta_1\delta_2+\delta_3\delta_4\right)\\
&\quad +\left(a_1^2+a_5^2\right)\left(a_3^2+a_7^2\right)\delta_1\delta_3+\left(a_1^2+a_5^2\right)^2\delta_2\delta_4,\\
d_5 &= \left[\left(a_1^2+a_5^2\right)^2 p_1\delta_3^2+3\left(a_1^2+a_5^2\right)^4\delta_2^2\right]\delta_4^2\\
&\quad +6\left(a_1^2+a_5^2\right)^3\left(a_1a_3+a_5a_7\right)\delta_2\left(\delta_1\delta_2\delta_4+\delta_3\delta_4^2\right)\\
&\quad +6\left(a_3^2+a_7^2\right)\left(a_1^2+a_5^2\right)^2\left(a_1a_3+a_5a_7\right)\delta_1\left(\delta_1\delta_2\delta_3+\delta_3^2\delta_4\right)\\
&\quad +3\left(a_3^2+a_7^2\right)^2\left(a_1^2+a_5^2\right)^2\delta_1^2\delta_3^2+\left(a_1^2+a_5^2\right)^2 p_1\delta_1^2\delta_2^2\\
&\quad +4\left(a_1^2+a_5^2\right)^2 p_1\delta_1\delta_2\delta_3\delta_4
\end{aligned}
$$

with p_i, $1\le i\le 4$ given by (3.7).

A direct symbolic computation can show that these two sets of the parameters are the same, since they can be solved from each other. Moreover, one has

$$
a_1a_7-a_3a_5 = -\frac{(a_1^2+a_5^2)(\delta_1\delta_2-\delta_3\delta_4)(a_1a_6-a_2a_5)}{(a_1\delta_3+a_2\delta_1)^2+(a_5\delta_3+a_6\delta_1)^2},
$$

and therefore, when

$$\delta_1(\delta_1\delta_2-\delta_3\delta_4)\neq 0,$$

the two sets determine the exactly same values for all the parameters and thus the same classes of associated lump solutions.

5. Two Illustrative Examples

Let us first consider the case of $\delta_6=0$ and take

$$\alpha=1,\quad \alpha_2=2,\quad \alpha_3=-3,\quad \delta_1=1,\quad \delta_2=0,\quad \delta_3=2,\quad \delta_4=2,\quad \delta_5=5,$$

which leads to a specific combined nonlinear equation

$$\begin{aligned}&u_{xxxx}+6u_xu_{xx}+2\left[3(u_xu_y)_x+u_{xxxy}\right]\\&\quad-3\left(4u_yu_{xy}+u_xu_{yy}+u_{xx}v+u_{xxyy}\right)\\&\quad+u_{yt}-2u_{xt}+2u_{xy}+5u_{yy}=0,\end{aligned}\tag{5.1}$$

where $v_x=u_{yy}$. This has a Hirota bilinear form

$$\left(D_x^4+2D_x^3D_y-3D_x^2D_y^2+D_yD_t-2D_xD_t+2D_xD_y+5D_y^2\right)f\cdot f=0$$

under the logarithmic transformations in (2.3). Upon further taking

$$a_1=2,\quad a_2=-2,\quad a_4=2,\quad a_5=2,\quad a_6=6,\quad a_8=5$$

the transformations in (2.3) with (3.1) present a pair of lump solutions to the first specific combined nonlinear equation (5.1):

$$\begin{aligned}u_1&=\frac{4(-110t+5x+10y+12)}{(-26t+x-2y+2)^2+(-42t+2x+6y+5)^2+55/8},\\v_1&=\frac{160}{(-26t+x-2y+2)^2+(-42t+2x+6y+5)^2+55/8}\\&\quad-\frac{32(-100t+5x+20y+13)}{[(-26t+x-2y+2)^2+(-42t+2x+6y+5)^2+55/8]^2}.\end{aligned}$$

Three three-dimensional plots and contour plots of the lump solution u_1 at three different times are made by using Maple in Fig. 1.

Let us second consider the case of $\delta_5=0$ and take

$$\alpha=1,\quad \alpha_2=2,\quad \alpha_3=-3,\quad \delta_1=1,\quad \delta_2=0,\quad \delta_3=2,\quad \delta_4=2,\quad \delta_5=5,$$

which leads to another specific combined nonlinear equation

$$\begin{aligned}&2(u_{xxxx}+6u_xu_{xx})-3[3(u_xu_y)_x+u_{xxxy}]\\&\quad-2(4u_yu_{xy}+u_xu_{yy}+u_{xx}v+u_{xxyy})\\&\quad+u_{yt}+2u_{xt}+u_{xy}+u_{tt}=0,\end{aligned}\tag{5.2}$$

where $v_x = u_{yy}$. This has a Hirota bilinear form

$$\left(2D_x^4 - 3D_x^3D_y - 2D_x^2D_y^2 + D_yD_t + 2D_xD_t + D_xD_y + D_t^2\right) f \cdot f = 0$$

under the logarithmic transformations in (2.3). Upon further taking

$$a_1 = 1, \quad a_3 = 2, \quad a_4 = 10, \quad a_5 = 3, \quad a_7 = -2, \quad a_8 = 5$$

the transformations in (2.3) with (3.1) present a pair of lump solutions to the second specific combined nonlinear equation (5.2):

$$\begin{aligned} u_2 =& \frac{4(-4t + 10x + 25)}{(2t + x - (24/5)y + 10)^2 + (-2t + 3x + (8/5)y + 5)^2 + 55/4}, \\ v_2 =& \frac{512}{5[(2t + x - (24/5)y + 10)^2 + (-2t + 3x + (8/5)y + 5)^2 + 55/4]} \\ & - \frac{512(-(8/5)t + (16/5)y - 5)^2}{[(2t + x - (24/5)y + 10)^2 + (-2t + 3x + (8/5)y + 5)^2 + 55/4]^2}. \end{aligned}$$

Similarly, three three-dimensional plots and density plots of the lump solution v_2 at three different times are made through Maple in Fig. 2.

6. Concluding Remarks

With Maple symbolic computation, we have computed two classes of lump solutions to a combined fourth-order nonlinear equation involving three types of nonlinear terms in (2+1)-dimensions. The computed lump solutions were explicitly presented in terms of the coefficients in the combined model equation. The presented results provide one new example of nonlinear equations in dispersive waves, which possess lump solutions. A few three-dimensional plots, contour plots and density plots of two specific lumps were made by using Maple plot tools.

We remark that the adopted ansatz on lump solutions is increasingly being used in computations, and all such obtained solutions provide valuable insights into related studies on soliton solutions and dromion-type solutions in soliton theory, generated through effective techniques including the Wronskian technique — cf. [40, 60], the generalised bilinear approach — cf. [22], Darboux transformations — cf. [61, 63, 69], — cf. [20, 29], the Riemann-Hilbert technique — cf. [28], symmetry reductions — cf. [8, 53], and symmetry constraints — cf. [19, 38] for the continuous case and [6, 35] for the discrete case.

We also remark that on one hand, many recent studies exhibit the striking richness of lump solutions to linear PDEs [29, 30, 33], besides nonlinear PDEs in (2+1)-dimensions [36, 45, 54, 56, 73, 80] and (3+1)-dimensions [7, 9, 12, 26, 48, 55, 65, 77, 78]. Based on the Hirota bilinear form and the generalised bilinear forms, some general formulations have also been established for lump solutions [2, 42, 43]. Different lump solutions also supplement the existing theories of solutions through other kinds of combinations [34, 52, 62, 81] and rogue wave ansätze [11, 57–59, 76], and can yield meaningful Lie-Bäcklund

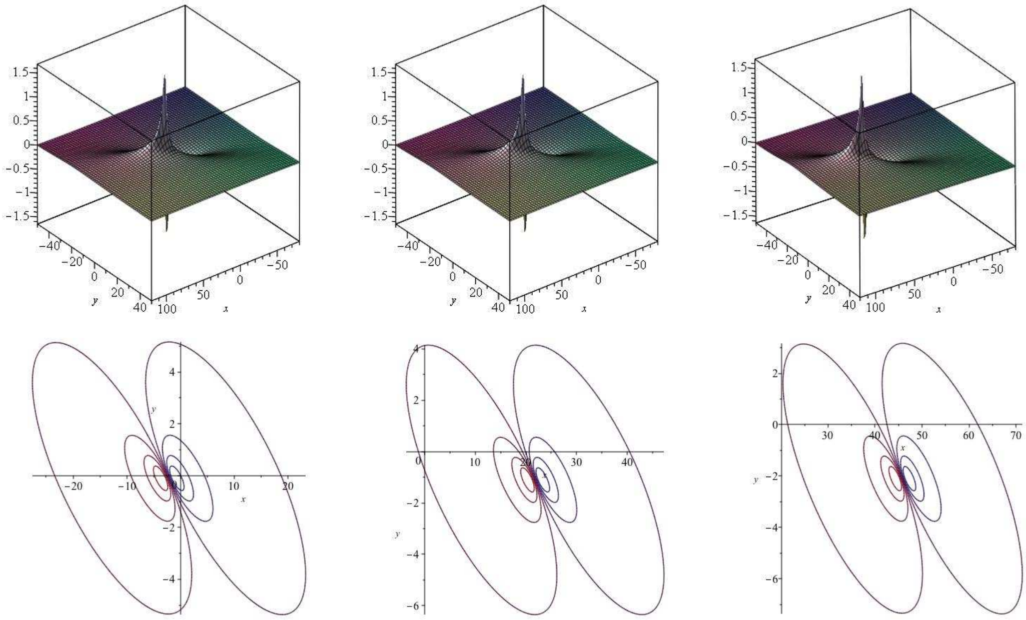

Figure 1: Profiles of u_1 when $t = 0, 1, 2$: 3d plots (top) and contour plots (bottom).

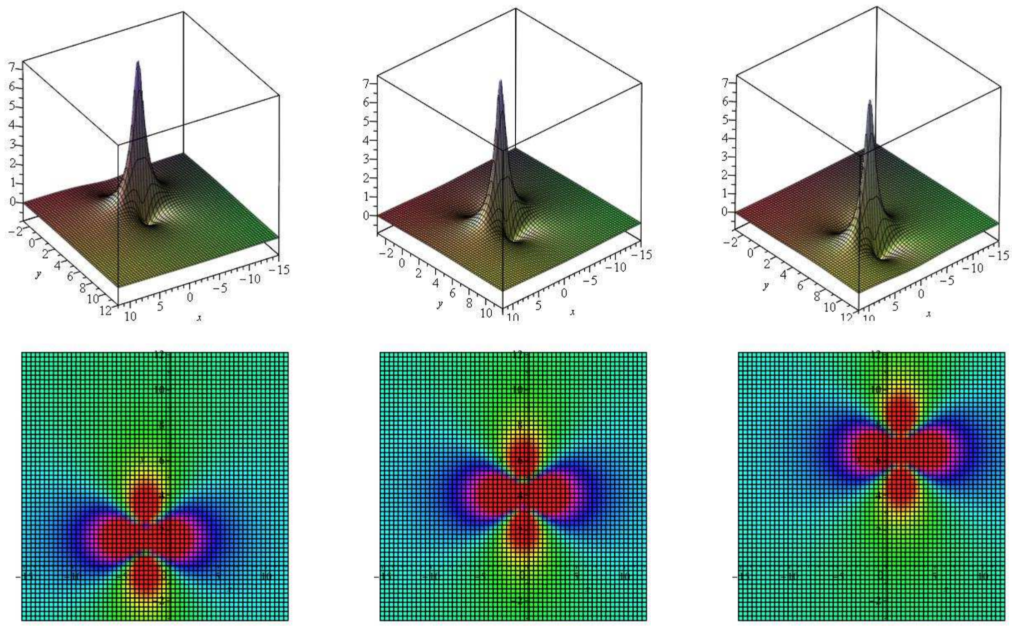

Figure 2: Profiles of v_2 when $t = 0, 5, 10$: 3d plots (top) and density plots (bottom).

symmetries, from taking derivatives with respect to some involved parameters. Further using those symmetries, one can formulate interesting conservation laws by working with adjoint symmetries [15, 25, 27]. On the other hand, various classes of interaction solutions between lumps and other kinds of dispersive waves [32, 39, 51, 70] have been computed for integrable equations in (2+1)-dimensions, and they can be classified into homoclinic interaction solutions [66–68] and heteroclinic interaction solutions [18, 50, 75, 79].

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