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Research paper

Abundant exact solutions to the discrete complex mKdV equation by Darboux transformation

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ABSTRACT

In this paper, an N-fold Darboux transformation is constructed for the discrete complex modified Korteweg-de Vries equation of focusing type, in terms of determinants. Through the obtained one-fold and two-fold Darboux transformations, a variety of new exact solutions, including an anti-dark soliton solution, a breather solution, a periodic solution, and a two-soliton solution, are derived from a nonzero constant and plane-wave seed solution. Via numerical simulation, a new kind of dynamical behavior of the two-soliton solution is exhibited, which tells that the two-soliton solution includes an anti-dark solitary wave and a w-shaped solitary wave.

1. Introduction

The complex modified Korteweg-de Vries (cmKdV) equation

$$q_\tau = q_{xxx} \pm 6|q|^2 q_x, \tag{1.1}$$

has a wide range of physical applications in the propagation transverse-magnetic waves in nematic optical fibers [1] and few-cycle optical pulses [2]. There are many other achievements for this model (1.1) including the inverse scattering transform, conserved quantities, stability of solitary wave solutions, numerical simulations, symmetry constraints, Darboux transformation (DT), and various kinds of solutions (see, e.g., [2–11]). Recently, the following discrete version of (1.1), proposed in Refs. [12,13],

$$\begin{aligned}\frac{dq_n}{dt} = {}& (1+\varepsilon|q_n|^2)\big[q_{n+2} - q_{N-2} + 2q_{N-1} - 2q_{n+1} + \varepsilon q_n(q_{N-1}^* q_{n+1} - q_{N-1}q_{n+1}^*) \\ & +\varepsilon q_n^*(q_{n+1}^2 - q_{N-1}^2) + \varepsilon|q_{n+1}|^2 q_{n+2} - \varepsilon|q_{N-1}|^2 q_{N-2}\big],\end{aligned} \tag{1.2}$$

has attracted a great deal of attention. Here q_n is a complex-valued function of a spatial integer variable $n \in \mathbb{Z}$ and a temporal continuous variable $t \in \mathbb{R}$, and q_n^* denotes the complex conjugate of q_n. The cases of $\varepsilon = \pm 1$ correspond to the focusing

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and defocusing cases, respectively. One can check that under the transformation

$$x = nh, \qquad \tau = 2h^3 t, \qquad q_n(t) = hq(x, \tau), \tag{1.3}$$

the discrete cmKdV Eq. (1.2) yields the cmKdV Eq. (1.1) with the 2-nd order preecision $O(h^2)$ as $h \to 0$.

The discrete cmKdV Eq. (1.2) can be viewed as the discrete coupled mKdV system researched in [14] by setting $q_n = a_n + ib_n$. Indeed, Eq. (1.2) is integrable in the sense of discrete Lax pairs obtained by using Ablowitz-Ladik's formulation [15]. For the discrete cmKdV$^+$ equation, namely, (1.2) with $\varepsilon = 1$, the authors of Ref. [16] have studies some types of exact solutions derived by an one-fold DT. When the spectrum parameter λ is a double eigenvalue, a w-shaped rational soliton and a rogue wave solution are obtained from a nonzero seed solution. For other cases of λ, a space-periodic breather solution has also been constructed.

DT is a powerful tool for constructing exact solutions to integrable systems, both in the continuous and discrete cases. Usually, N-soliton solutions can be presented in terms of particular determinants (see, e.g., [17]), including the Wronskian and Grammian determinants, through iterating a DT. In this paper, we will construct an N-fold DT in terms of determinants for the discrete cmKdV$^+$ Eq. (1.2). Through the obtained one-fold and two fold DTs, with a nonzero constant and plane-wave solution as a seed, a variety of new exact solutions including an anti-dark soliton, a breather solution, a periodic solution and a two-soliton solution, are derived. Via numerical simulation, a new dynamical property of the two-soliton solution generated from the two-fold DT is explored.

2. Darboux transformation in terms of determinants

In this section, we construct an N-fold DT by virtue of determinants for the discrete cmKdV$^+$ Eq. (1.2). Eq. (1.2) admits the following Lax pair

$$E\varphi_n = U_n \varphi_n, \quad \varphi_{n,t} = V_n \varphi_n, \quad \varphi_n = (\varphi_{n,1}, \varphi_{n,2})^T, \tag{2.1}$$

where the shift operator E is defined by $E\varphi_n = \varphi_{n+1}$, and the matrices U_n and V_n take the forms

$$\begin{aligned} U_n &= \begin{pmatrix} \lambda & q_n \\ -\varepsilon q_n^* & \lambda^{-1} \end{pmatrix}, \\ V_n &= \begin{pmatrix} A_n(\lambda, \lambda^{-1}, q_n) & B_n(\lambda, \lambda^{-1}, q_n) \\ -\varepsilon B_n(\lambda^{-1}, \lambda, q_n^*) & A_n(\lambda^{-1}, \lambda, q_n^*) \end{pmatrix}, \end{aligned} \tag{2.2}$$

with

$$\begin{aligned} A_n(\lambda, \lambda^{-1}, q_n) &= \frac{\lambda^4 - \lambda^{-4}}{2} + \lambda^2(\varepsilon q_n q_{N-1}^* - 1) - \lambda^{-2}(\varepsilon q_n^* q_{N-1} - 1) - 2\varepsilon q_n q_{N-1}^* + q_n^2 q_{N-1}^{*2} \\ &\quad + \varepsilon(1 + \varepsilon|q_{N-1}|^2) q_n q_{N-2}^* + \varepsilon(1 + \varepsilon|q_n|^2) q_{n+1} q_{N-1}^*, \\ B_n(\lambda, \lambda^{-1}, q_n) &= \lambda^3 q_n + \lambda^{-3} q_{N-1} + \lambda\left[(1 + \varepsilon|q_n|^2) q_{n+1} + \varepsilon q_n^2 q_{N-1}^* - 2q_n\right] \\ &\quad + \lambda^{-1}\left[(1 + \varepsilon|q_{N-1}|^2) q_{N-2} + \varepsilon q_n^* q_{N-1}^2 - 2q_{N-1}\right]. \end{aligned}$$

One can directly verify that the discrete zero curvature condition $U_{n,t} = (EV_n)U_n - U_n V_n$ of the linear spectral equations (2.1) yields (1.2).

The N-fold DT can be written as

$$\varphi_n[N] = T_n[N]\varphi_n, \tag{2.3}$$

where the Darboux matrix $T_n[N]$ is

$$T_n[N] = \begin{pmatrix} \lambda^N + \sum\limits_{k=1}^{N} T_{n,1}^{(N-2k)} \lambda^{N-2k} & \sum\limits_{k=1}^{N} T_{n,2}^{(N-2k+1)} \lambda^{N-2k+1} \\ (-1)^{N+1} \sum\limits_{k=1}^{N} T_{n,2}^{(N-2k+1)*} \lambda^{-N+2k-1} & (-1)^N \left(\lambda^{-N} + \sum\limits_{k=1}^{N} T_{n,1}^{(N-2k)*} \lambda^{-N+2k} \right) \end{pmatrix}. \tag{2.4}$$

Assuming that q_n is a solution of the discrete cmKdV$^+$ Eq. (1.2) and for $j = 1, 2, \ldots, N$, $\varphi_n^{(j)} = (\varphi_{n,1}^{(j)}, \varphi_{n,2}^{(j)})^T$ is an eigenfunction of the linear problems (2.1) with $\lambda = \lambda_j$, one can check that $\psi_n^{(j)}) = (\varphi_{n,2}^{(j)*}, -\varphi_{n,1}^{(j)*})^T$ is also the eigenfunction when $\lambda = (\lambda_j^*)^{-1}$. Furthermore, two column vectors in $T_n[N](\lambda_j)(\varphi_n^{(j)}, \psi_n^{(j)})$ are linearly dependent when $\det T_n[N](\lambda_j) = 0$. Therefore, $T_{n,1}^{(N-2k)}$ and $T_{n,2}^{(N-2k+1)}$ can be determined by

$$\left(\lambda_j^N + \sum_{k=1}^{N} T_{n,1}^{(N-2k)} \lambda_j^{N-2k} \right) \varphi_{n,1}^{(j)} + \left(\sum_{k=1}^{N} T_{n,2}^{(N-2k+1)} \lambda^{N-2k+1} \right) \varphi_{n,2}^{(j)} = 0,$$

$$\left((\lambda_j^*)^{-N} + \sum_{k=1}^{N} T_{n,1}^{(N-2k)} (\lambda_j^*)^{-N+2k}\right)\varphi_{n,2}^{(j)*} - \left(\sum_{k=1}^{N} T_{n,2}^{(N-2k+1)} (\lambda_j^*)^{-N+2k-1}\right)\varphi_{n,1}^{(j)*} = 0. \tag{2.5}$$

Under the above transformation (2.3), one can prove that the new linear problems

$$E\varphi_n[N] = U_n[N]\varphi_n[N], \quad \varphi_{n,t}[N] = V_n[N]\varphi_n[N], \tag{2.6}$$

where

$$U_n[N] = T_{n+1}[N]U_nT_n^{-1}[N], \quad V_n[N] = (T_{n,t}[N] + T_n[N]V_n)T_n^{-1}[N], \tag{2.7}$$

has the same form as the linear eigenfunction Eq. (2.1), except that q_n, q_n^* in U_n, V_n are replaced by $q_n[N], q_n^*[N]$ in $U_n[N]$, $V_n[N]$, respectively.

The relation between a new potential $q_n[N]$ and the old potential q_n is

$$q_n[N] = -q_nT_{n+1,1}^{(-N)} - T_{n+1,2}^{(-N+1)}, \tag{2.8}$$

where

$$T_{n,1}^{(-N)} = -\frac{\Omega_1[N]}{\Omega[N]}, \quad T_{n,2}^{(-N+1)} = -\frac{\Omega_2[N]}{\Omega[N]}, \tag{2.9}$$

with

$$\Omega[N] = \begin{vmatrix}
\lambda_1^{-N}\varphi_{n,1}^{(1)} & \lambda_1^{-N+1}\varphi_{n,2}^{(1)} & \lambda_1^{-N+2}\varphi_{n,1}^{(1)} & \lambda_1^{-N+3}\varphi_{n,2}^{(1)} & \cdots & \lambda_1^{N-2}\varphi_{n,1}^{(1)} & \lambda_1^{N-1}\varphi_{n,2}^{(1)} \\
\lambda_2^{-N}\varphi_{n,1}^{(2)} & \lambda_2^{-N+1}\varphi_{n,2}^{(2)} & \lambda_2^{-N+2}\varphi_{n,1}^{(2)} & \lambda_2^{-N+3}\varphi_{n,2}^{(2)} & \cdots & \lambda_2^{N-2}\varphi_{n,1}^{(2)} & \lambda_2^{N-1}\varphi_{n,2}^{(2)} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\lambda_N^{-N}\varphi_{n,1}^{(N)} & \lambda_N^{-N+1}\varphi_{n,2}^{(N)} & \lambda_N^{-N+2}\varphi_{n,1}^{(N)} & \lambda_N^{-N+3}\varphi_{n,2}^{(N)} & \cdots & \lambda_N^{N-2}\varphi_{n,1}^{(N)} & \lambda_N^{N-1}\varphi_{n,2}^{(N)} \\
(\lambda_1^*)^{N}\varphi_{n,2}^{(1)*} & -(\lambda_1^*)^{N-1}\varphi_{n,1}^{(1)*} & (\lambda_1^*)^{N-2}\varphi_{n,2}^{(1)*} & -(\lambda_1^*)^{N-3}\varphi_{n,1}^{(1)*} & \cdots & (\lambda_1^*)^{-N+2}\varphi_{n,2}^{(1)*} & -(\lambda_1^*)^{-N+1}\varphi_{n,1}^{(1)*} \\
(\lambda_2^*)^{N}\varphi_{n,2}^{(2)*} & -(\lambda_2^*)^{N-1}\varphi_{n,1}^{(2)*} & (\lambda_2^*)^{N-2}\varphi_{n,2}^{(2)*} & -(\lambda_2^*)^{N-3}\varphi_{n,1}^{(2)*} & \cdots & (\lambda_2^*)^{-N+2}\varphi_{n,2}^{(2)*} & -(\lambda_2^*)^{-N+1}\varphi_{n,1}^{(2)*} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
(\lambda_N^*)^{N}\varphi_{n,2}^{(N)*} & -(\lambda_N^*)^{N-1}\varphi_{n,1}^{(N)*} & (\lambda_N^*)^{N-2}\varphi_{n,2}^{(N)*} & -(\lambda_N^*)^{N-3}\varphi_{n,1}^{(N)*} & \cdots & (\lambda_N^*)^{-N+2}\varphi_{n,2}^{(N)*} & -(\lambda_N^*)^{-N+1}\varphi_{n,1}^{(N)*}
\end{vmatrix}.$$

It is noted that, here, the expression of $\Omega_1[N]$ and $\Omega_2[N]$ can be gotten easily, if one replaces $(\lambda_1^N\varphi_{n,1}^{(1)}, \lambda_2^N\varphi_{n,1}^{(2)}, \cdots, \lambda_N^N\varphi_{n,1}^{(N)}, (\lambda_1^*)^{-N}\varphi_{n,2}^{(1)*}, (\lambda_2^*)^{-N}\varphi_{n,2}^{(2)*}, \cdots, (\lambda_N^*)^{-N}\varphi_{n,2}^{(N)*})^{\mathrm{T}}$ for the first and second column in $\Omega[N]$, respectively.

Let us first give the one-, two-fold DT of the discrete cmKdV$^+$ Eq. (1.2). As $N=1$, the one-fold Darboux matrix $T_n[1]$ is written as

$$T_n[1] = \begin{pmatrix} \lambda_1 + T_{n,1}^{(-1)}\lambda_1^{-1} & T_{n,2}^{(0)} \\ T_{n,2}^{(0)*} & -(\lambda_1^{-1} + T_{n,1}^{(-1)*}\lambda_1) \end{pmatrix}, \tag{2.10}$$

where

$$T_{n,1}^{(-1)} = -\frac{\begin{vmatrix} \lambda_1\varphi_{n,1}^{(1)} & \varphi_{n,2}^{(1)} \\ (\lambda_1^*)^{-1}\varphi_{n,2}^{(1)*} & -\varphi_{n,1}^{(1)*} \end{vmatrix}}{\begin{vmatrix} \lambda_1^{-1}\varphi_{n,1}^{(1)} & \varphi_{n,2}^{(1)} \\ \lambda_1^*\varphi_{n,2}^{(1)*} & -\varphi_{n,1}^{(1)*} \end{vmatrix}}, \quad T_{n,2}^{(0)} = -\frac{\begin{vmatrix} \lambda_1^{-1}\varphi_{n,1}^{(1)} & \lambda_1\varphi_{n,1}^{(1)} \\ \lambda_1^*\varphi_{n,2}^{(1)*} & (\lambda_1^*)^{-1}\varphi_{n,2}^{(1)*} \end{vmatrix}}{\begin{vmatrix} \lambda_1^{-1}\varphi_{n,1}^{(1)} & \varphi_{n,2}^{(1)} \\ \lambda_1^*\varphi_{n,2}^{(1)*} & -\varphi_{n,1}^{(1)*} \end{vmatrix}}.$$

Suppose q_n is a solution and $\varphi_n^{(1)} = (\varphi_{n,1}^{(1)}, \varphi_{n,2}^{(1)})^T$ is an eigenfunction of the linear problems (2.1) with $\lambda = \lambda_1$. Then $\psi_n^{(1)} = (\varphi_{n,2}^{(1)*}, -\varphi_{n,1}^{(1)*})^T$ is also the eigenfunction when $\lambda = (\lambda_1^*)^{-1}$. Assume that $\det T_n[1](\lambda_1) = 0$, then two column vectors in $T_n[1](\lambda_1)(\varphi_n^{(1)}, \psi_n^{(1)})$ are linearly dependent. So $q_n[1]$ is exactly expressed as

$$q_n[1] = -q_n T^{(-1)}_{n+1,1} - T^{(0)}_{n+1,2} = \frac{\lambda_1(|\varphi^{(1)}_{n,1}|^2 + |\lambda_1|^2|\varphi^{(1)}_{n,2}|^2)}{\lambda_1^*(|\lambda_1|^2|\varphi^{(1)}_{n,1}|^2 + |\varphi^{(1)}_{n,2}|^2)} q_n + \frac{\lambda_1(|\lambda_1|^4 - 1)\varphi^{(1)}_{n,1}\varphi^{(1)*}_{n,2}}{\lambda_1^{*2}(|\lambda_1|^2|\varphi^{(1)}_{n,1}|^2 + |\varphi^{(1)}_{n,2}|^2)}. \tag{2.11}$$

By means of (2.8) in the form of fourth-order determinant as $N = 2$, we have

$$q_n[2] = -q_n T^{(-2)}_{n+1,1} - T^{(-1)}_{n+1,2} \tag{2.12}$$

where

$$T^{(-2)}_{n,1} = -\frac{\Omega_1[2]}{\Omega[2]}, \quad T^{(-1)}_{n,2} = -\frac{\Omega_2[2]}{\Omega[2]}, \tag{2.13}$$

with

$$\Omega[2] = \begin{vmatrix} \lambda_1^{-2}\varphi^{(1)}_{n,1} & \lambda_1^{-1}\varphi^{(1)}_{n,2} & \varphi^{(1)}_{n,1} & \lambda_1\varphi^{(1)}_{n,2} \\ \lambda_2^{-2}\varphi^{(2)}_{n,1} & \lambda_2^{-1}\varphi^{(2)}_{n,2} & \varphi^{(2)}_{n,1} & \lambda_2\varphi^{(2)}_{n,2} \\ \lambda_1^{*2}\varphi^{(1)*}_{n,2} & -\lambda_1^*\varphi^{(1)*}_{n,1} & \varphi^{(1)*}_{n,2} & -(\lambda_1^*)^{-1}\varphi^{(1)*}_{n,1} \\ \lambda_2^{*2}\varphi^{(2)*}_{n,2} & -\lambda_2^*\varphi^{(2)*}_{n,1} & \varphi^{(2)*}_{n,2} & -(\lambda_2^*)^{-1}\varphi^{(2)*}_{n,1} \end{vmatrix},$$

and

$$\Omega_1[2] = \begin{vmatrix} \lambda_1^{2}\varphi^{(1)}_{n,1} & \lambda_1^{-1}\varphi^{(1)}_{n,2} & \varphi^{(1)}_{n,1} & \lambda_1\varphi^{(1)}_{n,2} \\ \lambda_2^{2}\varphi^{(2)}_{n,1} & \lambda_2^{-1}\varphi^{(2)}_{n,2} & \varphi^{(2)}_{n,1} & \lambda_2\varphi^{(2)}_{n,2} \\ (\lambda_1^*)^{-2}\varphi^{(1)*}_{n,2} & -\lambda_1^*\varphi^{(1)*}_{n,1} & \varphi^{(1)*}_{n,2} & -(\lambda_1^*)^{-1}\varphi^{(1)*}_{n,1} \\ (\lambda_2^*)^{-2}\varphi^{(2)*}_{n,2} & -\lambda_2^*\varphi^{(2)*}_{n,1} & \varphi^{(2)*}_{n,2} & -(\lambda_2^*)^{-1}\varphi^{(2)*}_{n,1} \end{vmatrix},$$

$$\Omega_2[2] = \begin{vmatrix} \lambda_1^{-2}\varphi^{(1)}_{n,1} & \lambda_1^{2}\varphi^{(1)}_{n,1} & \varphi^{(1)}_{n,1} & \lambda_1\varphi^{(1)}_{n,2} \\ \lambda_2^{-2}\varphi^{(2)}_{n,1} & \lambda_2^{2}\varphi^{(2)}_{n,1} & \varphi^{(2)}_{n,1} & \lambda_2\varphi^{(2)}_{n,2} \\ \lambda_1^{*2}\varphi^{(1)*}_{n,2} & (\lambda_1^*)^{-2}\varphi^{(1)*}_{n,2} & \varphi^{(1)*}_{n,2} & -(\lambda_1^*)^{-1}\varphi^{(1)*}_{n,1} \\ \lambda_2^{*2}\varphi^{(2)*}_{n,2} & (\lambda_2^*)^{-2}\varphi^{(2)*}_{n,2} & \varphi^{(2)*}_{n,2} & -(\lambda_2^*)^{-1}\varphi^{(2)*}_{n,1} \end{vmatrix}.$$

3. Exact solutions from one-fold DT

In this section, we construct a few types of exact solutions through the one-fold DT obtained in Section 2.

3.1. Solutions with a constant background

For the seed solution $q_n = \rho$, $\rho \in \mathbb{R}$, solving the linear spectral equations (2.2) yields

$$\varphi^{(1)}_{n,1} = e^{-\frac{a\Delta\left(1+\lambda_1^2\right)t}{2\lambda_1^4} + 3\rho^4 t}\left(A^n + e^{\frac{a\Delta\left(1+\lambda_1^2\right)t}{\lambda_1^4}} B^n\right), \tag{3.1}$$

$$\varphi^{(1)}_{n,2} = \frac{1}{2\lambda_1\rho} e^{-\frac{a\Delta\left(1+\lambda_1^2\right)t}{2\lambda_1^4} + 3\rho^4 t}\left(A^n(1 - \Delta - \lambda_1^2) + e^{\frac{a\Delta\left(1+\lambda_1^2\right)t}{\lambda_1^4}} B^n(1 + \Delta - \lambda_1^2)\right), \tag{3.1}$$

where

$$A = \frac{1 - \Delta + \lambda_1^2}{2\lambda_1}, \quad B = \frac{1 + \Delta + \lambda_1^2}{2\lambda_1},$$
$$\Delta = \sqrt{(\lambda_1^2 - 1)^2 - 4\rho^2\lambda_1^2}, \quad a = (\lambda_1^2 - 1)^2 + 2\rho^2\lambda_1^2. \tag{3.2}$$

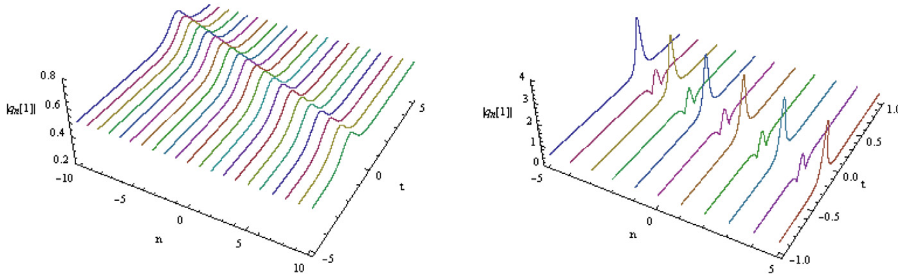
To illustrate the dynamical behavior of $q_n[1]$, we consider the following special cases.

Case 1: Anti-dark soliton and breather solution

Suppose that $|\lambda_1^2 - 1| > 2|\rho\lambda_1|$. As $A > 0$, $B > 0$, $\lambda_1 \in R$, inserting (3.1) into Eq. (2.11), we get an anti-dark soliton solution

$$q_n[1] = -\rho\frac{(\lambda_1^2 + 1)(\Delta^2 + 2\rho^2\lambda_1^2) + (\lambda_1^2 - 1)\left((\lambda_1^2 + 2\rho^2\lambda_1^2 - 1)\cosh\xi - \Delta\sinh\xi\right)}{2\rho^2\lambda_1^2(\lambda_1^2 + 1) + (\lambda_1^2 - 1)\left((\lambda_1^2 + 2\rho^2\lambda_1^2 - 1)\cosh\xi - \Delta\sinh\xi\right)}, \tag{3.3}$$

where $\xi = n\ln\frac{B}{A} + \frac{a\Delta(1+\lambda_1^2)}{\lambda_1^4}t$. This means that $q_n[1]$ is a solitary wave with the velocity $v = -\frac{a\Delta(1+\lambda_1^2)}{\lambda_1^4\ln\frac{B}{A}}$, which travels to the left. The amplitude is $|q_n[1]|_{max} = \frac{(\lambda_1^4-1)\sqrt{1+\rho^2}-\rho(\lambda_1^4+1)}{2\lambda_1^2}$. The dynamical profile of this anti-dark soliton solution is described in Fig. 1(a) when $\lambda_1 = \sqrt{3}$, $\rho = 1/2$.



(a) Anti-dark soliton solution (b) Breather solution

Fig. 1. (a): Anti-dark soliton solution (3.3) with $\lambda_1=\sqrt{3}$, $\rho=1/2$; (b): Breather solution (3.4) with $\lambda_1=2i$, $\rho=1/2$.

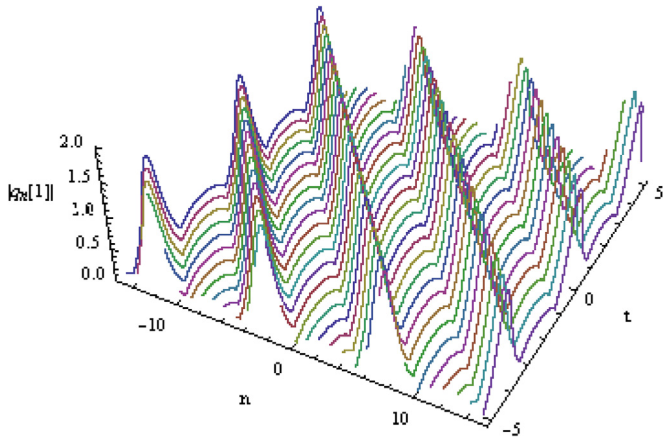


Fig. 2. Periodic solution (3.5) with $\lambda_1=\sqrt{2}$, $\rho=1/2$.

As $\lambda_1\in C$ and Im $A>0$. Let $\lambda_1=il \quad (l\neq 0,1)$. By using (2.11) and (3.1), a breather solution is given by

$$q_n[1]=\rho\frac{(l^2-1)((\Delta^2-2l^2\rho^2)\cos(n\pi)+i(l^2+1)\Delta\sin(n\pi))-(l^2+1)(\delta\cosh\xi+\Delta\sinh\xi)}{2\rho^2l^2(l^2-1)\cos(n\pi)+(l^2+1)(\delta\cosh\xi+\Delta\sinh\xi)}, \tag{3.4}$$

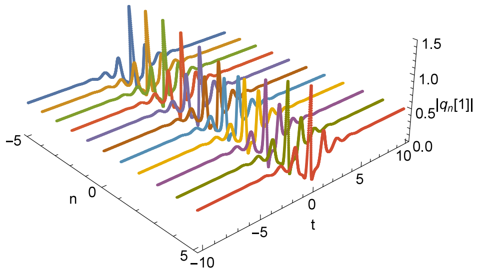
where $\xi=n\ln|\frac{B}{A}|+\frac{a\Delta(1+\lambda_1^2)}{\lambda_1^4}t$ and $\delta=l^2+2\rho^2l^2+1$. The dynamical profile of this breather solution (3.4) is shown in Fig. 1(b) for the parameter $\lambda_1=2i$, $\rho=1/2$. We can see that when n is odd, i.e., $n=2m+1, m\in\mathbb{Z}$, we get $|q_n[1]|_{max}=\frac{(b^4-1)\sqrt{1+\rho^2}+\rho(b^4+1)}{2b^2}$. When n is even, i.e., $n=2m$, there exist a local maximum values $|q_n[1]|=\frac{-(b^4-1)\sqrt{1+\rho^2}+\rho(b^4+1)}{2b^2}$ and two local minimum values that approach zero.

Case 2: Periodic solution

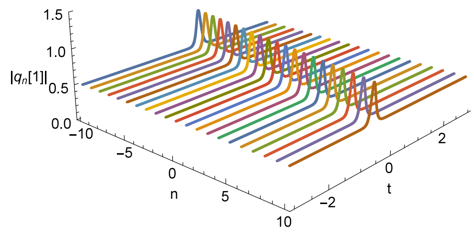
Suppose that $|\lambda_1^2-1|<2|\rho\lambda_1|$ and $\lambda_1\in R$. For this case, let $\Delta=i\Omega$, where $\Omega=\sqrt{4\rho^2\lambda_1^2-(\lambda_1^2-1)^2}$, $\eta=n\arg A-\frac{a\Omega(1+\lambda_1^2)}{2\lambda_1^4}t$ and $\arg A=-\arctan\frac{\Omega}{1+\lambda_1^2}$. Using (2.11) and (3.1), we obtain a periodic solution

$$q_n[1]=\rho\frac{(\lambda_1^2+1)(\Omega^2-2\rho^2\lambda_1^2)+(\lambda_1^2-1)\big((\lambda_1^2+2\rho^2\lambda_1^2-1)\cos(2\eta)-\Delta\sin(2\eta)\big)}{2\rho^2\lambda_1^2(\lambda_1^2+1)+(\lambda_1^2-1)\big((\lambda_1^2+2\rho^2\lambda_1^2-1)\cos(2\eta)-\Delta\sin(2\eta)\big)}, \tag{3.5}$$

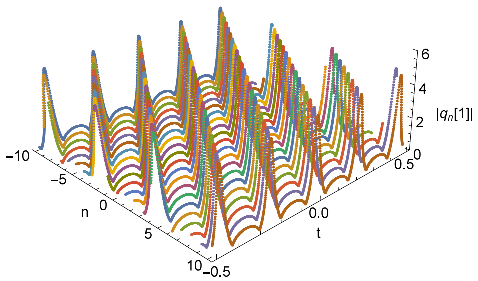
with the period $T_{space}=\frac{\pi}{|\arg A|}$ and $T_{time}=\frac{2\lambda_1^4\pi}{a\Omega(\lambda_1^2+1)}$ in space and time, respectively. When $\lambda_1=\sqrt{2}$, $\rho=1/2$, the dynamical profile of periodic solution (3.5) is shown in Fig. 2. $T_{space}=\pi/\arctan\frac{1}{3}\approx 9.76406$, $T_{time}=\frac{4\pi}{3}\approx 4.18879$.



(a) Breather solution



(b) Anti-dark soliton solution



(c) Periodic solution

Fig. 3. (a): Breather solution of discrete cmKdV$^+$ Eq. (1.2) with $k=\pi/2, \lambda_1=1+i, \rho=1/2, \alpha_1=1, \beta_1=1$; (b): Anti-dark soliton solution with $k=\pi, \lambda_1=2i, \rho=1/2, \alpha_1=1, \beta_1=1$; (c) Periodic solution with $k=\pi, \lambda_1=2i, \rho=1, \alpha_1=1, \beta_1=1$.

3.2. Solutions with a plane-wave background

Let us start with a plane-wave solution $q_n=\rho e^{i(kn+wt)}$, where the dispersion relation satisfies $w=4(1+\rho^2)[-1+(3\rho^2+1)\cos k]\sin k$, $\rho\in R$. Solving the linear spectral equations (2.2), we get eigenfunctions

$$\begin{aligned}\varphi_{n,1}^{(1)}&=e^{-\frac{c(a\Delta-\lambda_1^4 b)}{2\lambda_1^4}t}\left(\alpha_1 A^n+\beta_1 e^{\frac{ac\Delta}{\lambda_1^4}t}B^n\right),\\ \varphi_{n,2}^{(1)}&=\frac{1}{2\lambda_1\rho}e^{-ikN-\frac{c(a\Delta+\lambda_1^4 d)}{2\lambda_1^4}t}\left(\alpha_1 A^n(e^{ik}-\Delta-\lambda_1^2)+\beta_1 e^{\frac{ac\Delta}{\lambda_1^4}t}B^n(e^{ik}+\Delta-\lambda_1^2)\right),\end{aligned}\tag{3.6}$$

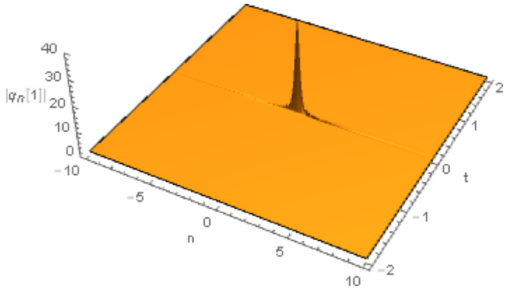
where

$$\begin{aligned}&A=\frac{e^{ik}-\Delta+\lambda_1^2}{2\lambda_1},\quad B=\frac{e^{ik}+\Delta+\lambda_1^2}{2\lambda_1},\quad \Delta=\sqrt{(\lambda_1^2-e^{ik})^2-4\rho^2\lambda_1^2e^{ik}},\\ &a=e^{ik}\left(1-2\lambda_1^2\right)+e^{2ik}\lambda_1^4\left(-2+\lambda_1^2\right)+\lambda_1^2\left(1+2\rho^2\right)(1+\lambda_1^2e^{3ik}),\\ &b=e^{4ik}(1+6\rho^2+6\rho^4)-(1+2e^{3ik})(1+2\rho^2)+2e^{ik},\quad c=e^{-2ik},\\ &d=e^{ik}(1+2\rho^2)(2+e^{3ik})-2e^{3ik}-(1+6\rho^2+6\rho^4).\end{aligned}$$

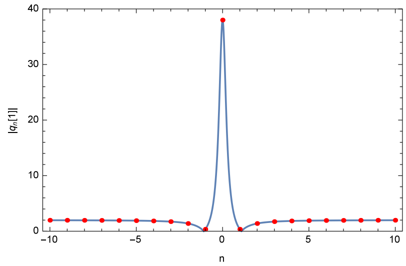
Then, substituting (3.6) into (2.11), we can derive $q_n[1]$ of the discrete cmKdV$^+$ equation. Here we omit the expression because it is too long and complicated. For this case, assume that $\Lambda=(\lambda_1^2-e^{ik})^2-4\rho^2\lambda_1^2e^{ik}$. When $\Lambda>0$, i.e., $|\lambda_1^2-e^{ik}|>2|\rho\lambda_1e^{ik/2}|$, if we choose $k=\pi/2, \lambda_1=1+i, \rho=1/2, \alpha_1=1, \beta_1=1$, a breather solution of the discrete cmKdV$^+$ Eq. (1.2) is obtained. If the parameters is given by $k=\pi, \lambda_1=2i, \rho=1/2, \alpha_1=1, \beta_1=1$, we get an anti-dark soliton solution. When $\Lambda<0$, i.e., $|\lambda_1^2-e^{ik}|<2|\rho\lambda_1e^{ik/2}|$, if we take $k=\pi, \lambda_1=2i, \rho=1, \alpha_1=1, \beta_1=1$, a periodic solution is derived. The dynamical profiles of these two solutions are displayed in Fig. 3.

Remark 1. We also can construct rogue wave solutions by the Darboux transformation. For example, we have

$$q_n[1]=\rho e^{i(kn+wt+k)}\left(1-\frac{2\rho^3}{\rho+\sqrt{1+\rho^2}}\frac{A_n}{B_n}\right),\tag{3.7}$$



(a)



(b)

Fig. 4. The rogue wave solution $q_n[1]$ with the parameters $\rho = 2, k = 1, s = -\frac{2+\sqrt{5}}{4}$.

where

$$\begin{aligned}
A_n = {} & 2n((1+\rho^2)(1+4(s+1)\rho^2) + \rho\sqrt{1+\rho^2}(1+2(s+1)(1+2\rho^2))) + (1+\rho^2)(1+2\rho(\rho+(1+\sqrt{1+\rho^2})) \\
& + 2\rho(\sqrt{1+\rho^2} + 2\rho(1+\rho^2+\rho\sqrt{1+\rho^2}))(n^2+s^2) + 2s(\rho(3+4\rho^2)\sqrt{1+\rho^2} + (1+\rho^2)(1+4\rho^2)) \\
& + 8(1+\rho^2)\Big[((1+\rho^2)(1+4(n+s+1)\rho^2) + \rho\sqrt{1+\rho^2}(1+2(n+s+1)(1+2\rho^2))) \\
& (-\cos k + (1+3\rho^2\cos 2k)) - i\rho(\sqrt{1+\rho^2} + 2\rho(1+\rho^2+\rho\sqrt{1+\rho^2}))(-1+2(2+3\rho^2)\cos k)\sin k\Big]t \\
& + 16\rho(1+\rho^2)(\sqrt{1+\rho^2} + 2\rho(1+\rho^2+\rho\sqrt{1+\rho^2}))\big[-2(1+6\rho^2(1+\rho^2))\cos k + \cos 2k \\
& + (1+2\rho^2)(2+9\rho^2(1+\rho^2) - 2\cos 3k) + (1+3\rho^2(1+\rho^2))\cos 4k\big]t^2,
\end{aligned}$$

$$\begin{aligned}
B_n = {} & (1+2\rho^2(n+s+1)^2)(1+\rho^2) + 2(n+s+1)\rho\sqrt{1+\rho^2}(1+(n+s+1)\rho^2) \\
& + 8\rho(1+\rho^2)\Big(\sqrt{1+\rho^2} + 2\rho(1+\rho^2+\rho\sqrt{1+\rho^2})(n+s+1)\Big)((1+3\rho^2)\cos 2k - \cos k)t \\
& + 16\rho^2(1+\rho^2)(1+\rho^2+\rho\sqrt{1+\rho^2})\big[-2(1+6\rho^2(1+\rho^2))\cos k + \cos 2k \\
& + (1+2\rho^2)(2+9\rho^2(1+\rho^2) - 2\cos 3k) + (1+3\rho^2(1+\rho^2))\cos 4k\big]t^2.
\end{aligned}$$

If we choose the parameters $\rho = 2, k = 1, s = -\frac{2+\sqrt{5}}{4}$, this rogue wave solution is exhibited in Fig. 4. The hight peak is $|q_n[1]|_{max} = \rho(3+4\rho^2) = 38$ and the lowest is $|q_n[1]|_{min} = 0.352941$. The peak amplitude of the rogue wave solution is at least three times the background ρ.

4. Two-soliton solution from two-fold DT

In this section, we study the dynamical behavior of the exact solution $q_n[2]$ (2.12) of the discrete cmKdV$^+$ Eq. (1.2) via numerical simulations. Because the expression of $q_n[2]$ is very long and complicated, we consider a constant background, i.e., a seed solution $q_n = \rho, \rho \in \mathbb{R}$. We take the eigenfunctions $\varphi_{n,k}^{(j)} (k, j = 1, 2)$ as follows

$$\begin{aligned}
\varphi_{n,1}^{(j)} &= e^{-\frac{a_j\Delta_j(1+\lambda_j^2)t}{2\lambda_j^4} + 3\rho^4 t}\left(\alpha_j A_j^n + \beta_j e^{\frac{a_j\Delta_j(1+\lambda_j^2)t}{\lambda_j^4}} B_j^n\right), \\
\varphi_{n,2}^{(j)} &= \frac{1}{2\lambda_j\rho} e^{-\frac{a_j\Delta_j(1+\lambda_j^2)t}{2\lambda_j^4} + 3\rho^4 t}\left(\alpha_j A_j^n(1-\Delta_j-\lambda_j^2) + \beta_j e^{\frac{a_j\Delta_j(1+\lambda_j^2)t}{\lambda_j^4}} B_j^n(1+\Delta_j-\lambda_j^2)\right),
\end{aligned} \tag{4.1}$$

where A_j, B_j, Δ_j, a_j is given by (3.2) with the corresponding spectrum parameter λ_j. Then, upon inserting (4.1) into (2.12) and (2.13), the obtained exact solution $q_n[2]$ of the discrete cmKdV$^+$ equation presents a two-soliton solution.

As the parameters are $\lambda_1 = \sqrt{3}, \lambda_2 = 2, \rho = 1/2, \alpha_1 = 1, \beta_1 = 1, \alpha_2 = 1, \beta_2 = 1,$ the discrete two-soliton solution $q_n[2]$ contains a anti-dark solitary wave (ξ_1 wave) and a w-shaped solitary wave (ξ_2 wave), which is depicted in Fig. 5. This is a new and interesting property of the discrete nonlinear cmKdV$^+$ equation. It is shown that $q_n[2]$ consists of two left travelling solitary waves with the velocities $v_j = -\frac{a_j\Delta_j(1+\lambda_j^2)}{\lambda_j^4 \ln\frac{B_1}{A_1}} (j = 1, 2)$. For the above parameters, the velocities of the solitary waves ξ_1 and ξ_2 are $v_1 \approx 4.785$ and $v_2 \approx 7.987$. Fig. 6 describes the evolution of the two-soliton solution $|q_n[2]|$ of the

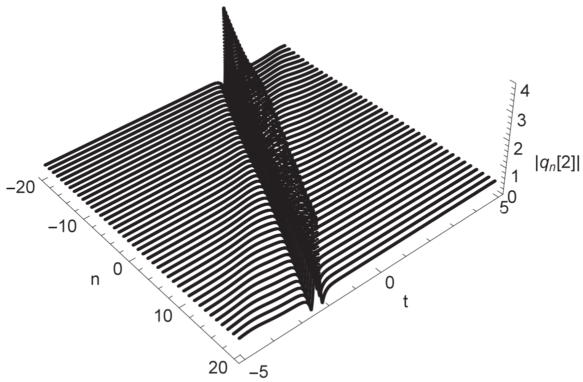
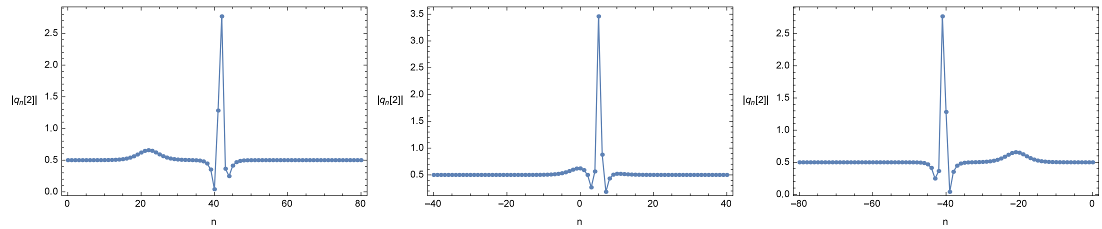


Fig. 5. Two-soliton solution $q_n[2]$ with $\lambda_1 = \sqrt{3}, \lambda_2 = 2, \rho = 1/2, \alpha_1 = 1, \beta_1 = 1, \alpha_2 = 1, \beta_2 = 1$.



(a) $t = -5$ (b) $t = -1/2$ (c) $t = 5$

Fig. 6. Evolution plot of two-soliton solution $|q_n[2]|$ with parameters $\lambda_1 = \sqrt{3}, \lambda_2 = 2, \rho = 1/2, \alpha_1 = 1, \beta_1 = 1, \alpha_2 = 1, \beta_2 = 1$. It consist of two left travelling waves, keeping velocities and amplitudes unchanged.

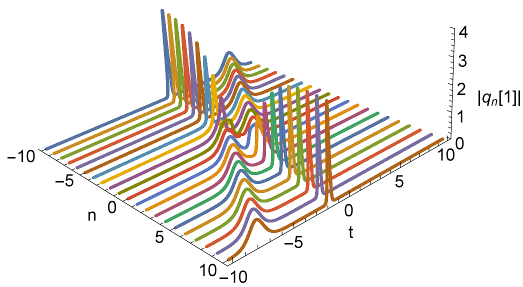


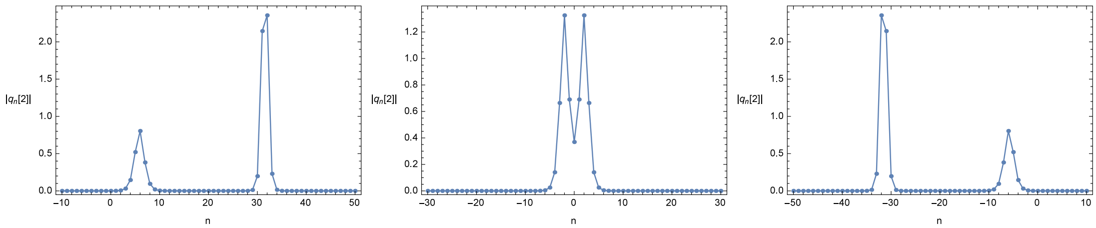
Fig. 7. Two-soliton solution $q_n[2]$ with $\lambda_1 = \frac{2}{3}, \lambda_2 = 2, c_1^{(1)} = 1, c_2^{(1)} = \frac{2}{3}, c_1^{(2)} = 1, c_2^{(2)} = 2$.

discrete cmKdV$^+$ Eq. (1.2). We can see that it presents an elastic collision. The amplitudes of the solitary wave ξ_1 and ξ_2 are $|\xi_1| = \frac{(\lambda_1^4-1)\sqrt{1+\rho^2}-\rho(\lambda_1^4+1)}{2\lambda_1^2} \approx 0.657$ and $|\xi_2| \approx 2.733$.

Suppose that $\rho = 0$, and the eigenfunction $\varphi_{n,k}^{(j)}(k, j = 1, 2)$ in (4.1) reduces to

$$\varphi_{n,1}^{(j)} = c_1^{(j)} e^{\xi(\lambda_j)}, \qquad \varphi_{n,2}^{(j)} = c_2^{(j)} e^{-\xi(\lambda_j)}, \tag{4.2}$$

where $\xi(\lambda_j) = n\ln(\lambda_j) + W(\lambda_j)t, W(\lambda_j) = \frac{\lambda_j^4-\lambda_j^{-4}}{2} - \lambda_j^2 + \lambda_j^{-2}$. By using the two-fold DT, we get the two-soliton solution $q_n[2]$. If we choose parameters $\lambda_1 = \frac{2}{3}, \lambda_2 = 2, c_1^{(1)} = 1, c_2^{(1)} = \frac{2}{3}, c_1^{(2)} = 1, c_2^{(2)} = 2$, where $c_2^{(j)} = \lambda_j c_1^{(j)}$, the two-soliton solution $q_n[2]$ is shown in Fig. 7. The velocities are $v_j = -\frac{W(\lambda_j)}{\ln(\lambda_j)}(j = 1, 2)$, which implies that this two-soliton consists of two left travelling waves. We can see it possesses an elastic interaction property. Fig. 8 describes the evolution of the two-soliton solution $|q_n[2]|$ of the discrete cmKdV$^+$ Eq. (1.2).



(a) $t=-5$ (b) $t=0$ (c) $t=5$

Fig. 8. Evolution plot of two-soliton solution $|q_n[2]|$.

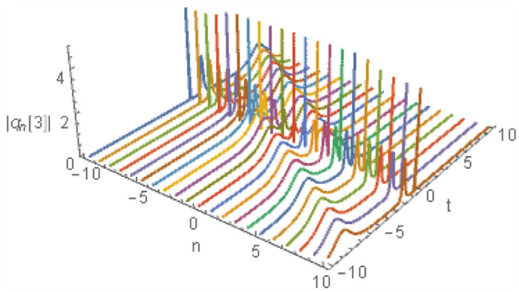
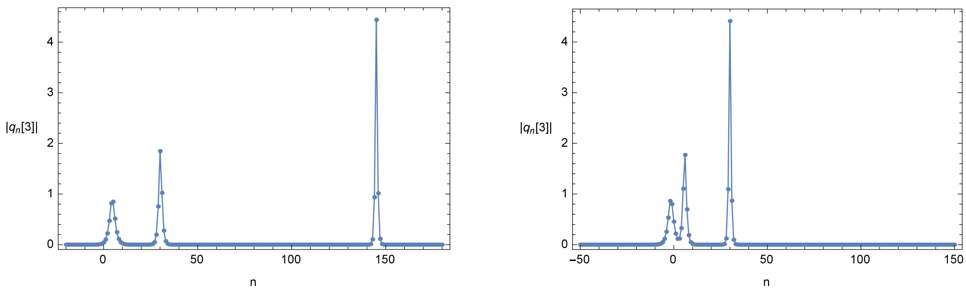
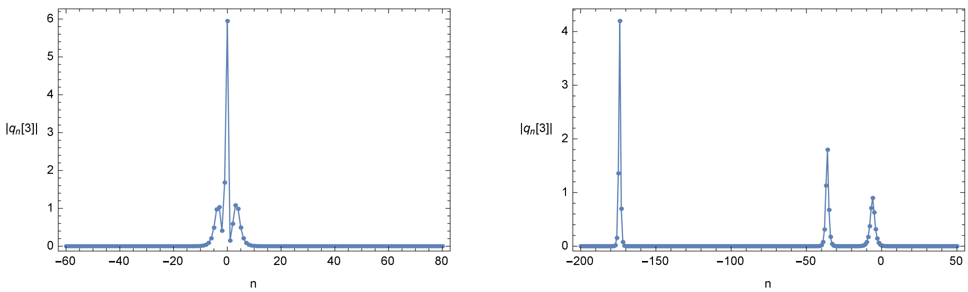


Fig. 9. Three-soliton solution $q_n[3]$ with $\lambda_1=\frac{2}{3}, \lambda_2=2, \lambda_3=3, c_1^{(1)}=1, c_2^{(1)}=\frac{2}{3}, c_1^{(2)}=1, c_2^{(2)}=2, c_1^{(3)}=1, c_2^{(3)}=2$.



(a) $t=-5$ (b) $t=-1$



(c) $t=0$ (d) $t=6$

Fig. 10. Evolution plot of three-soliton solution $|q_n[3]|$.

Remark 2. Similarly, we can analyze the the dynamic property of three-soliton solution by the Darboux transformation. However, the exact expression of $q_n[3]$ is omitted due to its complexity. For example, we can see that the three-soliton solution displayed in Figs. 9 and 10 is a left travelling wave and keeps elastic collision.

5. Conclusions and discussions

In this paper, we have constructed the N-fold DT in terms of determinants for the integrable discrete cmKdV$^+$ equation. The obtained N-fold DT (2.9) amends the results in Ref. [16]. Through one-fold DT, we have derived a few kinds of new exact solutions, including the anti-dark soliton solutions, the breather solutions and the periodic solutions, from a nonzero constant and plane-wave seed solution. We have also studied the dynamical property of the two-soliton solution via numerical simulation, and showed that the two-soliton solution includes an anti-dark solitary wave and a w-shaped solitary wave, a new and interesting solution phenomenon for the discrete nonlinear cmKdV$^+$ equation. Through some complicated and tedious computation, we can present higher-order soliton solutions in terms of determinants analogously.

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